Null controllability of a highly coupled fourth-order parabolic system with one internal control[*]

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Abstract

Controlling systems of partial differential equations with fewer controls than states has been a central topic in the last decades. The nature of the equations and existing couplings are key factors to consider. Even if we can find several works for parabolic systems, there are only a few for fourth-order equations. In this paper, we use the fictitious control method and the algebraic resolution method, as introduced by Coron and Lissy, in order to prove the null controllability of a highly coupled fourth-order parabolic system with only one control input.

Keywords: Controllability, fictitious control method, parabolic system

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Contents

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1 Introduction

During the last decades, the control community has been interested in studying control properties of coupled systems of partial differential equations, paying special attention to the underactuated case, where there are fewer controls than states. A fairly complete study has been performed for second-order parabolic systems, considering different couplings with internal or boundary controls [1, 20, 21, 11]. Still concerning parabolic systems, an important body of work has dealt with the Stokes and Navier-Stokes equations. Four equations (counting the divergence-free condition) can be controlled with three, two, or just one input, depending on the systems [23, 10, 4, 32]. Roughly speaking, the main strategy applied for internal control has been first to reduce by duality the null controllability to an observability inequality. Next, to apply a Carleman inequality to each state of the system and summing them up. Then, to try to eliminate the undesirable observations by using the equations of the system. A key breakthrough has been introduced in [12] by Coron and Lissy using the algebraic resolution method. This approach starts once a control result with the same number of controls as states is already proved. Then, by using algebraic relations coming from the equations themselves, the idea is getting ride of the undesirable components of the control. This strategy, introduced in [12] for a Navier-Stokes system, has also been applied in [13, 14, 30, 31] for other parabolic systems and in [28, 15] for other equations.

In this paper, we are interested in the controllability of fourth-order parabolic systems. Let us recall the known results in this context. The controllability of this kind of equations has been studied in [5, 6] when the control acts on the boundary and in [7, 19, 6, 24] for internal controls. All these papers are in dimension one for the space variable. The first results in higher dimensions are [22], [25]. Regarding the control of systems involving fourth-order parabolic equations, the first references are [9, 2] for internal controls and [8, 3, 26] for boundary controls.

Let us now focus on the specific problem that we are dealing with in this work. This is the control of a system that couples two fourth-order parabolic equations. Let $T > 0$, $L > 0$ and ω be a non-empty open subset of $(0, L)$. We introduce the following sets, $Q = (0, L) \times (0, T)$ and $Q_\omega = \omega \times (0, T)$. Our problem is to prove the null controllability of the following parabolic system of two equations coupled up to third-order terms with coupling coefficients depending on space and time and with only one internal control acting on the first equation. The system is given by

$$\begin{cases} \partial_t u_1 + \partial_x^4 u_1 = \displaystyle\sum_{j=1}^{2} (a_{1j}\partial_x^3 u_j + b_{1j}\partial_x^2 u_j + c_{1j}\partial_x u_j + d_{1j} u_j) + 1_\omega h, & \text{in } Q, \\ \partial_t u_2 + \partial_x^4 u_2 = \displaystyle\sum_{j=1}^{2} (a_{2j}\partial_x^3 u_j + b_{2j}\partial_x^2 u_j + c_{2j}\partial_x u_j + d_{2j} u_j), & \text{in } Q, \\ (u_1, u_2)(0, t) = (u_1, u_2)(L, t) = 0, & \text{in } (0, T), \\ \partial_x (u_1, u_2)(0, t) = \partial_x (u_1, u_2)(L, t) = 0, & \text{in } (0, T), \\ (u_1, u_2)(x, 0) = (u_1, u_2)^0(x), & \text{in } (0, L), \end{cases}$$

where $(u_1, u_2)^0 \in L^2(0, L)^2$ is the initial condition and $h \in L^2(Q)$ is the control. For any $i, j = 1, 2$, the coupling terms $(a_{ij}, b_{ij}, c_{ij}, d_{ij})$ are assumed to be in $C_c^\infty(Q)$. We can rewrite this system in a matrix form

$$\begin{cases} \partial_t U + \partial_x^4 U = A\partial_x^3 U + B\partial_x^2 U + C\partial_x U + DU + 1_\omega G h, & \text{in } Q, \\ U(0, t) = U(L, t) = 0, & \text{in } (0, T), \\ \partial_x U(0, t) = \partial_x U(L, t) = 0, & \text{in } (0, T), \\ U(x, 0) = U^0(x), & \text{in } (0, L), \end{cases} \tag{1.1}$$

with $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $A = (a_{ij})_{1\leq i,j\leq 2}$, $B = (b_{ij})_{1\leq i,j\leq 2}$, $C = (c_{ij})_{1\leq i,j\leq 2}$, $D = (d_{ij})_{1\leq i,j\leq 2}$ and $G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where (A, B, C, D) are supposed to be 2×2 matrices with coefficients in $C_c^\infty(Q)$.
In this paper, we are able to get the internal null controllability of this system. As our control input acts on the first equation only, we will require a condition on the coefficients of the system to get our result. The precise condition needs more context and notation to be clearly stated, and thus we only mention it in our main result by making reference to Section 5.

Theorem 1 *Let $T > 0$, $L > 0$ and ω be a non-empty open subset of $(0, L)$. Let assume that the condition (5.3) in Section 5 holds. For any $U^0 \in L^2(Q)^2$, there exists a control $h \in L^2(Q)$ such that the solution to (1.1) satisfies $U(\cdot, T) = 0$ in $(0, L)$.*

To prove this result, we proceed in different steps. First, in Section 2 we state some well-posedness and regularity results that we need. Next, in Section 3 we obtain some Carleman inequalities for a single equation and systems with non-homogeneous Neumann-like boundary conditions. Section 4 is devoted to proving the desired null controllability result when we are allowed to control both equations. This step is known as the fictitious control method as we added one input control. Finally, Section 5 uses the algebraic resolution method in order to remove the additional control input and proves our main result.

2 Well-posedness

In this section, we give some well-posedness and regularity results. We study the existence and uniqueness of direct and adjoint systems related to (1.1). We consider the direct system

$$\begin{cases} \partial_t U + \partial_x^4 U = A\partial_x^3 U + B\partial_x^2 U + C\partial_x U + DU + F, & \text{in } Q, \\ U(0,t) = U(L,t) = 0, & \text{in } (0,T), \\ \partial_x U(0,t) = \partial_x U(L,t) = 0, & \text{in } (0,T), \\ U(x,0) = U^0(x), & \text{in } (0,L), \end{cases} \tag{2.1}$$

and its adjoint given by

$$\begin{cases} -\partial_t \Phi + \partial_x^4 \Phi = -\partial_x^3(A^*\Phi) + \partial_x^2(B^*\Phi) - \partial_x(C^*\Phi) + D^*\Phi + \eta, & \text{in } Q, \\ \Phi(0,t) = \Phi(L,t) = 0, & \text{in } (0,T), \\ \partial_x \Phi(0,t) = \partial_x \Phi(L,t) = 0, & \text{in } (0,T), \\ \Phi(x,T) = \Phi^T(x), & \text{in } (0,L), \end{cases}$$

where * stands for the usual transpose operator on matrices. Introducing the matrices $\Pi = -A^*$, $\Lambda = -3\partial_x A^* + B^*$, $\Gamma = -3\partial_x^2 A^* + 2\partial_x B^* - C^*$, $\Theta = -\partial_x^3 A^* + \partial_x^2 B^* - \partial_x C^* + D^*$, we can rewrite last system as

$$\begin{cases} -\partial_t \Phi + \partial_x^4 \Phi = \Pi\partial_x^3 \Phi + \Lambda\partial_x^2 \Phi + \Gamma\partial_x \Phi + \Theta\Phi + \eta, & \text{in } Q, \\ \Phi(0,t) = \Phi(L,t) = 0, & \text{in } (0,T), \\ \partial_x \Phi(0,t) = \partial_x \Phi(L,t) = 0, & \text{in } (0,T), \\ \Phi(x,T) = \Phi^T(x), & \text{in } (0,L). \end{cases} \tag{2.2}$$

Theorem 2 *Let (A, B, C, D) in $C_c^\infty(Q)$. If $U^0 \in L^2(0, L)^2$ and $F \in L^1(0, T; L^2(0, L)^2)$, then there* *exists a unique solution* $U \in C([0,T];L^2(0,L)^2) \cap L^2(0,T;H_0^2(0,L)^2)$ *of system* (2.1)*. Moreover, there exists* $C > 0$ *such that*

$$||U||_{C([0,T];L^2(0,L)^2)\cap L^2(0,T;H^2(0,L)^2)} \leq C\left\{||U^0||_{L^2(0,L)^2} + ||F||_{L^1(0,T;L^2(0,L)^2)}\right\}. \tag{2.3}$$

Proof. This result can be proved by a usual Galerkin method as done in [16, 27]. ■

Theorem 3 *Let* $\Phi^T \in H_0^2(0,L)^2$ *and* $\eta \in L^2(Q)^2$*. Then, the solution* Φ *of* (2.2) *satisfies*

$$\Phi \in [C(0,T;H_0^2(0,L)) \cap L^2(0,T;H^4(0,L))]^2.$$

Let us write the system (2.2) *as* $-\Phi_t := L(t)\Phi + \eta$*, where* $L(t) = L(t,x,\partial_x)$ *and let* $d \in \mathbb{N}$*. If* $\Phi^T \in H^{4d+2}(0,L)^2$ *and* $\eta \in L^2(0,T;H^{4d}(0,L)^2) \cap H^d(0,T;L^2(0,L)^2)$ *satisfy the compatibility conditions*

$$\left\{\begin{array}{l} \bar{g}_0 := \Phi_T \in H_0^2(0,L)^2, \\ \bar{g}_1 := -L(T)\bar{g}_0 - \eta(T,.) \in H_0^2(0,L)^2, \\ \vdots \\ \bar{g}_d := -\left(\sum_{k=0}^{d-1} \left(\begin{array}{c} d-1 \\ k \end{array}\right) \partial_t^k L(T)\bar{g}_{d-1-k}\right) - \partial_t^{d-1}\eta(T,.) \in H_0^2(0,L)^2, \end{array}\right.$$

then, the solution Φ *of* (2.2) *satisfies*

$$\Phi \in [C(0,T;H^{4d+2}(0,L)) \cap L^2(0,T;H^{4d+4}(0,L)) \cap H^{d+1}(0,T;L^2(0,L))]^2,$$

and we have the existence of $C > 0$ *such that*

$$\|\Phi\|_{L^2(0,T;H^{4d+4}(0,L))\cap H^{d+1}(0,T;L^2(0,L))} \leq C\left(\|\eta\|_{L^2(0,T;H^{4d}(0,L)^2)\cap H^d(0,T;L^2(0,L)^2)} + \|\Phi_T\|_{H^{4d+2}(0,L)^2}\right).$$

Proof. This can be easily deduced from [29, section 6.4]. ■

3 Carleman estimates

In this section, we want to get new Carleman estimates for general fourth-order parabolic equations with non-homogeneous Neumann boundary conditions and right-hand side in $L^2(0,T;H^{-2}(0,L))$. We consider the system

$$\left\{\begin{array}{ll} -\varphi_t + \varphi_{4x} = B_0 + \partial_x B_1 + \partial_{xx} B_2, & \text{in } Q, \\ \varphi_{2x}(0,t) = b_1(t),\ \varphi_{2x}(L,t) = b_2(t), & \text{in } (0,T), \\ \varphi_{3x}(0,t) = b_3(t),\ \varphi_{3x}(L,t) = b_4(t), & \text{in } (0,T), \\ \varphi(x,T) = \varphi^T(x), & \text{in } (0,L). \end{array}\right. \tag{3.1}$$

The idea to prove this Carleman inequality is to follow [17] where the authors obtain a new Carleman estimate for the heat equation with non-homogeneous Neumann boundary conditions. First, we introduce some weight functions, as in [2]. Let $\eta \in C^4([0,L])$ be a function satisfying,

$$\left\{\begin{array}{l} \eta(x) > 0,\ \forall x \in (0,L),\ \eta(0) = \eta(L) = 0, \\ |\eta'(x)| \geq \delta > 0,\ \forall x \in [0,L] \setminus \omega_0, \end{array}\right. \tag{3.2}$$

for some $\omega_0 \Subset \omega$ where this means that $\bar{\omega}_0 \subset \omega$. Thus, we have

$$\eta'(0) \geq \delta \text{ and } -\eta'(L) \geq \delta. \tag{3.3}$$

We define the usual exponential weight functions given by

$$\begin{aligned} &\alpha(x,t) := \frac{e^{k\frac{m+1}{m}\lambda\|\eta\|_\infty} - e^{\lambda(k\|\eta\|_\infty+\eta(x))}}{t^m(T-t)^m}, \qquad \xi(x,t) := \frac{e^{\lambda(k\|\eta\|_\infty+\eta(x))}}{t^m(T-t)^m}, \\ &\alpha^*(t) := \max_{x\in[0,L]} \alpha(x,t) = \alpha(0,t) = \alpha(L,t), \quad \xi^*(t) := \min_{x\in[0,L]} \xi(x,t) = \xi(0,t) = \xi(L,t), \end{aligned} \tag{3.4}$$

where $\lambda > 1$, $k > m > 0$. We can remark that we have the following estimates

$$|\alpha_t| \leq CT\xi^{1+1/m},\ 1 \leq CT^{2m}\xi,\ |\partial_x^n \xi| \leq C\lambda^n \xi, \tag{3.5}$$

for any $n \in \mathbb{N}$, and some positive constant C independent of λ, and also, for any $a \in \mathbb{R}$,

$$\begin{aligned} &|\partial_x(\xi^a e^{\pm 2s\alpha})| \leq Cs\lambda\xi^{a+1}e^{\pm 2s\alpha}, \\ &|\partial_{xx}(\xi^a e^{\pm 2s\alpha})| \leq Cs^2\lambda^2\xi^{a+2}e^{\pm 2s\alpha}, \\ &|\partial_t(\xi^a e^{\pm 2s\alpha})| \leq CTs\xi^{a+1+1/m}e^{\pm 2s\alpha}. \end{aligned} \tag{3.6}$$

We first get the following Carleman estimate for the adjoint equation with homogeneous Neumann boundary conditions.

