Control problems for the Navier-Stokes system with nonlocal spatial terms

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Abstract

We consider the local null controllability of a modified Navier-Stokes system where we include nonlocal spatial terms. We generalize a previous work where the nonlocal spatial term is given by the linearization of a Ladyzhenskaya model for a viscous incompressible fluid. Here the nonlocal spatial term is more general and we consider a control with one vanishing component. The proof of the result is based on a Carleman estimate where the main difficulty consists in handling the nonlocal spatial terms. One key point corresponds to a particular decomposition of the solution of the adjoint system that allows us to overcome regularity issues. With a similar approach, we also show the existence of insensitizing controls for the same system.

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Contents

1 Introduction

Assume Ω is a domain of $\mathbb{R}^d$, $d = 2, 3$ with a smooth boundary $\partial\Omega$. We are interested in the control of the Navier-Stokes system with nonlocal spatial terms:

$$\begin{cases} \partial_t z - \Delta z + \displaystyle\sum_{i=1}^{n} \left(\int_\Omega b^{(i)} \cdot z \, dx \right) a^{(i)} + \nabla p + (z \cdot \nabla) z = f + 1_\omega u & \text{in } (0,T) \times \Omega, \\ \operatorname{div} z = 0 & \text{in } (0,T) \times \Omega, \\ z = 0 & \text{on } (0,T) \times \partial\Omega, \\ z(0,\cdot) = z^0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here, the state of the system is the velocity $z(t,x) \in \mathbb{R}^d$ and the pressure $p(t,x) \in \mathbb{R}^d$ of the fluid. Note that $a^{(i)}(t,x), b^{(i)}(t,x) \in \mathbb{R}^d$ are given vector fields and that the control $u = u(t,x) \in \mathbb{R}^d$ is localized in a nonempty domain $\omega \subset \Omega$. A similar system was considered in [18] for controllability issues, and in that article, the nonlocal spatial term appears from the linearization of a model considered by Ladyzhenskaya [21]. Some authors have also considered the controllability of partial differential equations with nonlocal spatial terms in [2, 15, 16, 19, 20, 22, 23, 24, 27]. Note in particular that in [16], the authors consider a general nonlocal term in the heat and in the wave equation of the form

$$\int_\Omega k(x,y) z(t,y) \, dy. \tag{1.2}$$

The nonlocal term in (1.1) can be written as above by setting

$$k(x,y) = \sum_{i=1}^{n} a^{(i)}(x) \otimes b^{(i)}(y), \text{ where } \quad a^{(i)}, b^{(i)} : \Omega \to \mathbb{R}^d, \quad (i = 1, \dots, n) \tag{1.3}$$

are given functions. In [16], the authors do not need this particular form of the kernel and show the null-controllability of the heat equation but their method is based on a compactness-uniqueness argument that does not permit to deduce directly a controllability result on the nonlinear systems. Moreover, their kernel has to satisfy analytical conditions. Let us also emphasize that in [15], the authors consider a nonlinear heat equation with a nonlocal term similar to the one here. They first show the approximate controllability of the linearized system by using a compactness-uniqueness argument and then deduce the approximate controllability of the nonlinear system by using a Kakutani fixed point argument. They obtain the local exact controllability to trajectories with a passage to the limit. In [18], the controllability of a Ladyzhenskaya model (see [21]) is considered: the corresponding equations are the Navier-Stokes system with a viscosity depending on the $L^2(\Omega)$ norm of the curl of the fluid velocity. By linearizing around a trajectory, one obtains a local term of the above form but with $n = 1$. The controllability of this linear system is proved in [18] by using Carleman estimates and leads to the controllability to the trajectories for the Ladyzhenskaya model. Here, we aim at extending this result in the case where the control has one vanishing component and also for the insensitizing control problem. Nevertheless, the controllability of the Navier-Stokes system for controls with one vanishing component is known only for the null-controllability (if there is no condition on the domain of the control). We thus consider here only the null-controllability of (1.1) (and the corresponding insensitizing control problem). We also replace the nonlocal spatial term coming from the linearization of the Ladyzhenskaya model by a more complex spatial term, that is (1.3) with $n \geqslant 1$.

Controllability of fluid systems with vanishing components for the control has been studied in several articles, see, for instance, [4], [3], [7] and [14]. Note that in [8], the authors obtain the local null controllability of the Navier-Stokes system in dimension 3 with a control having two vanishing components. Their method is different than the one used in the previous works and is based on a particular linearization and on results of Gromov. In the case of insensitizing controls, the problem associated with vanishing components for the control is tackled in [6] and in [5]. Let us remark that with the method developed here we slightly improve the result obtained in [6]. Indeed their observability results are obtained with more regularity on the source terms than what is needed here.