Lemma 4 *Let $r \in \mathbb{R}$, $f \in L^2(Q)$, $m \geq 1$, and $\omega \subset (0,L)$. Then, there exist $\lambda_0 > 0$ and $C > 0$ depending only on ω such that for every $\lambda \geq \lambda_0$, $s \geq C(T^{2m} + T^{2m-1})$ and $q^T \in L^2(0,L)$, the weak solution q of*

$$\begin{cases} -q_t + q_{4x} = f, & \text{in } Q, \\ q_{2x}(0,t) = 0,\ q_{2x}(L,t) = 0, & \text{in } (0,T), \\ q_{3x}(0,t) = 0,\ q_{3x}(L,t) = 0, & \text{in } (0,T), \\ q(x,T) = q^T(x), & \text{in } (0,L), \end{cases} \tag{3.7}$$

satisfies

$$\begin{aligned} &\iint_Q e^{-2s\alpha}(s^{11+r}\lambda^{12+r}\xi^{11+r}|q|^2 + s^{9+r}\lambda^{10+r}\xi^{9+r}|q_x|^2 + s^{7+r}\lambda^{8+r}\xi^{7+r}|q_{xx}|^2 + s^{5+r}\lambda^{6+r}\xi^{5+r}|q_{3x}|^2)\,dxdt \\ &\qquad + \iint_Q s^{3+r}\lambda^{4+r}\xi^{3+r}e^{-2s\alpha}(|q_t|^2 + |q_{4x}|^2)\,dxdt \leq C\iint_Q s^{4+r}\lambda^{4+r}\xi^{4+r}e^{-2s\alpha}|f|^2\,dxdt \\ &\qquad\qquad + C\iint_{\omega\times(0,T)} e^{-2s\alpha}s^{11+r}\lambda^{12+r}\xi^{11+r}|q|^2\,dxdt. \end{aligned} \tag{3.8}$$

Proof. Observe that, using a density argument, we can assume that q is smooth enough on $\overline{Q}$, so the following computations are justified. Let $p = q_{xx}$ so p is the solution of

$$
\begin{cases}
-p_t + p_{4x} = f_{xx}, & \text{in } Q, \\
p(0,t) = 0,\ p(L,t) = 0, & \text{in } (0,T), \\
p_x(0,t) = 0,\ p_x(L,t) = 0, & \text{in } (0,T), \\
p(x,T) = q^T_{xx}(x), & \text{in } (0,L).
\end{cases}
\tag{3.9}
$$

We apply the Carleman estimate given in [19], and get

$$
\begin{aligned}
&\iint_Q e^{-2s\alpha}(s^7\lambda^8\xi^7|p|^2 + s^5\lambda^6\xi^5|p_x|^2 + s^3\lambda^4\xi^3|p_{xx}|^2)\,dxdt \\
&\qquad\qquad \le C\iint_Q s^4\lambda^4\xi^4 e^{-2s\alpha}|f|^2\,dxdt + C\iint_{\omega\times(0,T)} s^7\lambda^8\xi^7 e^{-2s\alpha}|p|^2\,dxdt.
\end{aligned}
\tag{3.10}
$$

So by replacing p by q_{xx}, we obtain

$$
\begin{aligned}
&\iint_Q e^{-2s\alpha}(s^7\lambda^8\xi^7|q_{xx}|^2 + s^5\lambda^6\xi^5|q_{3x}|^2 + s^3\lambda^4\xi^3|q_{4x}|^2)\,dxdt \\
&\qquad\qquad \le C\iint_Q s^4\lambda^4\xi^4 e^{-2s\alpha}|f|^2\,dxdt + C\iint_{\omega\times(0,T)} s^7\lambda^8\xi^7 e^{-2s\alpha}|q_{xx}|^2\,dxdt.
\end{aligned}
$$

In order to include lower order terms, we recall this well-known lemma.

Lemma 5 (see [10]) *Let $r \in \mathbb{R}$. Then, there exists $C := C(r,\omega,L) > 0$ such that, for every $T > 0$ and every $u \in L^2(0,T;H^1(0,L))$,*

$$
s^2\lambda^2\iint_Q e^{-2s\alpha}\xi^{r+2}|u|^2dxdt \le C\left(\iint_Q e^{-2s\alpha}\xi^r|u_x|^2dxdt + s^2\lambda^2\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^{r+2}|u|^2dxdt\right),
$$

for every s and λ sufficiently large.

Thanks to this lemma, we easily get,

$$
\begin{aligned}
&\iint_Q e^{-2s\alpha}(s^{11}\lambda^{12}\xi^{11}|q|^2 + s^9\lambda^{10}\xi^9|q_x|^2 + s^7\lambda^8\xi^7|q_{xx}|^2 + s^5\lambda^6\xi^5|q_{3x}|^2)\,dxdt \\
&\qquad + \iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}(|q_t|^2 + |q_{4x}|^2)\,dxdt \le C\iint_Q s^4\lambda^4\xi^4 e^{-2s\alpha}|f|^2\,dxdt \\
&\qquad\qquad + C\iint_{\omega\times(0,T)} e^{-2s\alpha}(s^{11}\lambda^{12}\xi^{11}|q|^2 + s^9\lambda^{10}\xi^9|q_x|^2 + s^7\lambda^8\xi^7|q_{xx}|^2)\,dxdt,
\end{aligned}
$$

and with a standard localization argument, we have

$$
\begin{aligned}
&\iint_Q e^{-2s\alpha}(s^{11}\lambda^{12}\xi^{11}|q|^2 + s^9\lambda^{10}\xi^9|q_x|^2 + s^7\lambda^8\xi^7|q_{xx}|^2 + s^5\lambda^6\xi^5|q_{3x}|^2)\,dxdt \\
&\qquad + \iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}(|q_t|^2 + |q_{4x}|^2)\,dxdt \le C\iint_Q s^4\lambda^4\xi^4 e^{-2s\alpha}|f|^2\,dxdt \\
&\qquad\qquad + C\iint_{\omega\times(0,T)} e^{-2s\alpha}s^{11}\lambda^{12}\xi^{11}|q|^2\,dxdt.
\end{aligned}
$$

To obtain (3.8), it is enough to apply (3.10) to $\tilde{p} = \xi^{r/2}p$, where $r \in \mathbb{R}$, and follow the same steps as above. ■

Let us work now on the Carleman estimate for the equation with non-homogeneous boundary conditions.

Theorem 6 *Let* $B_0 \in L^2(Q)$, $B_1 \in L^2(0,T;H^{1/2}(0,L))$, $B_2 \in L^2(0,T;H^{3/2}(0,L))$, b_1, b_2, b_3, $b_4 \in L^2(0,T)$, $m \geq 1/2$ *and* $\omega \subset (0,L)$. *There exist* $\lambda_0 > 0$ *and* $C > 0$ *such that any solution* φ *of*

$$\begin{cases} -\varphi_t + \varphi_{4x} = B_0 + \partial_x B_1 + \partial_{xx} B_2, & \text{in } Q, \\ \varphi_{2x}(0,t) = b_1(t),\ \varphi_{2x}(L,t) = b_2(t), & \text{in } (0,T), \\ \varphi_{3x}(0,t) = b_3(t),\ \varphi_{3x}(L,t) = b_4(t), & \text{in } (0,T), \\ \varphi(x,T) = \varphi^T(x), & \text{in } (0,L), \end{cases} \tag{3.11}$$

satisfies, for every $\lambda \geq \lambda_0$ *and* $s \geq C(T^{2m} + T^{2m-1})$, *that*

$$\begin{aligned} &\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2 + s^5\lambda^6\xi^5 e^{-2s\alpha}|\varphi_x|^2 + s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2\,dxdt \\ &\quad\leq C\left(\iint_Q e^{-2s\alpha}(|B_0|^2 + s^2\lambda^2\xi^2|B_1|^2 + s^4\lambda^4\xi^4|B_2|^2)dxdt + \iint_{Q_\omega} s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2\,dxdt\right. \\ &\qquad + \int_0^T s^3\lambda^3\xi^{*3}e^{-2s\alpha^*}(|b_2(t)|^2 + |b_1(t)|^2 + |B_2(L,t)|^2 + |B_2(0,t)|^2)dt \\ &\left. + \int_0^T s\lambda\xi^* e^{-2s\alpha^*}(|b_3(t)|^2 + |b_4(t)|^2 + |B_1(L,t)|^2 + |B_1(0,t)|^2 + |\partial_x B_2(L,t)|^2 + |\partial_x B_2(0,t)|^2)dt\right). \end{aligned} \tag{3.12}$$

Proof. The proof of this Carleman result is rather long and is postponed to the Appendix A. ■

Remark 7 *We do not use the Carleman estimate proven in [2] for a fourth-order equation with non-homogeneous Dirichlet boundary conditions. In fact, in that article, the authors get a Carleman estimate with only* $s^7\lambda^8\xi^7e^{-2s\alpha}|\varphi|^2$ *on the left-hand side. To get* $s^3\lambda^4\xi^3|\varphi_{xx}|^2$ *on the left-hand side, we need to make some integrations by parts, and we can not get rid of the boundary terms appearing if we use non-homogeneous Dirichlet boundary conditions. That is the reason why we choose to deal with non-homogeneous Neumann boundary conditions.*

We can easily deduce from Theorem 6, the following Carleman inequality for the backward system of two coupled fourth-order equations with non-homogeneous Neumann boundary conditions.

Theorem 8 *Let* $\Psi^T \in L^2(0,L)^2$, $\bar{\eta} \in L^2(Q)^2$, $(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4) \in L^2(Q)^2$, *let* $m > 1/2$ *and* $\omega \subset (0,L)$. *There exists* $\lambda_0 > 0$ *and* $C > 0$ *such that any solution* Ψ *of*

$$\begin{cases} -\partial_t\Psi + \partial_x^4\Psi = \tilde{\Pi}\partial_x^3\Psi + \tilde{\Lambda}\partial_x^2\Psi + \tilde{\Gamma}\partial_x\Psi + \tilde{\Theta}\Psi + \bar{\eta}, & \text{in } Q, \\ \Psi_{2x}(0,t) = \bar{b}_1(t),\ \Psi_{2x}(L,t) = \bar{b}_2(t), & \text{in } (0,T), \\ \Psi_{3x}(0,t) = \bar{b}_3(t),\ \Psi_{3x}(L,t) = \bar{b}_4(t), & \text{in } (0,T), \\ \Psi(x,T) = \Psi^T(x), & \text{in } (0,L), \end{cases}$$

satisfies for every $\lambda \geq \lambda_0$ *and* $s \geq C(T^{2m} + T^{2m-1})$,

$$\begin{aligned} &\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\Psi_{2x}|^2 + s^7\lambda^8\xi^7 e^{-2s\alpha}|\Psi|^2 dxdt \\ &\qquad\leq C\left(\iint_Q e^{-2s\alpha}|\bar{\eta}|^2 dxdt + s^7\lambda^8\iint_{Q_\omega} e^{-2s\alpha}\xi^7|\Psi|^2 dxdt\right. \\ &\qquad\left. + s^3\lambda^3\int_0^T e^{-2s\alpha^*}(\xi^*)^3(|\bar{b}_1|^2 + |\bar{b}_2|^2)dt + s\lambda\int_0^T e^{-2s\alpha^*}\xi^*(|\bar{b}_3|^2 + |\bar{b}_4|^2)dt\right). \end{aligned}$$

Now we are ready to prove the following proposition.

Proposition 9 *There exists* $C := C(L,\omega)$, $\lambda_0 > 0$ *such that for all* $\Phi^T \in L^2(0,L)^2$, $m > 1/2$ *the corresponding solution* Φ *of system* (2.2) *satisfies*

$$\iint_Q e^{-2s\alpha} \sum_{r=0}^{10} (s\xi)^{23-2r} \lambda^{24-2r} |\partial_x^r \Phi|^2 dxdt \leq C \iint_{Q_\omega} e^{-2s\alpha} s^{23} \lambda^{24} \xi^{23} |\Phi|^2 dxdt, \tag{3.13}$$

for every $s \geq C(T^{2m} + T^{2m-1})$ *and* $\lambda \geq \lambda_0$.

Proof. Let Φ^T in $C_c^\infty(0,L)^2$ then owing to Theorem 3, we can prove by recurrence that the solution of (2.2) verifies $\Phi \in [C(0,T;H^{4d+2}(0,L)) \cap L^2(0,T;H^{4d+4}(0,L)) \cap H^{d+1}(0,T;L^2(0,L))]^2$ for all $d \in \mathbb{N}$. We derive our system (2.2) eight times in space and get the following problem with $\Psi = \Phi_{8x}$,

$$\begin{cases} -\partial_t \Psi + \partial_x^4 \Psi = \tilde{\Pi} \partial_x^3 \Psi + \tilde{\Lambda} \partial_x^2 \Psi + \tilde{\Gamma} \partial_x \Psi + \tilde{\Delta} \Psi + \bar{\eta}, & \text{in } Q, \\ \Psi_{2x}(0,t) = \Phi_{10x}(0,t),\ \Psi_{2x}(L,t) = \Phi_{10x}(L,t), & \text{in } (0,T), \\ \Psi_{3x}(0,t) = \Phi_{11x}(0,t), \Psi_{3x}(L,t) = \Phi_{11x}(L,t), & \text{in } (0,T), \\ \Psi(x,T) = \Psi^T(x) := \partial_{8x} \Phi^T(x), & \text{in } (0,L), \end{cases}$$

where $\bar{\eta} = \bar{\eta}(\Phi, \partial_x \Phi, \ldots, \partial_{7x} \Phi) \in L^2(Q)$. Furthermore, we get $\Phi_{10x}(0,t)$, $\Phi_{10x}(L,t)$, $\Phi_{11x}(0,t)$, $\Phi_{11x}(L,t) \in L^2(0,T)$. Thus, we can apply the Carleman estimate given in Theorem 8 with $\omega_8 \Subset \omega$ to get

$$\begin{aligned} &\iint_Q s^3 \lambda^4 \xi^3 e^{-2s\alpha} |\Psi_{2x}|^2 + s^7 \lambda^8 \xi^7 e^{-2s\alpha} |\Psi|^2 dxdt \\ &\quad \leq C \Bigg(\iint_Q e^{-2s\alpha} |\bar{\eta}|^2 dxdt + s^7 \lambda^8 \iint_{Q_{\omega_8}} e^{-2s\alpha} \xi^7 |\Psi|^2 dxdt \\ &\quad + s^3 \lambda^3 \int_0^T e^{-2s\alpha^*} (\xi^*)^3 (|\Phi_{10x}(0,t)|^2 + |\Phi_{10x}(L,t)|^2) dt \\ &\quad + s\lambda \int_0^T e^{-2s\alpha^*} \xi^* (|\Phi_{11x}(0,t)|^2 + |\Phi_{11x}(L,t)|^2) dt \Bigg). \end{aligned} \tag{3.14}$$

We follow [10, 13] and divide the proof of Proposition 9 into:

- Step 1. We estimate the boundary terms on the right-hand side of (3.14).
- Step 2. We compare $\iint_Q s^3 \lambda^4 \xi^3 e^{-2s\alpha} |\Psi_{2x}|^2 + s^7 \lambda^8 \xi^7 e^{-2s\alpha} |\Psi|^2 dxdt$ with the left-hand side wanted in (3.13).
- Step 3. We re-estimate the right-hand side.