In order to study the controllability of (1.1), we linearize it as follows

$$\begin{cases} \partial_t z - \Delta z + \displaystyle\sum_{i=1}^{n} \left(\int_\Omega b^{(i)} \cdot z \, dx \right) a^{(i)} + \nabla p = f + 1_\omega u & \text{in } (0,T)\times\Omega, \\ \operatorname{div} z = 0 & \text{in } (0,T)\times\Omega, \\ z = 0 & \text{on } (0,T)\times\partial\Omega, \\ z(0,\cdot) = z^0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

and we show the observability of the corresponding adjoint system that writes as follows:

$$\begin{cases} -\partial_t \varphi - \Delta \varphi + \displaystyle\sum_{i=1}^{n} \left(\int_\Omega a^{(i)} \cdot \varphi \, dx \right) b^{(i)} + \nabla \pi = g & \text{in } (0,T)\times\Omega, \\ \operatorname{div} \varphi = 0 & \text{in } (0,T)\times\Omega, \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,\cdot) = \varphi_T & \text{in } \Omega. \end{cases} \tag{1.5}$$

A standard method to obtain such an observability inequality on the above system was introduced in [17] and is based on Carleman estimates (see, [12] for a general presentation and [13] for the use of Carleman estimate to deal with the Navier-Stokes system).

In what follows, we assume that $d = 2$ and we suppose that our control u in (1.1) satisfies the following condition

$$u \cdot e_2 = 0, \tag{1.6}$$

where (e_1, e_2) is the canonical basis of $\mathbb{R}^2$. One can obtain the same results as the ones stated in this article for $d = 3$ or for $u \cdot e = 0$ for any vector $e \in \mathbb{R}^3$.

Let us present here our hypotheses on the functions $a^{(i)}$ and $b^{(i)}$, $i = 1, \ldots, n$. First we assume the following regularity conditions on $a^{(i)}$ and $b^{(i)}$:

$$a^{(i)} \in H^2(0,T;L^2(\Omega)), \quad b^{(i)} \in H^2(0,T;L^2(\Omega)) \cap L^2(0,T;H^4(\Omega)). \tag{1.7}$$

Then, we introduce

$$\breve{b}^{(i)} := \left(\Delta b_1^{(i)} - \partial_{x_1} \operatorname{div} b^{(i)} \right) = \partial_{x_2} \left(\partial_{x_2} b_1^{(i)} - \partial_{x_1} b_2^{(i)} \right). \tag{1.8}$$

Note that from Sobolev embedding (since $d = 2$, but it would also holds for $d = 3$), we have

$$\breve{b}^{(i)} \in C^0([0,T] \times \overline{\Omega}).$$

We assume the existence of nonempty open sets ω_i $(i = 1, \ldots, n)$ and of a constant $c > 0$ such that

$$\bigcup_{i=1}^{n} \omega_i \Subset \omega \tag{1.9}$$

and such that for any $i = 1, \ldots, n$,

$$\left| \breve{b}^{(i)} \right| \geqslant c, \quad \breve{b}^{(j)} \equiv 0 \quad (j > i) \quad \text{in } (0,T) \times \omega_i. \tag{1.10}$$

Remark 1.1. Before considering the observability of (1.5), one can consider the corresponding unique continuation property: let us consider φ a solution of (1.5) with $g \equiv 0$. If we assume that

$$\varphi_1 \equiv 0 \quad \text{in } (0,T) \times \omega, \tag{1.11}$$

do we have $\varphi \equiv 0$ in $(0,T) \times \Omega$? It seems difficult to find a necessary and sufficient condition for such a property. We can deduce from (1.11) that φ satisfies

$$\sum_{i=1}^{n} \left(\int_\Omega a^{(i)} \cdot \varphi \, dx \right) b_1^{(i)} + \frac{\partial \pi}{\partial x_1} \equiv 0 \quad \text{in } (0,T) \times \omega,$$

but even for $n = 1$, this equation involving the pressure may be difficult to use. One way to get rid of the pressure consists in applying Δ on the first equation of (1.5) and it leads to the heat equation

$$-\partial_t (\Delta \varphi_1) - \Delta (\Delta \varphi_1) + \sum_{i=1}^{n} \left(\int_\Omega a^{(i)} \cdot \varphi \, dx \right) \breve{b}^{(i)} = 0 \quad \text{in } (0,T) \times \Omega.$$

From (1.11), this implies that

$$\sum_{i=1}^{n} \left(\int_\Omega a^{(i)} \cdot \varphi \, dx \right) \breve{b}^{(i)} \equiv 0 \quad \text{in } (0,T) \times \omega.$$

In particular, if for all t the family $\left(\breve{b}^{(i)}(t,\cdot)\right)_i$ is linearly independent in ω, then we deduce that $\int_\Omega a^{(i)} \cdot \varphi \, dx = 0$ for all i and for all t and we see that φ is a solution of the standard Stokes system. Using [7], this implies that $\varphi \equiv 0$ in $(0,T) \times \Omega$. In our result, we have used the stronger hypothesis (1.10) that implies the linear independence of the family $\left(\breve{b}^{(i)}\right)_i$. We could expect that such a condition of linear independence should be sufficient for the observability of (1.5) but this is an open question at the moment. Note that in our Carleman estimates (see Section 3.2), we follow the above strategy to get rid of the pressure. Let us also remark that the linear independence of the family $\left(\breve{b}^{(i)}(t,\cdot)\right)_i$ in ω implies the linear independence of the family $\left(b^{(i)}(t,\cdot)\right)_i$ in ω.

Remark 1.2. In the proof of the main results, we can see that if $\breve{b}^{(i)} \equiv 0$ in $(0,T) \times \Omega$ for some i, then the corresponding local term does not play any role and can be removed at the beginning of the proof (see Section 3.2). Thus, the condition (1.10) can be assumed only for the subfamily of $(b^{(i)})$ such that $\breve{b}^{(i)} \not\equiv 0$. For such a family, there exist nonempty open sets $\widetilde{\omega}_i$ $(i = 1, \dots, n)$ and a constant $c > 0$ such that

$$\left|\breve{b}^{(i)}\right| \geqslant c \quad \text{in } (0,T) \times \widetilde{\omega}_i.$$

The hypothesis (1.10) is stronger than the above condition and, as explained above, it is not clear if it is necessary for the controllability of (1.1). This technical condition allows us to deal with the nonlocal terms.

Notation. In the whole paper, we use C as a generic positive constant that does not depend on the other terms of the inequality. The value of the constant C may change from one appearance to another. We also use the notation $X \lesssim Y$ if there exists a constant $C > 0$ such that we have the inequality $X \leqslant CY$.

Let us state our first main result:

Theorem 1.3. *Let us consider $T > 0$ and assume (1.7), (1.9) and (1.10). There exist continuous functions $\sigma_i : [0,T] \to \mathbb{R}_+$, $i = 1,2,3$, with $\sigma_i > 0$ in $[0,T)$, $\sigma_i(T) = 0$ such that any solution φ of (1.5) satisfies*

$$\|\sigma_3 \varphi\|_{L^2(0,T;L^2(\Omega))} + \|\varphi(0,\cdot)\|_{L^2(\Omega)} \lesssim \|\sigma_2 \varphi_1\|_{L^2(0,T;L^2(\omega))} + \|\sigma_1 g\|_{L^2(0,T;L^2(\Omega))}. \tag{1.12}$$

The functions $\sigma_1, \sigma_2, \sigma_3$ are defined precisely by (3.1), (3.2) in Section 3 by using the Carleman weights that we describe in Section 2.2.

Remark 1.4. The above result implies in particular the unique continuation property discussed in Remark 1.1 if we assume the sufficient conditions (1.7), (1.9) and (1.10). It also implies the final state observability of the system (1.5) with $g \equiv 0$ (see [26, Definition 6.1.1, p.173] for a precise definition of this concept). Following the method in [15], [16], for instance, it might be possible to show directly this inequality by using a compactness-uniqueness argument if we can show the unique continuation property by a different method. Using a standard duality argument, this would allow us to obtain the null controllability of (1.4). However, without the weight functions in (1.12), the null-controllability of the nonlinear system (1.1) seems difficult to reach and that is why we rely here on a proof based on Carleman estimates.

By standard methods (see, for instance, Section 4 in [18]), we deduce from Theorem 1.3 the following result:

Corollary 1.5. *Assume* (1.7), (1.9) *and* (1.10). *There exists a continuous function* $\sigma_0 : [0,T] \to \mathbb{R}_+$, *with* $\sigma_0 > 0$ *in* $[0,T)$, $\sigma_0(T) = 0$ *such that the following property holds: for any*

$$z^0 \in H^1_0(\Omega), \quad \operatorname{div} z^0 = 0, \qquad \frac{f}{\sigma_3} \in L^2((0,T)\times\Omega),$$

there exists $u \in L^2(0,T;L^2(\omega))$ *satisfying* (1.6) *and such that the solution* z *of* (1.4) *satisfies*

$$\left\| \frac{z}{\sigma_0} \right\|_{L^2(0,T;H^2(\Omega))\cap C^0([0,T];H^1(\Omega))\cap H^1(0,T;L^2(\Omega))} \leqslant C \left(\left\| \frac{f}{\sigma_3} \right\|_{L^2((0,T)\times\Omega)} + \|z^0\|_{H^1(\Omega)} \right). \tag{1.13}$$

In particular, $z(T,\cdot) = 0$.

Moreover, there exists a constant c_0 *such that if*

$$\|z^0\|_{H^1(\Omega)} + \left\| \frac{f}{\sigma_3} \right\|_{L^2((0,T)\times\Omega)} \leqslant c_0,$$

there exists $u \in L^2(0,T;L^2(\omega))$ *such that the solution* z *of* (1.1) *satisfies* (1.13) *and in particular,* $z(T,\cdot) = 0$.