Step 1. Estimation of the boundary terms. By using Lemma 15 in Appendix A, we get

$$\begin{aligned} &s^3 \lambda^3 \int_0^T e^{-2s\alpha^*} (\xi^*)^3 (|\Phi_{10x}(0,t)|^2 + |\Phi_{10x}(L,t)|^2) dt \\ &\quad \leq \frac{2}{L} s^3 \lambda^3 \int_0^T e^{-2s\alpha^*} (\xi^*)^3 \int_0^L |\Phi_{10x}|^2 dxdt + 4 s^3 \lambda^3 \int_0^T e^{-2s\alpha^*} (\xi^*)^3 \int_0^L |\Phi_{10x}| |\Phi_{11x}| dxdt. \end{aligned}$$

Thus,

$$s^3\lambda^3\int_0^T e^{-2s\alpha^*}(\xi^*)^3(|\Phi_{10x}(0,t)|^2+|\Phi_{10x}(L,t)|^2)dt$$
$$\leq Cs^4\lambda^4\int_0^T e^{-2s\alpha^*}(\xi^*)^4\int_0^L|\Phi_{10x}|^2dxdt+Cs^2\lambda^2\int_0^T e^{-2s\alpha^*}(\xi^*)^2\int_0^L|\Phi_{11x}|^2dxdt.$$

In the same way we obtain

$$s\lambda\int_0^T e^{-2s\alpha^*}\xi^*(|\Phi_{11x}(0,t)|^2+|\Phi_{11x}(L,t)|^2)dt$$
$$\leq Cs^2\lambda^2\int_0^T e^{-2s\alpha^*}(\xi^*)^2\int_0^L|\Phi_{11x}|^2dxdt+C\int_0^T e^{-2s\alpha^*}\int_0^L|\Phi_{12x}|^2dxdt.$$

Let $\hat{\Phi}:=\rho\Phi$ with $\rho\in C^\infty([0,T])$ defined by $\rho:=(s\xi^*)^a e^{-s\alpha^*}$, with $a\in\mathbb{R}$ to be chosen later. We have that $\partial_t^i\rho(T)=0$ for all $i\in\mathbb{N}$. Thus, $\hat{\Phi}$ is solution to the system

$$\begin{cases} -\partial_t\hat{\Phi}+\partial_x^4\hat{\Phi}=\Pi\partial_x^3\hat{\Phi}+\Lambda\partial_x^2\hat{\Phi}+\Gamma\partial_x\hat{\Phi}+\Delta\hat{\Phi}-\rho_t\Phi & \text{in } Q,\\ \hat{\Phi}(0,t)=\hat{\Phi}(L,t)=0 & \text{in } (0,T),\\ \partial_x\hat{\Phi}(0,t)=\partial_x\hat{\Phi}(L,t)=0 & \text{in } (0,T),\\ \hat{\Phi}(x,T)=0, & \text{in } (0,L). \end{cases}$$

As the compatibility conditions are trivially satisfied, Theorem 3 gives us for any $d\in\mathbb{N}$

$$\|\hat{\Phi}\|_{L^2(0,T;H^{4d+4}(0,L))\cap H^{d+1}(0,T;L^2(0,L))}\leq C\|\rho_t\Phi\|_{L^2(0,T;H^{4d}(0,L))\cap H^d(0,T;L^2(0,L))}. \tag{3.15}$$

We estimate the derivatives in time of $\rho:=(s\xi^*)^a e^{-s\alpha^*}$, with $a\in\mathbb{R}$, α and ξ given in (3.4) and the estimates on derivatives in time (3.5),

$$\rho_t=(as(s\xi^*)^{a-1}\xi_t^*-s(s\xi^*)^a\alpha_t^*)e^{-s\alpha^*}.$$

Thus,

$$|\rho_t|\leq CT(s\xi^*)^{a+1+1/m}e^{-s\alpha^*},$$
$$|\rho_{tt}|\leq CT^2(s\xi^*)^{a+2+2/m}e^{-s\alpha^*},$$
$$|\rho_{ttt}|\leq CT^3(s\xi^*)^{a+3+3/m}e^{-s\alpha^*}.$$

First, we take $d=1$ in (3.15) with $\rho=(s\xi^*)^4e^{-s\alpha^*}$, and then $d=0$ with $\rho=\partial_t((s\xi^*)^4e^{-s\alpha^*})$ to get

$$\begin{aligned}\int_0^T(s\xi^*)^8e^{-2s\alpha^*}\|\Phi\|^2_{H^8(0,L)}dt&\leq C\|\rho_t\Phi\|^2_{L^2(0,T;H^4(0,L)\cap H^1(0,T;L^2(0,L))}\\&\leq C\|\rho_{tt}\Phi\|^2_{L^2(0,T;L^2(0,L))}\\&\leq CT^4\int_0^T(s\xi^*)^{12+4/m}e^{-2s\alpha^*}\|\Phi\|^2_{L^2(0,L)}dt.\end{aligned}$$

Secondly, we take $d = 2$, in (3.15) $d = 1$ and lastly $d = 0$, with successively $\rho = e^{-s\alpha^*}$, $\rho = \partial_t(e^{-s\alpha^*})$ and $\rho = \partial_{tt}(e^{-s\alpha^*})$ to get

$$\begin{aligned}\int_0^T e^{-2s\alpha^*}\|\Phi\|^2_{H^{12}(0,L)}dt &\leq C\|\rho_t\Phi\|^2_{L^2(0,T;H^8(0,L)\cap H^2(0,T;L^2(0,L))}\\ &\leq C\|\rho_{tt}\Phi\|^2_{L^2(0,T;H^4(0,L))\cap H^1(0,T;L^2(0,L))}\\ &\leq C\|\rho_{ttt}\Phi\|^2_{L^2(0,T;L^2(0,L))}\\ &\leq CT^6\int_0^T (s\xi^*)^{6+6/m}e^{-2s\alpha^*}\|\Phi\|^2_{L^2(0,L)}dt.\end{aligned}$$

By interpolation, we have for all $u \in H^{12}(0,L)$ that

$$\|u\|^2_{H^{10}(0,L)} \leq C\|u\|_{H^8(0,L)}\|u\|_{H^{12}(0,L)},$$

and consequently

$$\begin{aligned}\int_0^T (s\xi^*)^4e^{-2s\alpha^*}\|\Phi\|^2_{H^{10}(0,L)}dt &\leq C\int_0^T \|(s\xi^*)^4e^{-s\alpha^*}\Phi\|_{H^8(0,L))}\|e^{-s\alpha^*}\Phi\|_{H^{12}(0,L))}dt\\ &\leq CT^5\int_0^T e^{-2s\alpha^*}(s\xi^*)^{\max[12+4/m,6+6/m]}\|\Phi\|^2_{L^2(0,L)}dt.\end{aligned}$$

We also have by interpolation, for all $u \in H^{12}(0,L)$,

$$\|u\|^2_{H^{11}(0,L)} \leq C\|u\|_{H^{10}(0,L)}\|u\|_{H^{12}(0,L)},$$

implying

$$\begin{aligned}\int_0^T (s\xi^*)^2e^{-2s\alpha^*}\|\Phi\|^2_{H^{11}(0,L)} &\leq C\int_0^T \|(s\xi^*)^4e^{-s\alpha^*}\Phi\|_{H^{10}(0,L))}\|e^{-s\alpha^*}\Phi\|_{H^{12}(0,L))}\\ &\leq CT^{11/2}\int_0^T e^{-2s\alpha^*}(s\xi^*)^{\max[12+4/m,6+6/m]}\|\Phi\|^2_{L^2(0,L)}.\end{aligned}$$

Finally, for $s \geq C(T^{2m} + T^{2m-1})$ and $m \geq 1$ we obtain the estimation

$$\begin{aligned}(3.16)\quad &\iint_Q s^3\lambda^4\xi^3e^{-2s\alpha}|\Phi_{10x}|^2 + s^7\lambda^8\xi^7e^{-2s\alpha}|\Phi_{8x}|^2dxdt\\ &\quad\leq C\left(\iint_Q e^{-2s\alpha}|\bar{\eta}|^2dxdt + s^7\lambda^8\iint_{Q_{\omega_8}} e^{-2s\alpha}\xi^7|\Phi_{8x}|^2dxdt\right.\\ &\qquad\left.+s^{22}\lambda^4\iint_Q e^{-2s\alpha^*}(\xi^*)^{22}|\Phi|^2dxdt\right).\end{aligned}$$

Step 2. If we use Lemma 5 with $r = 21$ and Φ, we have

$$\begin{aligned}s^{23}\lambda^{24}\iint_Q e^{-2s\alpha}\xi^{23}|\Phi|^2dxdt &\leq C\left(s^{21}\lambda^{22}\iint_Q e^{-2s\alpha}\xi^{21}|\Phi_x|^2dxdt\right.\\ &\qquad\left.+s^{23}\lambda^{24}\iint_{Q_{\omega_8}} e^{-2s\alpha}\xi^{23}|\Phi|^2dxdt\right).\end{aligned}$$

We apply Lemma 5 successively with $r = 19, 17, ..., 7$ to $\Phi_x, \Phi_{xx}, ..., \Phi_{7x}$. Furthermore, we have

$$\iint_Q e^{-2s\alpha}|\bar{\eta}|^2 dxdt \leq C \iint_Q e^{-2s\alpha} \sum_{i=0}^{7} |\Phi_{ix}|^2 dxdt.$$

Then, using (3.16) we get

$$\begin{aligned}\iint_Q & e^{-2s\alpha} \sum_{r=0}^{10} (s\xi)^{23-2r} \lambda^{24-2r} |\partial_x^r \Phi|^2 \\ & \leq C \iint_{Q_{\omega_8}} e^{-2s\alpha} \sum_{i=0}^{8} (s\xi)^{23-2r} \lambda^{24-2r} |\partial_x^r \Phi|^2 dxdt \\ & \qquad + C s^{22} \lambda^4 \iint_Q e^{-2s\alpha^*} (\xi^*)^{22} |\Phi|^2 dxdt.\end{aligned}$$

Thus, for s and λ large enough, we can absorb the last right-hand term to obtain

$$\iint_Q e^{-2s\alpha} \sum_{r=0}^{10} (s\xi)^{23-2r} \lambda^{24-2r} |\partial_x^r \Phi|^2 \leq C \iint_{Q_{\omega_8}} e^{-2s\alpha} \sum_{i=0}^{8} (s\xi)^{23-2r} \lambda^{24-2r} |\partial_x^r \Phi|^2 dxdt. \tag{3.17}$$

Step 3. Local terms on the right-hand side. Let $(\omega_i)_{i=0,...,7}$ be a sequence of open subsets such that $\omega_8 \Subset \omega_7 \Subset \cdots \Subset \omega$. Let $\theta_7 \in C^2([0, L])$ such that

$$\begin{cases} \text{Supp}(\theta_7) \subset \omega_7, \\ \theta_7 = 1 \text{ in } \omega_8, \\ 0 \leq \theta_7 \leq 1 \text{ in } (0, L). \end{cases}$$

Notice that we have

$$|\partial_x(\theta_7 e^{-2s\alpha} \xi^7)| \leq C s \lambda e^{-2s\alpha} \xi^8,$$

and by integration by parts

$$\begin{aligned} s^7 \lambda^8 \iint_{Q_{\omega_8}} & e^{-2s\alpha} \xi^7 |\Phi_{8x}|^2 dxdt \leq s^7 \lambda^8 \iint_{Q_{\omega_7}} \theta_7 e^{-2s\alpha} \xi^7 |\Phi_{8x}|^2 dxdt \\ & \leq C s^8 \lambda^9 \iint_{Q_{\omega_7}} e^{-2s\alpha} \xi^8 |\Phi_{8x}||\Phi_{7x}| dxdt + C s^7 \lambda^8 \iint_{Q_{\omega_7}} e^{-2s\alpha} \xi^7 |\Phi_{9x}||\Phi_{7x}| dxdt \\ & \leq \epsilon s^7 \lambda^8 \iint_{Q_{\omega_7}} e^{-2s\alpha} \xi^7 |\Phi_{8x}|^2 dxdt + \epsilon s^5 \lambda^6 \iint_{Q_{\omega_7}} e^{-2s\alpha} \xi^5 |\Phi_{9x}|^2 dxdt \\ & \qquad + C_\epsilon s^9 \lambda^{10} \iint_{Q_{\omega_7}} e^{-2s\alpha} \xi^9 |\Phi_{7x}|^2 dxdt. \end{aligned}$$

The first two terms of the last inequality can be absorbed by the left-hand terms of (3.17) by choosing ϵ sufficiently small. Similarly, with $\theta_k \in C^2([0, L])$ for $k = 0, \ldots, 6$, satisfying

$$\begin{cases} \text{Supp}(\theta_k) \subset \omega_k, \\ \theta_k = 1 \text{ in } \omega_{k+1}, \\ 0 \leq \theta_k \leq 1 \text{ in } (0, L), \end{cases}$$

we obtain

$$\begin{aligned}
&s^{23-2(k+1)}\lambda^{24-2(k+1)}\iint_{Q_{\omega_{k+1}}} e^{-2s\alpha}\xi^{23-2(k+1)}|\Phi_{(k+1)x}|^2dxdt\\
&\qquad\leq s^{23-2(k+1)}\lambda^{24-2(k+1)}\iint_{Q_{\omega_k}} \theta_k e^{-2s\alpha}\xi^{23-2(k+1)}|\Phi_{(k+1)x}|^2dxdt\\
&\qquad\leq \epsilon s^{23-2(k+1)}\lambda^{24-2(k+1)}\iint_{Q_{\omega_k}} e^{-2s\alpha}\xi^{23-2(k+1)}|\Phi_{(k+1)x}|^2dxdt\\
&\qquad+ \epsilon s^{23-2(k+2)}\lambda^{24-2(k+2)}\iint_{Q_{\omega_k}} e^{-2s\alpha}\xi^{23-2(k+2)}|\Phi_{(k+2)x}|^2dxdt\\
&\qquad\qquad+ C_\epsilon s^{23-2k}\lambda^{24-2k}\iint_{Q_{\omega_k}} e^{-2s\alpha}\xi^{23-2k}|\Phi_{kx}|^2dxdt.
\end{aligned}$$

Again, the first two terms of the last inequality can be absorbed by the left-hand terms of (3.17) by choosing ϵ sufficiently small. Thus, we get the desired result of Proposition 9, i.e.,

$$\iint_Q e^{-2s\alpha}\sum_{r=0}^{10}(s\xi)^{23-2r}\lambda^{24-2r}|\partial_x^r\Phi|^2 \leq Cs^{23}\lambda^{24}\iint_{Q_{\omega_0}} e^{-2s\alpha}\xi^{23}|\Phi|^2dxdt.$$

■

As a usual consequence of Proposition 9, we can deduce the following observability inequality.