The precise definition of σ_0 is given by (3.3) by using the Carleman weights in Section 2.2, with the condition (3.5) for the null-controllability of (1.1).

Let us consider the existence of insensitizing controls for (1.1). More precisely, we assume that the initial condition contains uncertainties, and we consider the corresponding system:

$$\begin{cases} \partial_t z_\tau - \Delta z_\tau + \displaystyle\sum_{i=1}^n \left(\int_\Omega b^{(i)} \cdot z_\tau \, dx \right) a^{(i)} + \nabla p_\tau + (z_\tau \cdot \nabla) z_\tau = f + 1_{\omega_\natural} u & \text{in } (0,T)\times\Omega, \\ \operatorname{div} z_\tau = 0 & \text{in } (0,T)\times\Omega, \\ z_\tau = 0 & \text{on } (0,T)\times\partial\Omega, \\ z_\tau(0,\cdot) = z^0 + \tau \widehat{z}^0 & \text{in } \Omega, \end{cases}$$

where $\|\widehat{z}^0\|_{L^2(\Omega)} = 1$ and $\tau \in \mathbb{R}_+$ is assumed to be small. For $\tau = 0$, we write z instead of z_τ (so that z is the solution of (1.1) with ω replaced by $\omega_\natural$). We have replaced ω by $\omega_\natural$ since we need here a control region larger than for the controllability result of the previous section. The system is observed by a sentinel $\mathcal{J}$ chosen here as

$$\mathcal{J}(z_\tau) := \frac{1}{2} \iint_{(0,T)\times\mathcal{O}} |z_\tau|^2 \, dx \, dt,$$

where $\mathcal{O}$ is a nonempty open subset of Ω. The insensitization problem consists in finding a control u such that

$$\left. \frac{d\mathcal{J}(z_\tau)}{d\tau} \right|_{\tau=0} = 0 \quad \forall \widehat{z}^0 \in L^2(\Omega) \quad \text{with } \|\widehat{z}^0\|_{L^2(\Omega)} = 1.$$

However, by considering the system

$$\begin{cases} -\partial_t w - \Delta w + \displaystyle\sum_{i=1}^{n} \left(\int_\Omega a^{(i)} \cdot w \, dx \right) b^{(i)} + \nabla q + (\nabla z)^* w - (z \cdot \nabla) w = z 1_{\mathcal{O}} & \text{in } (0,T) \times \Omega, \\ \operatorname{div} w = 0 & \text{in } (0,T) \times \Omega, \\ w = 0 & \text{on } (0,T) \times \partial\Omega, \\ w(T, \cdot) = 0 & \text{in } \Omega, \end{cases} \tag{1.14}$$

one can write (after some standard computations)

$$\left. \frac{d\mathcal{J}(z_\tau)}{d\tau} \right|_{\tau=0} = \int_\Omega w(0,x) \cdot \hat{z}^0(x) \, dx.$$

Consequently, it is equivalent to search for a control u such that the solution w of (1.14) satisfies $w(0,\cdot) = 0$. Since the right-hand side of w is $z 1_{\mathcal{O}}$ where z is the solution of (1.1), we deal with a controllability problem for a cascade system with a forward equation and a backward equation. It is usual to avoid some possible issues at $t = 0$ by assuming $z^0 = 0$. In fact, it is a difficult problem to determine an adequate class for the initial conditions z^0 such that the result below remains valid. One can quote [9] where the authors tackle this issue for the heat equation. In order to show this null-controllability result, we consider the linearized system:

$$\begin{cases} \partial_t z - \Delta z + \displaystyle\sum_{i=1}^{n} \left(\int_\Omega b^{(i)} \cdot z \, dx \right) a^{(i)} + \nabla p = f^{(0)} + 1_{\omega_\natural} u & \text{in } (0,T) \times \Omega, \\ -\partial_t w - \Delta w + \displaystyle\sum_{i=1}^{n} \left(\int_\Omega a^{(i)} \cdot w \, dx \right) b^{(i)} + \nabla q = f^{(1)} + z 1_{\mathcal{O}} & \text{in } (0,T) \times \Omega, \\ \operatorname{div} z = \operatorname{div} w = 0 & \text{in } (0,T) \times \Omega, \\ z = w = 0 & \text{on } (0,T) \times \partial\Omega, \\ z(0,\cdot) = 0 \quad \text{in } \Omega, \quad w(T,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{1.15}$$

and its adjoint system:

$$\begin{cases} -\partial_t \varphi - \Delta \varphi + \displaystyle\sum_{i=1}^{n} \left(\int_\Omega a^{(i)} \cdot \varphi \, dx \right) b^{(i)} + \nabla \pi_\varphi = \psi 1_{\mathcal{O}} + g^{(0)} & \text{in } (0,T) \times \Omega, \\ \partial_t \psi - \Delta \psi + \displaystyle\sum_{i=1}^{n} \left(\int_\Omega b^{(i)} \cdot \psi \, dx \right) a^{(i)} + \nabla \pi_\psi = g^{(1)} & \text{in } (0,T) \times \Omega, \\ \operatorname{div} \varphi = \operatorname{div} \psi = 0 & \text{in } (0,T) \times \Omega, \\ \varphi = \psi = 0 & \text{on } (0,T) \times \partial\Omega, \\ \varphi(T,\cdot) = 0 \quad \text{in } \Omega, \quad \psi(0,\cdot) = \psi^0 & \text{in } \Omega. \end{cases} \tag{1.16}$$

We introduce

$$\breve{a}^{(i)} := \left(\Delta a_1^{(i)} - \partial_{x_1} \operatorname{div} a^{(i)} \right). \tag{1.17}$$

We assume that $a^{(i)}$ and $b^{(i)}$ satisfy (1.7), (1.10) and we add the following hypotheses: first we suppose the additional regularity on $a^{(i)}$

$$a^{(i)} \in L^2(0,T; H^4(\Omega)). \tag{1.18}$$

Note that from Sobolev embeddings, (since $d = 2$, but is also holds for $d = 3$), we have

$$\breve{a}^{(i)} \in C^0([0,T] \times \overline{\Omega}).$$

Then, we assume the existence of nonempty open sets $\widehat{\omega}_i$ $(i = 1, \dots, n)$, $\widehat{\omega}$ and a constant $c > 0$ such that

$$\bigcup_{i=1}^{n} \widehat{\omega}_i \Subset \widehat{\omega} \Subset \mathcal{O}, \quad \omega_\natural := \omega \cup \widehat{\omega} \tag{1.19}$$

with the following properties

$$\breve{b}_i \equiv 0 \quad \text{in } (0,T)\times\widehat{\omega} \quad (i=1,\dots,n), \tag{1.20}$$

and for any $i=1,\dots,n$,

$$\left|\breve{a}^{(i)}\right| \geqslant c, \quad \breve{a}^{(j)} \equiv 0 \quad (j>i) \quad \text{in } (0,T)\times\widehat{\omega}_i. \tag{1.21}$$

Note that condition (1.19) yields in particular that $\mathcal{O}\cap\omega_\natural \neq \emptyset$.

Remark 1.6. We don't need in (1.21) to have the same order as (1.10) and we can thus permute the indices i in (1.21).

Theorem 1.7. *Assume* (1.7)*,* (1.9)*,* (1.10)*,* (1.18)*,* (1.19)*,* (1.20) *and* (1.21)*. There exist continuous functions* $\widehat{\sigma}_i : [0,T] \to \mathbb{R}_+$*,* $i=1,2,3$*, with* $\widehat{\sigma}_i > 0$ *in* $(0,T]$*,* $\widehat{\sigma}_i(0)=0$ *such that any solution* (φ,ψ) *of* (1.16) *satisfies*

$$\|\widehat{\sigma}_3\varphi\|_{L^2(0,T;L^2(\Omega))} + \|\widehat{\sigma}_3\psi\|_{L^2(0,T;L^2(\Omega))} \lesssim \|\widehat{\sigma}_2\varphi_1\|_{L^2(0,T;L^2(\omega_\natural))} + \left\|\widehat{\sigma}_1 g^{(0)}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\widehat{\sigma}_1 g^{(1)}\right\|_{L^2(0,T;L^2(\Omega))}. \tag{1.22}$$

We give the precise definitions of $\widehat{\sigma}_i$, $i=1,2,3$ in Section 4.

Remark 1.8. Even if we remove the nonlocal spatial terms in our systems (1.15) and (1.16), the above result improves the insensitizing control result obtained in [6]. Indeed, in this previous work, a regularity condition for the source terms is $g^{(1)} \in L^2(0,T;H^1(\Omega))$ whereas, with our approach, we only need $g^{(1)} \in L^2(0,T;L^2(\Omega))$. A tool that we introduce here to deal with these regularity problems is an adequate decomposition of the solution of the adjoint system. In many articles devoted to the controllability of the Navier-Stokes system, the solution of the adjoint system is decomposed into two functions and here we show how to generalize such a decomposition by splitting the solution in several functions which allows us to work with a regular solution for the Carleman estimate. A similar decomposition is already proposed in [1] but they need to modify the Carleman weights so that they are regular at $t=0$. Such a strategy cannot be applied in the context of insensitizing controls since we have to work with a forward system and a backward system. We thus need to consider another decomposition as in [1].