Lemma 10 *There exists $C_{obs} > 0$ such that for every $(\phi_1^T, \phi_2^T) \in L^2(0,L)^2$, the solution to system* (2.2) *satisfies,*

$$\int_0^L |\Phi(x,0)|^2dx \leq C_{obs}\iint_{\omega_0\times(0,T)} e^{-2s_0\alpha}\xi^{23}|\Phi|^2dxdt.$$

Proof.

We first prove the dissipation of the adjoint problem (2.2). Let $(\varphi_1^T, \varphi_2^T) \in L^2(0,L)^2$. We multiply each equation of (2.2) by φ_1 and φ_2, respectively, and we integrate on $(0,L)$. Then, we get for $j = 1,2$

$$\begin{aligned}
\int_0^L (-\partial_t\varphi_j + \partial_x^4\varphi_j - \sum_{i=1}^2 \big[-a_{ij}\partial_x^3\varphi_i + (-3\partial_x a_{ij} + b_{ij})\partial_x^2\varphi_i + (-3\partial_x^2 a_{ij} + 2\partial_x b_{ij} - c_{ij})\partial_x\varphi_i\\
+(-\partial_x^3 a_{ij} + \partial_x^2 b_{ij} - \partial_x c_{ij} + d_{ij})\varphi_i\big])\varphi_j dx = 0.
\end{aligned}$$

Let us estimate each term. First, we have

$$\int_0^L (-\partial_t\varphi_j + \partial_x^4\varphi_j)\varphi_j dx = -\frac{1}{2}\partial_t\int_0^L \varphi_j^2dx + \int_0^L |\partial_x^2\varphi_j|^2dx.$$

Then, with $\epsilon > 0$ to be chosen later, we have by applying Young's inequality and Ehrling's Lemma that

$$\begin{aligned}
|\int_0^L -a_{ij}\partial_x^3\varphi_i\varphi_j dx| &= |\int_0^L \partial_x(a_{ij)}\partial_x^2\varphi_i\varphi_j dx + \int_0^L a_{ij}\partial_x^2\varphi_i\partial_x\varphi_j dx| \\
&\le \|\partial_x(a_{ij)}\|_\infty \int_0^L |\partial_x^2\varphi_i||\varphi_j| dx + \|a_{ij}\|_\infty \int_0^L |\partial_x^2\varphi_i||\partial_x\varphi_j| dx \\
&\le \epsilon \int_0^L |\partial_x^2\varphi_i|^2 dx + C\int_0^L |\varphi_j|^2 dx + C\int_0^L |\partial_x\varphi_j|^2 dx \\
&\le \epsilon \int_0^L |\partial_x^2\varphi_i|^2 dx + \epsilon\int_0^L |\partial_x^2\varphi_j|^2 dx + C\int_0^L |\varphi_j|^2 dx.
\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}
|\int_0^L (-3\partial_x a_{ij} + b_{ij})\partial_x^2\varphi_i\varphi_j dx| &\le (3\|\partial_x a_{ij}\|_\infty + \|b_{ij}\|_\infty)\int_0^L |\partial_x^2\varphi_i||\varphi_j| dx \\
&\le \epsilon\int_0^L |\partial_x^2\varphi_i|^2 dx + C\int_0^L |\varphi_j|^2 dx.
\end{aligned}$$

With Young's inequality and Ehrling's Lemma, we get

$$\begin{aligned}
|\int_0^L (-3\partial_x^2 a_{ij} + 2\partial_x b_{ij} - c_{ij})\partial_x\varphi_i\varphi_j dx| &\le (3\|\partial_x^2 a_{ij}\|_\infty + 2\|\partial_x b_{ij}\|_\infty + \|c_{ij}\|_\infty)\int_0^L |\partial_x\varphi_i||\varphi_j| dx \\
&\le \epsilon\int_0^L |\partial_x^2\varphi_i|^2 dx + C\int_0^L |\varphi_j|^2 dx.
\end{aligned}$$

And lastly, we can write

$$\begin{aligned}
|\int_0^L (-\partial_x^3 a_{ij} + \partial_x^2 b_{ij} - \partial_x c_{ij} + d_{ij})\varphi_i\varphi_j dx| &\le (\|\partial_x^3 a_{ij}\|_\infty + \|\partial_x^2 b_{ij}\|_\infty + \|\partial_x c_{ij}\|_\infty + \|d_{ij}\|_\infty)\int_0^L |\varphi_i||\varphi_j| dx \\
&\le C\int_0^L |\varphi_i|^2 dx + C\int_0^L |\varphi_j|^2 dx.
\end{aligned}$$

Then, by summing all those inequalities, we easily get

$$-\partial_t \int_0^L |\Phi|^2 dx \le C\int_0^L |\Phi|^2 dx,$$

from where we deduce that

$$\int_0^L |\Phi(x,0)|^2 dx e^{-Ct} \le \int_0^L |\Phi(x,t)|^2 dx.$$

We integrate this inequality between $T/4$ and $3T/4$ to get

$$\int_0^L |\Phi(x,0)|^2 dx \le C\int_{T/4}^{3T/4}\int_0^L |\Phi(x,t)|^2 dx dt.$$

The desired result can be obtained in the standard way from this inequality. ■

4 Fictitious control method: adding a control

As mentioned already, in order to obtain our null controllability result with one control input, we first obtain the same result but using two control inputs. To make this result useful later, we need to work with controls that are regular enough. Let us introduce $\theta \in C^2([0,L)]$ such that

$$\begin{cases} \mathrm{Supp}(\theta) \subset \omega, \\ \theta = 1 \text{ in } \omega_0, \\ 0 \leq \theta \leq 1 \text{ in } (0,L). \end{cases}$$

The previous result of observability, Lemma 10 enables us to obtain the following null controllability result with two controls.

Proposition 11 *The system*

$$(4.1) \qquad \begin{cases} \partial_t U + \partial_x^4 U = A\partial_x^3 U + B\partial_x^2 U + C\partial_x U + DU + \theta H, & \textit{in } Q, \\ U(0,t) = U(L,t) = 0, & \textit{in } (0,T), \\ \partial_x U(0,t) = \partial_x U(L,t) = 0, & \textit{in } (0,T), \\ U(x,0) = U^0(x), & \textit{in } (0,L), \end{cases}$$

is null controllable at any time T, that is, for any $U^0 \in L^2(0,L)^2$, there exists a control $H \in L^2(Q)^2$ such that the solution to system (4.1) satisfies $U(.,T) = 0$ in $(0,L)$. Moreover, for every $K \in (0,1)$, we have $e^{Ks_0\alpha^*}H \in L^2(0,T;H^{10}(0,L)^2) \cap H^2(0,T;H^2(0,L)^2)$ and the estimate

$$(4.2) \qquad \|e^{Ks_0\alpha^*}H\|_{L^2(0,T;H^{10}(0,L)^2)\cap H^2(0,T;H^2(0,L)^2)} \leq C\|U^0\|_{L^2(0,L)^2}.$$

Proof. To prove Proposition 11, we follow exactly the proof of Proposition 2.4 in [13]. We use the duality method of Fursikov and Imanuvilov [18]. Let $U^0 \in L^2(0,L)^2, k \in \mathbb{N}^*$, and $\rho := \xi^2 3e^{-2s\alpha}$. We consider the optimal control problem

$$(4.3) \qquad \begin{cases} \text{minimize } J_k(H) := \dfrac{1}{2}\displaystyle\iint_Q \rho^{-1}|H|^2 dxdt + \dfrac{k}{2}\int_0^L |U(x,T)|^2 dx, \\ H \in L^2(Q,\rho^{-1/2})^2, \end{cases}$$

where U is the solution in $C([0,T];L^2(0,L)^2) \cap L^2(0,T;H_0^2(0,L)^2)$ to system (4.1). The functional $J_k : L^2(Q,\rho^{-1/2})^2 \to \mathbb{R}^+$ is differentiable, coercive and strictly convex on $L^2(Q,\rho^{-1/2})^2$. Thus, there exists a unique solution to the optimal control problem (4.3), and the optimal control H_k is characterized thanks to the solution U_k to the primal system

$$(4.4) \qquad \begin{cases} \partial_t U_k + \partial_x^4 U_k = A\partial_x^3 U_k + B\partial_x^2 U_k + C\partial_x U_k + DU_k + \theta H_k, & \text{in } Q, \\ U_k(0,t) = U_k(L,t) = 0, & \text{in } (0,T), \\ \partial_x U_k(0,t) = \partial_x U_k(L,t) = 0, & \text{in } (0,T), \\ U_k(x,0) = U^0(x), & \text{in } (0,L), \end{cases}$$

the solution Φ_k to the dual system,

$$(4.5) \qquad \begin{cases} -\partial_t \Phi_k + \partial_x^4 \Phi_k = -\partial_x^3(A^*\Phi_k) + \partial_x^2(B^*\Phi_k) - \partial_x(C^*\Phi_k) + D^*\Phi_k, & \text{in } Q, \\ \Phi_k(0,t) = \Phi_k(L,t) = 0, & \text{in } (0,T), \\ \partial_x \Phi_k(0,t) = \partial_x \Phi_k(L,t) = 0, & \text{in } (0,T), \\ \Phi_k(x,T) = kU_k(x,T), & \text{in } (0,L), \end{cases}$$

and

$$(4.6)\qquad \begin{cases} H_k = -\rho\theta\Phi_k, \text{ in } Q, \\ H_k \in L^2(Q, \rho^{-1/2}). \end{cases}$$

We divide the rest of the proof into two steps. First, we prove that the sequence $(H_k)_{k\in\mathbb{N}^*}$ converges to a control $H \in L^2(Q, \rho^{-1/2})$ with an associated solution to (4.1) satisfying $U(.,T) = 0$ in $(0, L)$. Then, we prove that the control is regular enough and satisfies (4.2).
Due to (4.4), (4.5) and (4.6), we have

$$\begin{aligned} J_k(H_k) &= \frac{-1}{2}\iint_Q \theta\Phi_k^* H_k dxdt + \frac{1}{2}\int_0^L \Phi_k^*(x,T)U_k(x,T)dx \\ &= \frac{1}{2}\int_0^L \Phi_k^*(x,0)U^0(x)dx. \end{aligned}$$

Thanks to the observability inequality given in Lemma 10, we also have

$$\begin{aligned} \|\Phi_k(.,0)\|^2_{L^2(0,L)} &\leq C_{obs}\iint_Q \rho\theta^2|\Phi_k|^2 dxdt = C_{obs}\iint_Q \rho^{-1}|H_k|^2 dxdt \\ &\leq 2C_{obs}J_k(H_k) \leq 2C_{obs}\|\Phi_k(.,0)\|_{L^2(0,L)}\|U^0\|_{L^2(0,L)}. \end{aligned}$$

Thus, we get

$$\|\Phi_k(.,0)\|_{L^2(0,L)} \leq 2C_{obs}\|U^0\|_{L^2(0,L)},$$

and we easily deduce that,

$$(4.7)\qquad J_k(H_k) \leq C_{obs}\|U^0\|^2_{L^2(0,L)}.$$

Thanks to the well-posedness result in Theorem 2, we have

$$\begin{aligned} \|U_k\|_{C([0,T];L^2(0,L)^2)\cap L^2(0,T;H_0^2(0,L)^2)} &\leq C(\|U^0\|_{L^2(0,L)^2} + \|\theta H_k\|_{L^2(Q)}) \\ &\leq C(1 + C_{obs})\|U^0\|_{L^2(0,L)^2}, \end{aligned}$$

where C does not depend on U^0 nor k. We deduce then that we can extract some subsequences of H_k and U_k still denoted by H_k and U_k such that

$$\begin{cases} H_k \rightharpoonup H \text{ in } L^2(Q, \rho^{-1/2}), \\ U_k \rightharpoonup U \text{ in } C([0,T]; L^2(0,L)^2) \cap L^2(0,T; H_0^2(0,L)^2), \\ U_k(.,T) \rightharpoonup 0 \text{ in } L^2(0,L). \end{cases}$$

Passing to the limit in (4.4), U is solution to (4.1), and $U(.,T) = 0$ in $(0, L)$ with control $H \in L^2(Q, \rho^{-1/2})$. Using (4.7), we get that

$$\|H\|^2_{L^2(Q,\rho^{-1/2})} \leq C_{obs}\|U^0\|_{L^2(0,L)^2}.$$

Now, let us prove that this control H is sufficiently regular, i.e., belong to $L^2(0,T;H^{10}(0,L)^2) \cap H^2(0,T;H^2(0,L))$. First of all, for any $K \in (0,1)$ there exists a constant $C > 0$ such that

$$e^{2Ks_0\alpha^*} \leq C\xi^{-23}e^{2s_0\alpha}.$$

Thus, we have

$$\|e^{2Ks_0\alpha^*}H_k\|^2_{L^2(Q)^2} \leq C\iint_Q \xi^{-23}e^{2s_0\alpha}|H|^2 dxdt \leq J_k(H_k) \leq C_{obs}\|U^0\|^2_{L^2(0,L)^2}.$$

Furthermore, we know that for any $n \geq 0$, any $\eta \geq 0$,

$$\left\{ \begin{array}{l} |\partial_x^n(\xi^\eta e^{-2s_0\alpha}| \leq C\xi^{\eta+n}e^{-2s_0\alpha}, \\ |\partial_t^2\partial_x^2\xi^\eta e^{-2s_0\alpha}| \leq CT^2\xi^{\eta+4+2/m}e^{-2s_0\alpha}. \end{array} \right.$$

Then, we obtain for $n = 1, \dots, 10$

$$\|e^{2Ks_0\alpha^*}\partial_x^n H_k\|_{L^2(Q)^2}^2 \leq C \iint_Q e^{-4s_0\alpha+2s_0\alpha^*} \sum_{i=0}^{n} \xi^{46+2(n-i)}|\partial_x^i\Phi_k|^2 dxdt$$

and similarly, for $n, m = 0, \dots, 2$, we get

$$\|e^{2Ks_0\alpha^*}\partial_x^n\partial_t^m H_k\|_{L^2(Q)^2}^2 \leq C \iint_Q e^{-4s_0\alpha+2s_0\alpha^*} \sum_{i,j=0}^{n,m} T^{m-j}(\xi^{46+2(n-i)+(2+2/m)(m-j)}|\partial_x^i\partial_t^j\Phi_k|^2 dxdt.$$

As for any $\eta, \nu > 0$, there exists a constant $C_{\eta,\nu}$ such that,

$$|\xi^\eta e^{-4s_0\alpha+2s_0\alpha^*}| \leq C_{\eta,\nu}\xi^\nu e^{-2s\alpha},$$

we deduce from all those previous inequalities,

$$\|e^{Ks_0\alpha^*}H_k\|_{L^2(0,T;H^{10}(0,L)^2)\cap H^2(0,T;H^2(0,L)^2)}^2 \leq C \iint_Q e^{-2s\alpha} \sum_{r=0}^{10} (s\xi)^{23-2r}\lambda^{24-2r}|\partial_x^r\Phi_k|^2.$$

It remains to use our Carleman estimate (3.13) to get

$$\|e^{Ks_0\alpha^*}H_k\|_{L^2(0,T;H^{10}(0,L)^2)\cap H^2(0,T;H^2(0,L)^2)}^2 \leq C \iint_{Q_\omega} e^{-2s\alpha}\xi^{23}|\Phi_k|^2 dxdt \leq C\|H_k\|_{L^2(Q)^2}^2.$$

By using estimate (4.7), we obtain

$$\|e^{Ks_0\alpha^*}H_k\|_{L^2(0,T;H^{10}(0,L)^2)\cap H^2(0,T;H^2(0,L)^2)} \leq C\|U^0\|_{L^2(0,L)^2}.$$

We conclude by letting $k \to +\infty$. ■

5 Algebraic resolution method: removing a control

In the previous section, we obtain the null controllability with two controls, and now we will remove one of those to prove our main result, namely Theorem 1.