We deduce from the above result the existence of insensitizing controls for the linear and for the nonlinear problem:

Corollary 1.9. *Assume* (1.7)*,* (1.10)*,* (1.9)*,* (1.18)*,* (1.21)*,* (1.19)*, and* (1.20)*. There exists a continuous function* $\widehat{\sigma}_0 : [0,T] \to \mathbb{R}_+$*, with* $\widehat{\sigma}_0 > 0$ *in* $(0,T]$*,* $\widehat{\sigma}_0(0)=0$ *such that for any*

$$\frac{f^{(0)}}{\widehat{\sigma}_3}, \frac{f^{(1)}}{\widehat{\sigma}_3} \in L^2((0,T)\times\Omega),$$

there exists $u \in L^2(0,T;L^2(\omega))$ *satisfying* (1.6) *and such that the solution* (z,w) *of* (1.15) *satisfies*

$$\left\|\frac{(z,w)}{\widehat{\sigma}_0}\right\|_{L^2(0,T;H^2(\Omega))\cap C^0([0,T];H^1(\Omega))\cap H^1(0,T;L^2(\Omega))} \lesssim \left\|\frac{\left(f^{(0)},f^{(1)}\right)}{\widehat{\sigma}_3}\right\|_{L^2((0,T)\times\Omega)}.$$

In particular, $w(0,\cdot)=0$*.*

Moreover, there exists a constant c_0 *such that for any*

$$\left\|\frac{f}{\widehat{\sigma}_3}\right\|_{L^2((0,T)\times\Omega)} \leqslant c_0,$$

there exists $u \in L^2(0,T;L^2(\omega))$ *such that the solution* (z,w) *of* (1.1)*,* (1.14) *with* $z^0=0$ *satisfies*

$$\left\|\frac{(z,w)}{\widehat{\sigma}_0}\right\|_{L^2(0,T;H^2(\Omega))\cap C^0([0,T];H^1(\Omega))\cap H^1(0,T;L^2(\Omega))} \lesssim \left\|\frac{f}{\widehat{\sigma}_3}\right\|_{L^2((0,T)\times\Omega)}. \tag{1.23}$$

In particular, $w(0,\cdot)=0$*.*

The precise choice for $\widehat{\sigma}_0$ is given by (4.5) in Section 4.

Remark 1.10. A different strategy to show the null-controllability of (1.15) would consist in using a fictitious control method such as in [10]. In order to use this approach, one has to solve an algebraic problem for the Stokes system with nonlocal spatial terms, which is a difficult issue but that might give different conditions from (1.21).

The outline of the article is as follows: in the next section, we start by some well-posedness and regularity results for linear systems of the form (1.5) and (1.16). Then we introduce the Carleman weights we will use to show Theorem 1.3 and we recall some classical Carleman estimates. In Section 3, we show Theorem 1.3 by introducing our new decomposition for the solution of the adjoint system and then performing a Carleman estimate on one of the terms of this decomposition. Finally, Section 4 is devoted to the proof of Theorem 1.7 and Corollary 1.9 by using a similar decomposition as the one used in the proof of Theorem 1.3.

2 Preliminaries

2.1 Well-posedness results

Notation. To simplify the notation in the article, we write

$$\mathcal{L}(\psi, \pi) = (g, \psi^0) \tag{2.1}$$

if

$$\begin{cases} \partial_t \psi - \Delta\psi + \displaystyle\sum_{i=1}^n \left(\int_\Omega b^{(i)} \cdot \psi \, dx \right) a^{(i)} + \nabla\pi = g & \text{in } (0,T)\times\Omega, \\ \operatorname{div}\psi = 0 & \text{in } (0,T)\times\Omega, \\ \psi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \psi(0,\cdot) = \psi^0 & \text{in } \Omega, \end{cases}$$

and

$$\mathcal{L}^*(\varphi, \pi) = (g, \varphi_T) \tag{2.2}$$

if

$$\begin{cases} -\partial_t \varphi - \Delta\varphi + \displaystyle\sum_{i=1}^n \left(\int_\Omega a^{(i)} \cdot \varphi \, dx \right) b^{(i)} + \nabla\pi = g & \text{in } (0,T)\times\Omega, \\ \operatorname{div}\varphi = 0 & \text{in } (0,T)\times\Omega, \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \varphi(T,\cdot) = \varphi_T & \text{in } \Omega. \end{cases} \tag{2.3}$$

We also set

$$\begin{aligned} X_1 &:= L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)), \\ X_2 &:= L^2(0,T;H^4(\Omega)) \cap H^2(0,T;L^2(\Omega)), \\ X_3 &:= L^2(0,T;H^6(\Omega)) \cap H^3(0,T;L^2(\Omega)). \end{aligned}$$

Note that by interpolation results, we have

$$X_k = \left(\bigcap_{j=0}^{k} H^j(0,T;H^{2(k-j)}(\Omega)) \right) \bigcap \left(\bigcap_{j=0}^{k-1} C^j([0,T];H^{2(k-j)-1}(\Omega)) \right).$$

We have the following results that can be obtained in a standard way. For the sake of completeness, we give a sketch of the proof for the first proposition in Appendix A. The second one can be obtained in a similar way.