Let $U^0 \in L^2(0,L)^2$. From Proposition 11, we know that there exists a control $H \in L^2(Q)^2$ such that the solution $\tilde{U}$ to system (4.1) satisfies $\tilde{U}(\cdot,T) = 0$ in $(0,L)$ and also $e^{Ks_0\alpha^*}H \in L^2(0,T;H^{10}(0,L)^2) \cap H^2(0,T;H^2(0,L)^2$. Assume that we are able to built a trajectory $(\hat{U}, \hat{g})$ solution of

$$(5.1) \qquad \left\{ \begin{array}{ll} \partial_t\hat{U} + \partial_x^4\hat{U} = A\partial_x^3\hat{U} + B\partial_x^2\hat{U} + C\partial_x\hat{U} + D\hat{U} + H + G\hat{g}, & \text{in } Q_\omega, \\ \hat{U}(0,t) = \hat{U}(L,t) = 0, & \text{in } (0,T), \\ \partial_x\hat{U}(0,t) = \partial_x\hat{U}(L,t) = 0, & \text{in } (0,T), \\ \hat{U}(x,0) = 0, & \text{in } (0,L), \\ \hat{U}(x,T) = 0, & \text{in } (0,L). \end{array} \right.$$

Due to the well-posedness result in Theorem 2, we see that the solution has the regularity $\hat{U} \in C([0,T];L^2(0,L)^2) \cap L^2(0,T;H_0^2(0,L))$. Thus, we can define $(U,h) := (\tilde{U} - \hat{U}, -\hat{g})$ belonging to $C([0,T];L^2(0,L)^2) \cap L^2(0,T;H_0^2(0,L)) \times L^2(Q)$, which is the desired solution of (1.1) starting from the initial state U^0 and arriving at the null state at time T.

In this way, in order to end the proof of Theorem 1 we have to build the trajectory $(\hat{U}, \hat{g})$ solution of (5.1). This is where algebraic resolution comes in, as in [12, 13], among others. Notice that we have to find $(\hat{U}, \hat{g})$ for any given control H, because that control changes depending on the initial state U^0 to be steered to zero.

Let $H = (h_1, h_2)$ be given with $\mathrm{supp}(H)$ strictly included in Q_ω. We need for (5.1) a solution $(\hat{U}, \hat{g})$ (with appropriate regularity) with its support also strictly included in Q_ω. Denoting $\hat{U} = (\hat{u}_1, \hat{u}_2)$, we rewrite the problem as solving $\mathcal{L}(\hat{u}_1, \hat{u}_2, \hat{g}) = (h_1, h_2)$ where

$$\mathcal{L}(\hat{u}_1,\hat{u}_2,\hat{g}) := \begin{pmatrix} \mathcal{L}_1(\hat{u}_1,\hat{u}_2) - \hat{g} \\ \mathcal{L}_2(\hat{u}_1,\hat{u}_2) \end{pmatrix} := \begin{pmatrix} \partial_t \hat{u}_1 + \partial_x^4 \hat{u}_1 - \displaystyle\sum_{j=1}^{2} (a_{1j}\partial_x^3 \hat{u}_j + b_{1j}\partial_x^2 \hat{u}_j + c_{1j}\partial_x \hat{u}_j + d_{1j}\hat{u}_j) - \hat{g} \\ \partial_t \hat{u}_2 + \partial_x^4 \hat{u}_2 - \displaystyle\sum_{j=1}^{2} (a_{2j}\partial_x^3 \hat{u}_j + b_{2j}\partial_x^2 \hat{u}_j + c_{2j}\partial_x \hat{u}_j + d_{2j}\hat{u}_j) \end{pmatrix}$$

Now, the goal being to find a partial differential operator $\mathcal{M}$ such that

$$\mathcal{L} \circ \mathcal{M} = Id \text{ in } Q_\omega. \tag{5.2}$$

As we shall use later, once we have the operator $\mathcal{M}$, we just have to take $(\hat{u}_1, \hat{u}_2, \hat{g}) = \mathcal{M}(h_1, h_2)$. We can first just solve $\mathcal{L}_2(\hat{u}_1, \hat{u}_2) = h_2$ and then take $\hat{g} = \mathcal{L}_1(\hat{u}_1, \hat{u}_2) - h_1$ as is done, for example, in [14]. Thus, we look for a partial differential operator $\mathcal{M}_2$ such that

$$\mathcal{L}_2 \circ \mathcal{M}_2 = Id \text{ in } Q_\omega.$$

Remark 12 *Notice that the steps* $(\hat{u}_1, \hat{u}_2) = \mathcal{M}_2 h_2$ *and after* $\hat{g} = \mathcal{L}_1 \circ \mathcal{M}_2 h_2 - h_1$, *require* h_2 *has to be regular enough. We will derive the control in space and time.*

A key step is realizing that solving $\mathcal{L}_2 \circ \mathcal{M}_2 = Id$ in Q_ω is equivalent to solving

$$\mathcal{M}_2^* \circ \mathcal{L}_2^* = Id \text{ in } Q_\omega.$$

The adjoint $\mathcal{L}_2^*$ of the operator $\mathcal{L}_2$ is given by

$$\mathcal{L}_2^* \phi := \begin{pmatrix} \mathcal{A}_1 \phi \\ \mathcal{A}_2 \phi \end{pmatrix} = \begin{pmatrix} \partial_x^3(a_{21}\phi) - \partial_x^2(b_{21}\phi) + \partial_x(c_{21}\phi) - d_{21}\phi \\ -\partial_t \phi + \partial_x^4 \phi + \partial_x^3(a_{22}\phi) - \partial_x^2(b_{22}\phi) + \partial_x(c_{22}\phi) - d_{22}\phi \end{pmatrix}.$$

The goal is now to make some algebraic manipulations with $\mathcal{A}_1\phi$ and $\mathcal{A}_2\phi$ in order to get ϕ. We consider the operator

$$\mathcal{Q}(\phi) = \begin{pmatrix} \mathcal{A}_1\phi \\ \partial_x\mathcal{A}_1\phi \\ \partial_{2x}\mathcal{A}_1\phi \\ \partial_{3x}\mathcal{A}_1\phi \\ \partial_{4x}\mathcal{A}_1\phi \\ \partial_{5x}\mathcal{A}_1\phi \\ \partial_{6x}\mathcal{A}_1\phi \\ \partial_t\mathcal{A}_1\phi \\ \partial_{tx}\mathcal{A}_1\phi \\ \partial_{txx}\mathcal{A}_1\phi \\ \mathcal{A}_2\phi \\ \partial_x\mathcal{A}_2\phi \\ \partial_{2x}\mathcal{A}_2\phi \\ \partial_{3x}\mathcal{A}_2\phi \\ \partial_{4x}\mathcal{A}_2\phi \\ \partial_{5x}\mathcal{A}_2\phi \end{pmatrix} = \begin{pmatrix} I & 0 \\ \partial_x & 0 \\ \partial_{2x} & 0 \\ \partial_{3x} & 0 \\ \partial_{4x} & 0 \\ \partial_{5x} & 0 \\ \partial_{6x} & 0 \\ \partial_t & 0 \\ \partial_{tx} & 0 \\ \partial_{txx} & 0 \\ 0 & I \\ 0 & \partial_x \\ 0 & \partial_{2x} \\ 0 & \partial_{3x} \\ 0 & \partial_{4x} \\ 0 & \partial_{5x} \end{pmatrix} \circ \mathcal{L}_2^*\phi = \mathcal{S} \circ \mathcal{L}_2^*\phi := M \begin{pmatrix} \phi \\ \phi_x \\ \phi_{2x} \\ \phi_{3x} \\ \phi_{4x} \\ \phi_{5x} \\ \phi_{6x} \\ \phi_{7x} \\ \phi_{8x} \\ \phi_{9x} \\ \phi_t \\ \phi_{tx} \\ \phi_{txx} \\ \phi_{t3x} \\ \phi_{t4x} \\ \phi_{t5x} \end{pmatrix}.$$

We clearly see that $\mathcal{S}$ is an operator of degree 6 in space, 1 in time, and 1-2 in time-space, and M is a 16×16 square matrix depending on the coefficients $(a_{2,j}, b_{2,j}, c_{2,j}, d_{2,j})_{j=1,2}$ and their derivatives in time and space. Let us suppose that

$$|\det M(x,t)| > C, \quad \text{for every } (x,t) \in \mathcal{O} \times (T_1, T_2) \tag{5.3}$$

where $\mathcal{O}$ is an open subset of Q_ω. Then, with this hypothesis, the matrix M is invertible. By denoting P_1 the projection on the first component we get

$$P_1 M^{-1} M \left(\phi, \phi_x, \phi_{2x}, \phi_{3x}, \phi_{4x}, \phi_{5x}, \phi_{6x}, \phi_{7x}, \phi_{8x}, \phi_{9x}, \phi_t, \phi_{tx}, \phi_{txx}, \phi_{t3x}, \phi_{t4x}, \phi_{t5x}\right)^t = \phi$$

or

$$P_1 M^{-1} \mathcal{S} \circ \mathcal{L}_2^* \phi = \phi.$$

Thus, we can choose

$$\mathcal{M}_2^* := P_1 M^{-1} \mathcal{S}.$$

In order to get $(\hat{u}_1, \hat{u}_2) = \mathcal{M}_2 h_2$, we need to derive h_2, once in time, six times in space, and 1-2 times in time-space. After that, to get $\hat{g} = \mathcal{L}_1(\hat{u}_1, \hat{u}_2) - h_1$ we need to derive again h_2 at a maximum of 2 times in time, 10 times in space, and 1-6 and 2-2 times in time-space. Looking at the regularity we need, we obtain

$$h_2 \in H^2(L^2) \cap L^2(H^{10}) \cap H^1(H^6) \cap H^2(H^2) = L^2(H^{10}) \cap H^2(H^2).$$

The desired trajectory is given by $(\hat{U}, \hat{g}) = \mathcal{M}(\theta H)$ with $\mathcal{M}$ defined in (5.2) and $\mathcal{M} : L^2(0,T; H^{10}(0,L)^2) \cap H^2(0,T; H^2(0,L)^2 \to L^2(\omega \times (0,T))^3$.

Remark 13 *Of course, there are some limitations on the coefficients to manage to get ϕ at the end. The matrix M has to be invertible. The determinant of M can be explained in terms of the coefficients.*

Remark 14 *Case of constant coefficients. We can remark that if the coefficients are constant, the determinant of the matrix is null. This is also the case in the work [14]. In that case, we can follow [13] where the idea is to take* $(\hat{u}_1, \hat{u}_2, \hat{g}) = \mathcal{M}(h_1, h_2) := (h_2, 0, \mathcal{L}_1(h_2, 0) - h_1)$ *and then we get,*

$$\mathcal{L} \circ \mathcal{M}(h_1, h_2) = \mathcal{N}(h_1, h_2) := (h_1, -a_{21}\partial_x^3 h_2 + b_{21}\partial_x^2 h_2 - c_{21}\partial_x h_2 + d_{21}h_2).$$

However, we can remark that if we want to prove an observability inequality with an observation on $|\mathcal{N}^*\Phi|^2$ *then we must ask for* $a_{21} = 0$*, obtaining only a coupling term of order 2 on the second equation. Indeed, we need a sort of Poincaré's inequality in* $|\mathcal{N}^*\Phi|^2$ *for* $\phi \in H_0^2$*. Thus, we can only have up to second-order coupling terms.*

A Appendix

The proof of Theorem 6 is decomposed into two steps:

1. First we prove a Carleman estimate with $s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2$ on the left hand side by using a proof by transposition. This proof follows also [2] where a Carleman estimate for a Kuramoto-Sivashinsky equation with non-homogeneous Dirichlet boundary conditions is given.
2. Then we multiply the PDE in (3.11) by $s^3\lambda^4\xi^3 e^{-2s\alpha}\varphi$ and make some integration by parts to get $s^3\lambda^4\xi^3|\varphi_{xx}|^2$ on the left hand side.