Proposition 2.1. *Assume* (1.7)*. Then, for any* $\varphi_T \in H_0^1(\Omega)$*,* $\operatorname{div} \varphi_T = 0$ *in* Ω *and* $g \in L^2(0,T;L^2(\Omega))$*, there exists a unique solution of* (2.2) $(\varphi,\pi) \in X_1 \times L^2(0,T;H^1(\Omega)/\mathbb{R})$ *and we have the estimate*

$$\|\varphi\|_{X_1} + \|\nabla\pi\|_{L^2(0,T;L^2(\Omega))} \lesssim \|\varphi_T\|_{H^1(\Omega)} + \|g\|_{L^2(0,T;L^2(\Omega))}. \tag{2.4}$$

Assume $g \in X_1$*, with* $g(T,\cdot) = 0$*. Then, the solution of* $\mathcal{L}^*(\varphi,\pi) = (g,0)$ *satisfies* $\varphi \in X_2$ *with the estimate*

$$\|\varphi\|_{X_2} \lesssim \|g\|_{X_1}. \tag{2.5}$$

Assume $g \in X_2$*, with* $g(T,\cdot) = \partial_t g(T,\cdot) = 0$*. Then, the solution of* $\mathcal{L}^*(\varphi,\pi) = (g,0)$ *satisfies* $\varphi \in X_3$ *with the estimate*

$$\|\varphi\|_{X_3} \lesssim \|g\|_{X_2}. \tag{2.6}$$

Proposition 2.2. *Assume* (1.7) *and* (1.18)*. Then, for any* $\psi^0 \in H_0^1(\Omega)$ *such that* $\operatorname{div} \psi^0 = 0$ *in* Ω *and for any* $g \in L^2(0,T;L^2(\Omega))$*, there exists a unique solution of* (2.1) $(\psi,\pi) \in X_1 \times L^2(0,T;H^1(\Omega)/\mathbb{R})$ *and we have the estimate*

$$\|\psi\|_{X_1} + \|\nabla\pi\|_{L^2(0,T;L^2(\Omega))} \lesssim \left\|\psi^0\right\|_{H^1(\Omega)} + \|g\|_{L^2(0,T;L^2(\Omega))}.$$

Assume $g \in X_1$*, with* $g(0,\cdot) = 0$*. Then the solution of* $\mathcal{L}(\psi,\pi) = (g,0)$ *satisfies* $\psi \in X_2$ *with the estimate*

$$\|\psi\|_{X_2} \lesssim \|g\|_{X_1}.$$

Assume $g \in X_2$*, with* $g(0,\cdot) = \partial_t g(0,\cdot) = 0$*. Then, the solution of* $\mathcal{L}(\psi,\pi) = (g,0)$ *satisfies* $\psi \in X_3$ *with the estimate*

$$\|\psi\|_{X_3} \lesssim \|g\|_{X_2}.$$

2.2 Some standard Carleman estimates

We recall here some Carleman estimates that were obtained in previous articles. Let us first introduce the weights functions. Using [17] (see also [26, Theorem 9.4.3, p.299]), there exist for any $i = 1,\dots,n$, $\eta_i \in C^2(\overline{\Omega})$ satisfying

$$\eta_i > 0 \text{ in } \Omega, \quad \eta_i = 0 \text{ on } \partial\Omega, \quad \max_{\Omega} \eta_i = 1, \quad \nabla\eta_i \neq 0 \text{ in } \overline{\Omega \setminus \omega_i}.$$

We use η_i to define the following standard functions:

$$\alpha_i(t,x) = \frac{\exp\{\lambda(2m+2)\} - \exp\{\lambda(2m+\eta_i(x))\}}{t^m(T-t)^m}, \quad \xi_i(t,x) = \frac{\exp\{\lambda(2m+\eta_i(x))\}}{t^m(T-t)^m},$$

where

$$m \geqslant 9, \quad \lambda > 1.$$

The minimum and the maximum of these functions with respect to x do not depend on i:

$$\alpha_\sharp(t) = \max_{x\in\overline{\Omega}} \alpha_i(t,x) = \frac{\exp\{\lambda(2m+2)\} - \exp\{2\lambda m\}}{t^m(T-t)^m}, \quad \xi_\sharp(t) = \min_{x\in\overline{\Omega}} \xi_i(t,x) = \frac{\exp\{2\lambda m\}}{t^m(T-t)^m}, \tag{2.7}$$

$$\alpha_\flat(t) = \min_{x\in\overline{\Omega}} \alpha_i(t,x) = \frac{\exp\{\lambda(2m+2)\} - \exp\{\lambda(2m+1)\}}{t^m(T-t)^m}, \quad \xi_\flat(t) = \max_{x\in\overline{\Omega}} \xi_i(t,x) = \frac{\exp\{\lambda(2m+1)\}}{t^m(T-t)^m}, \tag{2.8}$$

so that

$$\xi_\sharp \leqslant \xi_i \leqslant \xi_\flat, \quad e^{-s\alpha_\sharp} \leqslant e^{-s\alpha_i} \leqslant e^{-s\alpha_\flat} \quad (i = 1,\dots,n). \tag{2.9}$$