We first view φ as a solution by transposition of (3.11), that is, φ is the unique solution in $L^2(Q)$ of

$$\text{(A.1)}\quad \iint_Q \varphi g\,dxdt = \iint_Q B_0 w\,dxdt - \iint_Q B_1 w_x\,dxdt + \iint_Q B_2 w_{xx}\,dxdt + \int_0^L \varphi^T w(x,T)\,dx$$
$$+ \int_0^T (b_2(t) - B_2(L,t))w_x(L,t)\,dt - \int_0^T (b_1(t) - B_2(0,t))w_x(0,t)\,dt$$
$$+ \int_0^T (B_1(L,t) + \partial_x B_2(L,t) - b_4(t))w(L,t)\,dt - \int_0^T (B_1(0,t) + \partial_x B_2(0,t) - b_3(t))w(0,t)\,dt,$$

where $g \in L^2(Q)$ and w is the solution of

$$\text{(A.2)}\qquad \begin{cases} w_t + w_{4x} = g, & \text{in } Q, \\ w_{2x}(0,t) = 0,\ w_{2x}(L,t) = 0, & \text{in } (0,T), \\ w_{3x}(0,t) = 0,\ w_{3x}(L,t) = 0, & \text{in } (0,T), \\ w(x,0) = 0, & \text{in } (0,L). \end{cases}$$

Then, we choose to solve the null controllability problem, i.e., to find (w, h) such that

$$\text{(A.3)}\qquad \begin{cases} w_t + w_{4x} = s^7\lambda^8\xi^7 e^{-2s\alpha}\varphi + h1_\omega, & \text{in } Q, \\ w_{2x}(0,t) = 0,\ w_{2x}(L,t) = 0, & \text{in } (0,T), \\ w_{3x}(0,t) = 0,\ w_{3x}(L,t) = 0, & \text{in } (0,T), \\ w(x,0) = 0, w(x,T) = 0, & \text{in } (0,L). \end{cases}$$

With the Carleman estimate given in Lemma 4, we solve the controllability problem (A.3) thanks to the Lax-Milgram theorem. We use the same type of proof and notation as in the proof of Theorem 3.5 in [2].

First we define the space

$$E_0 := \{q \in C^\infty(\bar{Q}),\ q_{2x}(0,t) = q_{2x}(L,t) = q_{3x}(0,t) = q_{3x}(L,t) = 0\}.$$

Let us define $P(q) := -q_t + q_{4x}$ and $\kappa : E_0 \times E_0 \to \mathbb{R}$ the bilinear form,

$$\kappa(q_1, q_2) := \iint_Q e^{-2s\alpha} P(q_1)P(q_2)dxdt + s^7\lambda^8 \iint_{Q_\omega} e^{-2s\alpha}\xi^7 q_1 q_2 dxdt,$$

and $\ell : E_0 \to \mathbb{R}$ the linear form

$$\ell(q) := s^7\lambda^8 \iint_Q e^{-2s\alpha}\xi^7 \varphi q dxdt.$$

We choose s and λ large enough so that the Carleman estimate (3.8) is satisfied, then $\kappa(.,.)^{1/2}$ defines a norm in E_0. We call E the completion of E_0 for this norm, then E is a Hilbert space for the scalar product $(.,.)_E := \kappa(.,.)$. By using the usual Cauchy-Schwartz inequality and (3.8), we get,

$$|\ell(q)| \le C(s^7\lambda^8 \iint_Q \xi^7 e^{-2s\alpha}|\varphi|^2 dxdt)^{1/2}\kappa(q,q)^{1/2}, \tag{A.4}$$

that is ℓ is a bounded operator on E and we can apply Lax-Milgram theorem, there exists a unique $\hat{q} \in E$ such that $\kappa(\hat{q}, q) = \ell(q)$ for all $q \in E$. Let us define

$$\hat{w} := e^{-2s\alpha}P(\hat{q}),\ \hat{h} := -s^7\lambda^8\xi^7 e^{-2s\alpha}\hat{q}1_\omega.$$

Then we easily get by using $\kappa(\hat{q}, \hat{q}) = \ell(\hat{q})$ with (A.4)

$$\iint_Q e^{2s\alpha}|\hat{w}|^2 dxdt + s^{-7}\lambda^{-8} \iint_{Q_\omega} e^{2s\alpha}\xi^{-7}|\hat{h}|^2 dxdt \le Cs^7\lambda^8 \iint_Q \xi^7 e^{-2s\alpha}|\varphi|^2 dxdt. \tag{A.5}$$

Furthermore, we can verify that $(\hat{w}, \hat{h})$ is a solution to the control problem (A.3). We take $g = s^7\lambda^8\xi^7 e^{-2s\alpha}\varphi + \hat{h}1_\omega$ in (A.1)-(A.2). That gives us

$$\begin{aligned}\iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2\,dxdt = &-\iint_{Q_\omega} \hat{h}\varphi dxdt + \iint_Q B_0\hat{w}\,dxdt - \iint_Q B_1\hat{w}_x\,dxdt\\ &+ \int_0^T b_2(t)\hat{w}_x(L,t)\,dt - \int_0^T b_1(t)\hat{w}_x(0,t)\,dt\\ &+ \int_0^T (B_1(L,t) - b_4(t))\hat{w}(L,t)\,dt - \int_0^T (B_1(0,t) - b_3(t))\hat{w}(0,t)\,dt.\end{aligned}$$

Then we have by using Young's inequality,

$$
\begin{aligned}
&\iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2\,dxdt\\
&\leq C\Bigg(\iint_{Q_\omega} s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2dxdt+\iint_Q e^{-2s\alpha}(|B_0|^2+s^2\lambda^2\xi^2|B_1|^2+s^4\lambda^4\xi^4|B_2|^2)\,dxdt\\
&\qquad\qquad+s^3\lambda^3\int_0^T e^{-2s\alpha^*}\xi^{*3}(|b_1(t)|^2+|b_2(t)|^2+|B_2(L,t)|^2+|B_2(0,t)|^2)dt\\
&+s\lambda\int_0^T e^{-2s\alpha^*}\xi^*(|b_3(t)|^2+|b_4(t)|^2+|B_1(L,t)|^2+|B_1(0,t)|^2+|\partial_x B_2(L,t)|^2+|\partial_x B_2(0,t)|^2)dt\Bigg)\\
&+\epsilon\Bigg(\iint_Q e^{2s\alpha}(|\hat{w}|^2+s^{-2}\lambda^{-2}\xi^{-2}|\hat{w}_x|^2+s^{-4}\lambda^{-4}\xi^{-4}|\hat{w}_{xx}|^2)\,dxdt+s^{-7}\lambda^{-8}\iint_{Q_\omega} e^{2s\alpha}\xi^{-7}|\hat{h}|^2dxdt\\
&\qquad\qquad+s^{-3}\lambda^{-3}\int_0^T e^{2s\alpha^*}\xi^{*-3}(|\hat{w}_x(L,t)|^2+|\hat{w}_x(0,t)|^2)\,dt\\
&\qquad\qquad\qquad+s^{-1}\lambda^{-1}\int_0^T e^{2s\alpha^*}\xi^{*-1}(\hat{w}(L,t)^2+\hat{w}(0,t)^2)\,dt\Bigg).
\end{aligned}
\tag{A.6}
$$

We already have estimate (A.5). The last two boundary terms of the previous inequality, $s^{-4}\lambda^{-4}\iint_Q \xi^{-4}e^{2s\alpha}|\hat{w}_{xx}|^2dxdt$, and $\iint_Q s^{-2}\lambda^{-2}\xi^{-2}e^{2s\alpha}|\hat{w}_x|^2dxdt$ in terms of $\iint_Q s^7\lambda^8\xi^7e^{-2s\alpha}|\varphi|^2$ remain to be estimated. We have first:

$$
\begin{aligned}
s^{-2}\lambda^{-2}\int_0^T[\partial_x(\xi^{-2}e^{2s\alpha})|\hat{w}|^2]_0^L dt &= s^{-2}\lambda^{-2}\int_0^T[e^{2s\alpha}\lambda\xi^{-1}(-2s-2\xi^{-1})\eta'|\hat{w}|^2]_0^L dt\\
&= 2s^{-2}\lambda^{-1}\int_0^T e^{2s\alpha^*}\xi^{*-1}(s+\xi^{*-1})(\eta'(0)|\hat{w}(0,t)|^2-\eta'(L)|\hat{w}(L,t)|^2)dt.
\end{aligned}
$$

Using the definition of the weight (3.3), we get

$$
s^{-2}\lambda^{-2}\int_0^T[\partial_x(\xi^{-2}e^{2s\alpha})|\hat{w}|^2]_0^L dt \geq 2\delta s^{-1}\lambda^{-1}\int_0^T e^{2s\alpha^*}\xi^{*-1}(|\hat{w}(0,t)|^2+|\hat{w}(L,t)|^2)dt.
$$

Thus we have with (3.6),

$$
\begin{aligned}
2\delta s^{-1}\lambda^{-1}\int_0^T e^{2s\alpha^*}\xi^{*-1}(|\hat{w}(0,t)|^2+|\hat{w}(L,t)|^2)dt &\leq s^{-2}\lambda^{-2}\int_0^T[\partial_x(\xi^{-2}e^{2s\alpha})|\hat{w}|^2]_0^L dt\\
&\leq s^{-2}\lambda^{-2}\iint_Q \partial_{xx}(\xi^{-2}e^{2s\alpha})|\hat{w}|^2dxdt+2s^{-2}\lambda^{-2}\iint_Q \partial_x(\xi^{-2}e^{2s\alpha})\hat{w}\hat{w}_x dxdt\\
&\leq C\iint_Q e^{2s\alpha}|\hat{w}|^2dxdt+\epsilon s^{-2}\lambda^{-2}\iint_Q \xi^{-2}e^{2s\alpha}|\hat{w}_x|^2dxdt.
\end{aligned}
\tag{A.7}
$$

Similarly we get,

$$
s^{-4}\lambda^{-4}\int_0^T[\partial_x(\xi^{-4}e^{2s\alpha})|\hat{w}_x|^2]_0^L dt \geq 2\delta s^{-3}\lambda^{-3}\int_0^T e^{2s\alpha^*}\xi^{*-3}(|\hat{w}_x(0,t)|^2+|\hat{w}_x(L,t)|^2)dt.
$$

Thus, we have

$$\begin{aligned}2\delta s^{-3}\lambda^{-3}\int_0^T e^{2s\alpha^*}\xi^{*-3}(|\hat{w}_x(0,t)|^2+|\hat{w}_x(L,t)|^2)dt &\leq s^{-4}\lambda^{-4}\int_0^T [\partial_x(\xi^{-4}e^{2s\alpha})|\hat{w}_x|^2]_0^L dt\\ &\leq s^{-4}\lambda^{-4}\iint_Q \partial_{xx}(\xi^{-4}e^{2s\alpha})|\hat{w}_x|^2dxdt+2s^{-4}\lambda^{-4}\iint_Q \partial_x(\xi^{-4}e^{2s\alpha})\hat{w}_x\hat{w}_{xx}dxdt\\ &\leq Cs^{-2}\lambda^{-2}\iint_Q \xi^{-2}e^{2s\alpha}|\hat{w}_x|^2dxdt+\epsilon s^{-4}\lambda^{-4}\iint_Q \xi^{-4}e^{2s\alpha}|\hat{w}_{xx}|^2dxdt.\end{aligned}\tag{A.8}$$

We estimate those integrals in several steps.

A.1 Estimating the second-order term

We multiply equation (A.3) by $s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}\hat{w}$ and integrate by parts in Q:

$$\iint_Q s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}\hat{w}(\hat{w}_t+\hat{w}_{4x}-s^7\lambda^8\xi^7e^{-2s\alpha}\varphi-\hat{h}1_\omega)dxdt=0.$$

We get

(A.9)

$$\begin{aligned}\iint_Q s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}|\hat{w}_{xx}|^2dxdt &= -\iint_Q s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}\hat{w}\hat{w}_tdxdt+\iint_Q s^3\lambda^4\xi^3e^{s\alpha}\hat{w}e^{-s\alpha}\varphi dxdt\\ &\quad+\iint_Q s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}\hat{w}\hat{h}1_\omega dxdt-\iint_Q s^{-4}\lambda^{-4}\partial_{xx}(\xi^{-4}e^{2s\alpha})\hat{w}\hat{w}_{xx}dxdt\\ &\quad-2\iint_Q s^{-4}\lambda^{-4}\partial_x(\xi^{-4}e^{2s\alpha})\hat{w}_x\hat{w}_{xx}dxdt.\end{aligned}$$

We estimate each of the five terms appearing in the right hand side of the previous inequality with Young's inequality, (3.5) and (3.6),

$$\begin{aligned}|\iint_Q s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}\hat{w}\hat{w}_tdxdt| &= |\frac{1}{2}\iint_Q s^{-4}\lambda^{-4}\partial_t(\xi^{-4}e^{2s\alpha})|\hat{w}|^2dxdt|\\ &\leq CT\iint_Q s^{-3}\lambda^{-4}\xi^{-3+1/m}e^{2s\alpha}|\hat{w}|^2dxdt\leq C\iint_Q e^{2s\alpha}|\hat{w}|^2dxdt,\end{aligned}$$

when $m>1/2$ and $s\geq CT^{2m}$. Furthermore, we get

$$|\iint_Q s^3\lambda^4\xi^3e^{s\alpha}\hat{w}e^{-s\alpha}\varphi dxdt|\leq Cs^6\lambda^8\iint_Q \xi^6e^{-2s\alpha}|\varphi|^2dxdt+C\iint_Q e^{2s\alpha}|\hat{w}|^2dxdt,$$

$$|\iint_Q s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}\hat{w}\hat{h}1_\omega dxdt|\leq Cs^{-8}\lambda^{-8}\iint_{Q_\omega} e^{2s\alpha}\xi^{-8}|\hat{h}|^2dxdt+C\iint_Q e^{2s\alpha}|\hat{w}|^2dxdt,$$

$$\begin{aligned}|\iint_Q s^{-4}\lambda^{-4}\partial_{xx}(\xi^{-4}e^{2s\alpha})\hat{w}\hat{w}_{xx}dxdt| &\leq C\iint_Q s^{-2}\lambda^{-2}\xi^{-2}e^{2s\alpha}|\hat{w}\hat{w}_{xx}|dxdt\\ &\leq C\iint_Q e^{2s\alpha}|\hat{w}|^2dxdt+\epsilon\iint_Q s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}|\hat{w}_{xx}|^2dxdt,\end{aligned}$$

$$|\iint_Q s^{-4}\lambda^{-4}\partial_x(\xi^{-4}e^{2s\alpha})\hat{w}_x\hat{w}_{xx}dxdt| \leq C\iint_Q s^{-3}\lambda^{-3}\xi^{-3}e^{2s\alpha}|\hat{w}_x\hat{w}_{xx}|dxdt$$
$$\leq C\iint_Q s^{-2}\lambda^{-2}\xi^{-2}e^{2s\alpha}|\hat{w}_x|^2dxdt + \epsilon\iint_Q s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}|\hat{w}_{xx}|^2dxdt.$$

Thus we get, with $\epsilon > 0$ sufficiently small, from (A.9) and the five previous estimates,

$$\iint_Q s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}|\hat{w}_{xx}|^2dxdt \leq C\iint_Q e^{2s\alpha}|\hat{w}|^2dxdt + Cs^6\lambda^8\iint_Q \xi^6e^{-2s\alpha}|\varphi|^2dxdt$$
$$+ Cs^{-8}\lambda^{-8}\iint_{Q_\omega} e^{2s\alpha}\xi^{-8}|\hat{h}|^2dxdt + C\iint_Q s^{-2}\lambda^{-2}\xi^{-2}e^{2s\alpha}|\hat{w}_x|^2dxdt.$$

By using (A.5), we have,

(A.10)
$$\iint_Q s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}|\hat{w}_{xx}|^2dxdt \leq Cs^7\lambda^8\iint_Q \xi^7e^{-2s\alpha}|\varphi|^2dxdt + C\iint_Q s^{-2}\lambda^{-2}\xi^{-2}e^{2s\alpha}|\hat{w}_x|^2dxdt.$$

A.2 Estimating the first-order term

By using some integration by parts, (3.5) and (3.6), we have

$$\iint_Q s^{-2}\lambda^{-2}\xi^{-2}e^{2s\alpha}|\hat{w}_x|^2dxdt = \int_0^T [s^{-2}\lambda^{-2}\xi^{-2}e^{2s\alpha}\hat{w}_x\hat{w}]_0^L dt - \iint_Q s^{-2}\lambda^{-2}\partial_x(\xi^{-2}e^{2s\alpha})\hat{w}_x\hat{w}dxdt$$
$$- \iint_Q s^{-2}\lambda^{-2}\xi^{-2}e^{2s\alpha}\hat{w}_{xx}\hat{w}dxdt$$
$$= \int_0^T s^{-2}\lambda^{-2}\xi^{*-2}e^{2s\alpha^*}[\hat{w}_x\hat{w}]_0^L dt - \iint_Q s^{-2}\lambda^{-2}\partial_x(\xi^{-2}e^{2s\alpha})\hat{w}_x\hat{w}dxdt$$
$$- \iint_Q s^{-2}\lambda^{-2}\xi^{-2}e^{2s\alpha}\hat{w}_{xx}\hat{w}dxdt$$
$$\leq |\int_0^T s^{-2}\lambda^{-2}\xi^{*-2}e^{2s\alpha^*}[\hat{w}_x\hat{w}]_0^L dt| + C\iint_Q e^{2s\alpha}|\hat{w}|^2dxdt$$
$$+ \epsilon\iint_Q s^{-2}\lambda^{-2}\xi^{-2}e^{2s\alpha}|\hat{w}_x|^2dxdt + \epsilon\iint_Q s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}|\hat{w}_{xx}|^2dxdt.$$

Thus we get, with $\epsilon > 0$ sufficiently small,

(A.11)
$$\iint_Q s^{-2}\lambda^{-2}\xi^{-2}e^{2s\alpha}|\hat{w}_x|^2dxdt \leq |\int_0^T s^{-2}\lambda^{-2}\xi^{*-2}e^{2s\alpha^*}[\hat{w}_x\hat{w}]_0^L dt| + C\iint_Q e^{2s\alpha}|\hat{w}|^2dxdt$$
$$+ \epsilon\iint_Q s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}|\hat{w}_{xx}|^2dxdt.$$

Lastly we estimate the boundary terms $\int_0^T s^{-2}\lambda^{-2}\xi^{*-2}e^{2s\alpha^*}[\hat{w}_x\hat{w}]_0^L dt$,

(A.12)
$$|\int_0^T s^{-2}\lambda^{-2}\xi^{*-2}e^{2s\alpha^*}[\hat{w}_x\hat{w}]_0^L dt| \leq C\int_0^T s^{-1}\lambda^{-1}\xi^{*-1}e^{2s\alpha^*}(|\hat{w}(0,t)|^2 + |\hat{w}(L,t)|^2)dt$$
$$+ \epsilon\int_0^T s^{-3}\lambda^{-3}\xi^{*-3}e^{2s\alpha^*}(|\hat{w}_x(0,t)|^2 + |\hat{w}_x(L,t)|^2)dt.$$

With (A.7) and (A.8) in (A.12) we have,

$$|\int_0^T s^{-2}\lambda^{-2}\xi^{*-2}e^{2s\alpha^*}[\hat{w}_x\hat{w}]_0^L dt| \leq C\iint_Q e^{2s\alpha}|\hat{w}|^2 dxdt + \epsilon s^{-2}\lambda^{-2}\iint_Q \xi^{-2}e^{2s\alpha}|\hat{w}_x|^2 dxdt + \epsilon s^{-4}\lambda^{-4}\iint_Q \xi^{-4}e^{2s\alpha}|\hat{w}_{xx}|^2 dxdt. \tag{A.13}$$

With (A.11) and (A.13) we have for ϵ sufficiently small,

$$\iint_Q s^{-2}\lambda^{-2}\xi^{-2}e^{2s\alpha}|\hat{w}_x|^2 dxdt \leq C\iint_Q e^{2s\alpha}|\hat{w}|^2 dxdt + \epsilon\iint_Q s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}|\hat{w}_{xx}|^2 dxdt. \tag{A.14}$$

A.3 Adding a zero-order term on the left-hand side

Now we use (A.14) with (A.10) to get for ϵ sufficiently small,

$$\iint_Q s^{-4}\lambda^{-4}\xi^{-4}e^{2s\alpha}|\hat{w}_{xx}|^2 dxdt + \iint_Q s^{-2}\lambda^{-2}\xi^{-2}e^{2s\alpha}|\hat{w}_x|^2 dxdt \leq Cs^7\lambda^8\iint_Q \xi^7 e^{-2s\alpha}|\varphi|^2 dxdt. \tag{A.15}$$

This estimate (A.15) with (A.5), (A.7) and (A.8) allows us to obtain,

$$s^{-1}\lambda^{-1}\int_0^T e^{2s\alpha^*}\xi^{*-1}(|\hat{w}(0,t)|^2 + |\hat{w}(L,t)|^2)dt \leq Cs^7\lambda^8\iint_Q \xi^7 e^{-2s\alpha}|\varphi|^2 dxdt. \tag{A.16}$$

and

$$s^{-3}\lambda^{-3}\int_0^T e^{2s\alpha^*}\xi^{*-3}(|\hat{w}_x(0,t)|^2 + |\hat{w}_x(L,t)|^2)dt \leq Cs^7\lambda^8\iint_Q \xi^7 e^{-2s\alpha}|\varphi|^2 dxdt. \tag{A.17}$$

It remains to apply all those estimates in (A.6) to obtain the first Carleman estimate, for ϵ sufficiently small,

$$\begin{aligned}\iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2\, dxdt \leq C\Bigg(&\iint_Q e^{-2s\alpha}(|B_0|^2 + s^2\lambda^2\xi^2|B_1|^2 + s^4\lambda^4\xi^4|B_2|^2)dxdt \\ &+ s\lambda\int_0^T e^{-2s\alpha^*}\xi^*(|b_3(t)|^2 + |b_4(t)|^2 + |B_1(L,t)|^2 + |B_1(0,t)|^2 + |\partial_x B_2(L,t)|^2 + |\partial_x B_2(0,t)|^2)dt \\ &+ s^3\lambda^3\int_0^T e^{-2s\alpha^*}\xi^{*3}(|b_1(t)|^2 + |b_2(t)|^2 + |B_2(L,t)|^2 + |B_2(0,t)|^2)dt + \iint_{Q_\omega} s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2\, dxdt\Bigg).\end{aligned} \tag{A.18}$$

This ends the first part of the proof of Theorem 3.12.

A.4 Adding a second-order term on the left-hand side

We multiply (3.11) by $s^3\lambda^4\xi^3 e^{-2s\alpha}\varphi$, we first estimate the following integral I :

$$\begin{aligned} I &:= \iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}\varphi\,(-\varphi_t + \varphi_{4x})\,dxdt, \\ &= I_1 + I_2 \end{aligned}$$

We first have

$$I_1 := -\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}\varphi\varphi_t dxdt = \frac{1}{2}\iint_Q s^3\lambda^4\partial_t(\xi^3 e^{-2s\alpha})|\varphi|^2 dxdt.$$

Thus for $s \geq C(T^{2m} + T^{2m-1})$ and $m > 1/2$, with (3.6) we obtain,

$$|I_1| \leq CT\iint_Q \lambda^4 (s\xi)^{4+1/m} e^{-2s\alpha}|\varphi|^2 dxdt \leq C\iint_Q \lambda^8 (s\xi)^7 e^{-2s\alpha}|\varphi|^2 dxdt. \tag{A.19}$$

$$\begin{aligned}
I_2 &:= \iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}\varphi\varphi_{4x} dxdt \\
&= \int_0^T [s^3\lambda^4\xi^3 e^{-2s\alpha}\varphi\varphi_{3x}]_0^L dt - \iint_Q s^3\lambda^4\partial_x(\xi^3 e^{-2s\alpha})\varphi\varphi_{3x} dxdt - \iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}\varphi_x\varphi_{3x} dxdt \\
&= \int_0^T [s^3\lambda^4\xi^3 e^{-2s\alpha}\varphi\varphi_{3x} - s^3\lambda^4\partial_x(\xi^3 e^{-2s\alpha})\varphi\varphi_{2x} - s^3\lambda^4\xi^3 e^{-2s\alpha}\varphi_x\varphi_{2x}]_0^L dt \\
&+ \iint_Q s^3\lambda^4\partial_{xx}(\xi^3 e^{-2s\alpha})\varphi\varphi_{2x} dxdt + 2\iint_Q s^3\lambda^4\partial_x(\xi^3 e^{-2s\alpha})\varphi_x\varphi_{2x} dxdt + \iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2 dxdt \\
&:= I_{21} + \iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2 dxdt.
\end{aligned}$$

Thus

$$\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2 dxdt = I - I_1 - I_{21} \leq |I| + |I_1| + |I_{21}| \tag{A.20}$$

We can first estimate the cross terms in $\varphi\varphi_{2x}$ appearing in I_{21} with the estimates on the weights given in (3.6).

$$\begin{aligned}
\iint_Q s^3\lambda^4|\partial_{xx}(\xi^3 e^{-2s\alpha})||\varphi\varphi_{2x}| dxdt &\leq C\iint_Q s^5\lambda^6\xi^5 e^{-2s\alpha}|\varphi\varphi_{2x}| dxdt \\
&\leq C\iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2 dxdt + \epsilon\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2 dxdt
\end{aligned}$$

and we also have for the term in $\varphi_x\varphi_{2x}$ appearing in I_{21},

$$\begin{aligned}
\iint_Q s^3\lambda^4|\partial_x(\xi^3 e^{-2s\alpha})||\varphi_x\varphi_{2x}| dxdt &\leq C\iint_Q s^4\lambda^5\xi^4 e^{-2s\alpha}|\varphi_x\varphi_{2x}| dxdt \\
&\leq C\iint_Q s^5\lambda^6\xi^5 e^{-2s\alpha}|\varphi_x|^2 dxdt + \epsilon\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2 dxdt.
\end{aligned} \tag{A.21}$$

Now we estimate the boundary terms appearing in I_{21}. For that, we first prove an easy technical lemma.

Lemma 15 *Let* $f \in H^1(0,L)$*, then* $|f(L)|^2 + |f(0)|^2 \leq \frac{2}{L}\|f\|^2_{L^2(0,L)} + 4\int_0^L |f(x)f'(x)|dx.$

Proof. For the right boundary term we have easily,

$$\begin{aligned}|f(L)|^2 = [\frac{x}{L}f^2(x)]_0^L = \int_0^L \partial_x(\frac{x}{L}f^2(x))dx = \frac{1}{L}\int_0^L f^2(x)dx + \int_0^L \frac{2x}{L}f(x)f'(x)dx \\ \leq \frac{1}{L}\|f\|^2_{L^2(0,L)} + 2\int_0^L |f(x)f'(x)|dx,\end{aligned}$$

and similarly we have,

$$\begin{aligned}|f(0)|^2 = [\frac{x-L}{L}f^2(x)]_0^L = \int_0^L \partial_x(\frac{x-L}{L}f^2(x))dx = \frac{1}{L}\int_0^L f^2(x)dx + \int_0^L \frac{2(x-L)}{L}f(x)f_x(x)dx \\ \leq \frac{1}{L}\|f\|^2_{L^2(0,L)} + 2\int_0^L |f(x)f'(x)|dx.\end{aligned}$$

■

We apply Lemma 15 to the boundary terms appearing in I_{21}.

$$\begin{aligned}|\int_0^T [s^3\lambda^4\xi^3 e^{-2s\alpha}\varphi\varphi_{3x}]_0^L dt| &= |\int_0^T s^3\lambda^4\xi^{*3}e^{-2s\alpha^*}(\varphi(L,t)b_4(t) - \varphi(0,t)b_3(t))dt| \\ &\leq C\int_0^T s^5\lambda^7\xi^{*5}e^{-2s\alpha^*}(|\varphi(L,t)|^2 + |\varphi(0,t)|^2)dt + C\int_0^T s\lambda\xi^* e^{-2s\alpha^*}(|b_3(t)|^2 + |b_4(t)|^2)dt \\ &\leq C\int_0^T s^5\lambda^7\xi^{*5}e^{-2s\alpha^*}(\int_0^L |\varphi|^2 + |\varphi\varphi_x|\, dx)dt + C\int_0^T s\lambda\xi^* e^{-2s\alpha^*}(|b_3(t)|^2 + |b_4(t)|^2)dt \\ &\leq C\iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2 + \epsilon\iint_Q s^5\lambda^6\xi^5 e^{-2s\alpha}|\varphi_x|^2 dt + C\int_0^T s\lambda\xi^* e^{-2s\alpha^*}(|b_3(t)|^2 + |b_4(t)|^2)dt.\end{aligned}$$

Similarly,

$$\begin{aligned}|\int_0^T [s^3\lambda^4\partial_x(\xi^3 e^{-2s\alpha})\varphi\varphi_{2x}]_0^L dt| &\leq C\int_0^T s^4\lambda^5\xi^{*4}e^{-2s\alpha^*}(|\varphi(L,t)b_2(t)| + |\varphi(0,t)b_1(t)|)dt \\ &\leq C\int_0^T s^5\lambda^7\xi^{*5}e^{-2s\alpha^*}(|\varphi(L,t)|^2 + |\varphi(0,t)|^2)dt + C\int_0^T s^3\lambda^3\xi^{*3}e^{-2s\alpha^*}(|b_2(t)|^2 + |b_1(t)|^2)dt \\ &\leq C\iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2 dxdt + \epsilon\iint_Q s^5\lambda^6\xi^5 e^{-2s\alpha}|\varphi_x|^2 dxdt \\ &\quad + C\int_0^T s^3\lambda^3\xi^{*3}e^{-2s\alpha^*}(|b_2(t)|^2 + |b_1(t)|^2)dt.\end{aligned}$$

Furthermore,

$$\begin{aligned}|\int_0^T [s^3\lambda^4\xi^3 e^{-2s\alpha}\varphi_x\varphi_{2x}]_0^L dt| &\leq \int_0^T s^3\lambda^4\xi^{*3}e^{-2s\alpha^*}(|\varphi_x(L,t)b_2(t)| + |\varphi_x(0,t)b_1(t)|)dt \\ &\leq C\int_0^T s^3\lambda^5\xi^{*3}e^{-2s\alpha^*}(|\varphi_x(L,t)|^2 + |\varphi_x(0,t)^2|) + C\int_0^T s^3\lambda^3\xi^{*3}e^{-2s\alpha^*}(|b_2(t)|^2 + |b_1(t)|^2)dt \\ &\leq C\int_0^T s^3\lambda^5\xi^{*3}e^{-2s\alpha^*}(\int_0^L |\varphi_x|^2 + \int_0^L |\varphi_x\varphi_{2x}|) + C\int_0^T s^3\lambda^3\xi^{*3}e^{-2s\alpha^*}(|b_2(t)|^2 + |b_1(t)|^2)dt \\ &\leq C\iint_Q s^5\lambda^6\xi^5 e^{-2s\alpha}|\varphi_x|^2 dxdt + \epsilon\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2) dxdt \\ &\quad + C\int_0^T s^3\lambda^3\xi^{*3}e^{-2s\alpha^*}(|b_2(t)|^2 + |b_1(t)|^2)dt.\end{aligned}$$

It remains to estimate the term $\iint_Q s^5\lambda^6\xi^5 e^{-2s\alpha}|\varphi_x|^2 dxdt$. By integration by parts, we first obtain

$$\begin{aligned}
&\iint_Q s^5\lambda^6\xi^5 e^{-2s\alpha}|\varphi_x|^2 dxdt\\
&\quad = \int_0^T [s^5\lambda^6\xi^5 e^{-2s\alpha}\varphi_x\varphi]_0^L dt - \iint_Q s^5\lambda^6\partial_x(\xi^5 e^{-2s\alpha})\varphi_x\varphi dxdt - \iint_Q s^5\lambda^6\xi^5 e^{-2s\alpha}\varphi_{2x}\varphi dxdt\\
&\quad = \int_0^T [s^5\lambda^6\xi^5 e^{-2s\alpha}\varphi_x\varphi]_0^L dt - \frac{1}{2}\int_0^T [s^5\lambda^6\partial_x(\xi^5 e^{-2s\alpha})|\varphi|^2]_0^L dt + \frac{1}{2}\iint_Q s^5\lambda^6\partial_{xx}(\xi^5 e^{-2s\alpha})|\varphi|^2 dxdt\\
&\quad - \iint_Q s^5\lambda^6\xi^5 e^{-2s\alpha}\varphi_{2x}\varphi dxdt.
\end{aligned}$$

We estimate each of those terms separately thanks to Lemma 15, Young's inequality and estimates (3.6).

$$\begin{aligned}
&\text{(A.22)}\quad |\int_0^T [s^5\lambda^6\xi^5 e^{-2s\alpha}\varphi_x\varphi]_0^L dt|\\
&\leq C\int_0^T s^6\lambda^7\xi^{*6}e^{-2s\alpha^*}(\int_0^L|\varphi|^2 + \int_0^L|\varphi\varphi_x|dx)dt + \epsilon\int_0^T s^4\lambda^5\xi^{*4}e^{-2s\alpha^*}(\int_0^L|\varphi_x|^2 + \int_0^L|\varphi_x\varphi_{2x}|dx)dt\\
&\quad \leq C\iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2 dxdt + \epsilon\iint_Q s^5\lambda^6\xi^5 e^{-2s\alpha}|\varphi_x|^2 dxdt + \epsilon\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2 dxdt,
\end{aligned}$$

$$\begin{aligned}
|\int_0^T [s^5\lambda^6\partial_x(\xi^5 e^{-2s\alpha})|\varphi|^2]_0^L dt| &\leq C\int_0^T s^6\lambda^7\xi^{*6}e^{-2s\alpha^*}(\int_0^L|\varphi|^2 + \int_0^L|\varphi\varphi_x|)dt\\
&\leq C\iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2 dxdt + \epsilon\iint_Q s^5\lambda^6\xi^5 e^{-2s\alpha}|\varphi_x|^2 dxdt,
\end{aligned}$$

$$|\iint_Q s^5\lambda^6\partial_{xx}(\xi^5 e^{-2s\alpha})|\varphi|^2 dxdt| \leq C\iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2 dxdt,$$

$$|\iint_Q s^5\lambda^6\xi^5 e^{-2s\alpha}\varphi_{2x}\varphi dxdt| \leq C\iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2 dxdt + \epsilon\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2 dxdt.$$

Thus, we get, for ϵ sufficiently small,

$$\text{(A.23)}\quad \iint_Q s^5\lambda^6\xi^5 e^{-2s\alpha}|\varphi_x|^2 dxdt \leq C\iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2 dxdt + \epsilon\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2 dxdt.$$

With all those previous estimates, we deduce that

(A.24)

$$\begin{aligned}
|I_{21}| \leq\, & C\iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2dxdt + \epsilon\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2dxdt + C\iint_Q s^5\lambda^6\xi^5 e^{-2s\alpha}|\varphi_x|^2dxdt \\
& + C\int_0^T s^3\lambda^3\xi^{*3}e^{-2s\alpha^*}(|b_2(t)|^2+|b_1(t)|^2)dt + C\int_0^T s\lambda\xi^* e^{-2s\alpha^*}(|b_3(t)|^2+|b_4(t)|^2)dt \\
\leq\, & C\iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2dxdt + \epsilon\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2dxdt \\
& + C\int_0^T s^3\lambda^3\xi^{*3}e^{-2s\alpha^*}(|b_2(t)|^2+|b_1(t)|^2)dt + C\int_0^T s\lambda\xi^* e^{-2s\alpha^*}(|b_3(t)|^2+|b_4(t)|^2)dt.
\end{aligned}$$

On the other side, with (3.11), we know that

$$\begin{aligned}
I =\, & \iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}\varphi(B_0+\partial_x B_1+\partial_{xx}^2 B_2)dxdt \\
=\, & \iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}\varphi B_0 dxdt - \iint_Q s^3\lambda^4\partial_x(\xi^3 e^{-2s\alpha})\varphi B_1 dxdt - \iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}\varphi_x B_1 dxdt \\
& + \iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}\varphi_{2x}B_2 dxdt + 2\iint_Q s^3\lambda^4\partial_x(\xi^3 e^{-2s\alpha})\varphi_x B_2\, dxdt + \iint_Q s^3\lambda^4\partial_{xx}(\xi^3 e^{-2s\alpha})\varphi B_2\, dxdt \\
& - \int_0^T s^3\lambda^4\xi^{*3}e^{-2s\alpha^*}(B_2(L,t)\varphi_x(L,t) - B_2(0,t)\varphi_x(0,t))dt \\
& - \int_0^T s^3\lambda^4[\partial_x(\xi^3 e^{-2s\alpha})B_2(x,t)\varphi(x,t)]_0^L\, dt \\
& + \int_0^T s^3\lambda^4\xi^{*3}e^{-2s\alpha^*}((B_1(L,t)+\partial_x B_2(L,t))\varphi(L,t) - (B_1(0,t)+\partial_x B_2(0,t))\varphi(0,t))dt.
\end{aligned}$$

By using (3.6), we have,

$$\begin{aligned}
|I| \leq C\Bigg(& \iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2dxdt + \iint_Q s^{-1}\xi^{-1}e^{-2s\alpha}|B_0|^2dxdt + \iint_Q s^4\lambda^5\xi^4 e^{-2s\alpha}|\varphi||B_1|dxdt \\
& + \iint_Q s^4\lambda^6\xi^4 e^{-2s\alpha}|\varphi_x|^2dxdt + \iint_Q s^2\lambda^2\xi^2 e^{-2s\alpha}|B_1|^2dxdt + \iint_Q s^5\lambda^6\xi^5 e^{-2s\alpha}|\varphi||B_2|dxdt \\
& + \iint_Q s^2\lambda^4\xi^2 e^{-2s\alpha}|\varphi_{2x}|^2dxdt + \iint_Q s^4\lambda^4\xi^4 e^{-2s\alpha}|B_2|^2dxdt + \iint_Q s^4\lambda^5\xi^4 e^{-2s\alpha}|\varphi_x||B_2|dxdt \\
& + \int_0^T s^5\lambda^7\xi^{*5}e^{-2s\alpha^*}(|\varphi(L,t)|^2+|\varphi(0,t)|^2)dt + \int_0^T s^3\lambda^5\xi^{*3}e^{-2s\alpha^*}(|\varphi_x(L,t)|^2+|\varphi_x(0,t)|^2)dt \\
& + \int_0^T s^3\lambda^3\xi^{*3}e^{-2s\alpha^*}(|B_2(L,t)|^2+|B_2(0,t)|^2)dt \\
& + \int_0^T s\lambda\xi^* e^{-2s\alpha^*}(|B_1(L,t)|^2+|B_1(0,t)|^2+|\partial_x B_2(L,t)|^2+|\partial_x B_2(0,t)|^2)dt\Bigg)
\end{aligned}$$

And then,

$$
\begin{aligned}
|I| \leq C \Bigg(& \iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2 dxdt + \iint_Q s^4\lambda^6\xi^4 e^{-2s\alpha}|\varphi_x|^2 dxdt + \iint_Q s^2\lambda^4\xi^2 e^{-2s\alpha}|\varphi_{2x}|^2 dxdt \\
& + \iint_Q s^{-1}\xi^{-1} e^{-2s\alpha}|B_0|^2 dxdt + \iint_Q s^2\lambda^2\xi^2 e^{-2s\alpha}|B_1|^2 dxdt + \iint_Q s^4\lambda^4\xi^4 e^{-2s\alpha}|B_2|^2 dxdt \\
& + \int_0^T s^5\lambda^7\xi^{*5} e^{-2s\alpha^*}(|\varphi(L,t)|^2 + |\varphi(0,t)|^2)dt + \int_0^T s^3\lambda^5\xi^{*3} e^{-2s\alpha^*}(|\varphi_x(L,t)|^2 + |\varphi_x(0,t)|^2)dt \\
& + \int_0^T s^3\lambda^3\xi^{*3} e^{-2s\alpha^*}(|B_2(L,t)|^2 + |B_2(0,t)|^2)dt \\
& + \int_0^T s\lambda\xi^* e^{-2s\alpha^*}(|B_1(L,t)|^2 + |B_1(0,t)|^2 + |\partial_x B_2(L,t)|^2 + |\partial_x B_2(0,t)|^2)dt \Bigg).
\end{aligned}
$$

Thus, with the help of (A.23) and Lemma 15, we obtain an estimation on $|I|$.

$$
\begin{aligned}
|I| \leq C \Bigg(& \iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2 dxdt + \iint_Q s^{-1}\xi^{-1} e^{-2s\alpha}|B_0|^2 dxdt \\
& + \iint_Q s^2\lambda^2\xi^2 e^{-2s\alpha}|B_1|^2 dxdt + \iint_Q s^4\lambda^4\xi^4 e^{-2s\alpha}|B_2|^2 dxdt \\
& + \int_0^T s^3\lambda^3\xi^{*3} e^{-2s\alpha^*}(|B_2(L,t)|^2 + |B_2(0,t)|^2)dt \\
& + \int_0^T s\lambda\xi^* e^{-2s\alpha^*}(|B_1(L,t)|^2 + |B_1(0,t)|^2 + |\partial_x B_2(L,t)|^2 + |\partial_x B_2(0,t)|^2)dt \Bigg) \\
& + \epsilon \iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2 dxdt.
\end{aligned} \tag{A.25}
$$

We are now ready to get the wanted estimate with (A.19), (A.20), (A.24) and (A.25),

$$
\begin{aligned}
\iint_Q & s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2 dxdt \leq |I| + |I_1| + |I_{21}| \\
& \leq C \Bigg(\iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2 dxdt + \iint_Q s^{-1}\xi^{-1} e^{-2s\alpha}|B_0|^2 dxdt + \iint_Q s^2\lambda^2\xi^2 e^{-2s\alpha}|B_1|^2 dxdt \\
& + \iint_Q s^4\lambda^4\xi^4 e^{-2s\alpha}|B_2|^2 dxdt \\
& + \int_0^T s^3\lambda^3\xi^{*3} e^{-2s\alpha^*}(|b_2(t)|^2 + |b_1(t)|^2 + |B_2(L,t)|^2 + |B_2(0,t)|^2)dt \\
& + \int_0^T s\lambda\xi^* e^{-2s\alpha^*}(|b_3(t)|^2 + |b_4(t)|^2 + |B_1(L,t)|^2 + |B_1(0,t)|^2 + |\partial_x B_2(L,t)|^2 + |\partial_x B_2(0,t)|^2)dt \Bigg) \\
& + \epsilon \iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|dxdt.
\end{aligned}
$$

Thus for ϵ sufficiently small we have :

(A.26)

$$\begin{aligned}
&\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_{2x}|^2 dxdt \\
&\quad \leq C\left(\iint_Q s^7\lambda^8\xi^7 e^{-2s\alpha}|\varphi|^2 dxdt + \iint_Q s^{-1}\xi^{-1}e^{-2s\alpha}|B_0|^2 dxdt + \iint_Q s^2\lambda^2\xi^2 e^{-2s\alpha}|B_1|^2 dxdt\right. \\
&\quad + \iint_Q s^4\lambda^4\xi^4 e^{-2s\alpha}|B_2|^2 dxdt \\
&\quad + \int_0^T s^3\lambda^3\xi^{*3}e^{-2s\alpha^*}(|b_2(t)|^2 + |b_1(t)|^2 + |B_2(L,t)|^2 + |B_2(0,t)|^2)dt \\
&\quad \left. + \int_0^T s\lambda\xi^* e^{-2s\alpha^*}(|b_3(t)|^2 + |b_4(t)|^2 + |B_1(L,t)|^2 + |B_1(0,t)|^2 + |\partial_x B_2(L,t)|^2 + |\partial_x B_2(0,t)|^2)dt\right).
\end{aligned}$$

And with (A.18) and (A.23), we have the desired Carleman estimate of Theorem 6. ■

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