We also have the following useful formula: for any $\varepsilon \in (0,1)$, if $\lambda \geqslant -\ln\varepsilon$, then

$$e^{-s\alpha_i} \leqslant e^{-s(1-\varepsilon)\alpha_\sharp} \leqslant e^{-s(1-\varepsilon)\alpha_j} \quad (i,j \in \{1,\dots,n\}). \tag{2.10}$$

Note that we have the following useful relations: for any $j \geqslant 1$, there exists a constant $C > 0$

$$\left|\frac{d^j \alpha_\sharp}{dt^j}\right| + \left|\frac{d^j \xi_\sharp}{dt^j}\right| \leqslant CT^j \left(\xi_\sharp\right)^{1+j/m}. \tag{2.11}$$

We also have

$$\xi_\sharp \geqslant \frac{4^m}{T^{2m}}. \tag{2.12}$$

In particular, for $s_0 > 0$, if $s \geqslant s_0 T^{2m}$,

$$s\xi_\sharp \geqslant s_0 4^m. \tag{2.13}$$

We can now state a Carleman estimate for the gradient operator (see, for instance, [7, Lemma 3]):

Lemma 2.3. *Assume $i \in \{1, \ldots, n\}$ and $r \in \mathbb{R}$. There exist $\lambda_0, s_0 > 0$ depending on the geometry and on η_i such that for any $\lambda \geqslant \lambda_0$, $s \geqslant s_0 T^{2m}$ and $u \in L^2(0,T;H^1(\Omega))$,*

$$\iint_{(0,T)\times\Omega} e^{-2s\alpha_i} \left(s\lambda\xi_i\right)^{r+2} |u|^2 \, dx\, dt \lesssim \iint_{(0,T)\times\Omega} \left(s\lambda\xi_i\right)^{r} e^{-2s\alpha_i} |\nabla u|^2 \, dx\, dt + \iint_{(0,T)\times\omega_i} \left(s\lambda\xi_i\right)^{r+2} e^{-2s\alpha_i} |u|^2 \, dx\, dt.$$

In particular, for any $u \in L^2(0,T)$,

$$\iint_{(0,T)\times\Omega} e^{-2s\alpha_i} |u|^2 \, dx\, dt \lesssim \iint_{(0,T)\times\omega_i} e^{-2s\alpha_i} |u|^2 \, dx\, dt.$$

The second Carleman estimate we recall here corresponds to the Laplace operator (see, for instance, [7, Lemma 4]):

Lemma 2.4. *Assume $i \in \{1, \ldots, n\}$ and $r \in \mathbb{R}$. There exist $\lambda_0, s_0 > 0$ depending on the geometry and on η_i such that for any $\lambda \geqslant \lambda_0$, $s \geqslant s_0 T^{2m}$ and $u \in L^2(0,T;H^2(\Omega) \cap H^1_0(\Omega))$,*

$$\iint_{(0,T)\times\Omega} s^{r+4}\lambda^{r+6}\xi_i^{r+4} e^{-2s\alpha_i} |u|^2 \, dx\, dt + \iint_{(0,T)\times\Omega} s^{r+2}\lambda^{r+4}\xi_i^{r+2} e^{-2s\alpha_i} |\nabla u|^2 \, dx\, dt \lesssim \iint_{(0,T)\times\Omega} s^{r+1}\lambda^{r+2}\xi_i^{r+1} e^{-2s\alpha_i} |\Delta u|^2 \, dx\, dt + \iint_{(0,T)\times\omega_i} s^{r+4}\lambda^{r+6}\xi_i^{r+4} e^{-2s\alpha_i} |u|^2 \, dx\, dt.$$

Finally, we recall a Carleman estimate for the heat equation with Neumann boundary conditions:

$$\left\{\begin{array}{rll} \partial_t u + \Delta u = f^{(1)} & \text{in } (0,T)\times\Omega, \\ -\dfrac{\partial u}{\partial n} = f^{(2)} & \text{on } (0,T)\times\partial\Omega. \end{array}\right. \tag{2.14}$$

The following lemma is obtained in [11] (see also [7, Lemma 5]):

Lemma 2.5. *Assume $i \in \{1, \ldots, n\}$. There exist $\lambda_0, s_0 > 0$ depending on the geometry and on η_i such that for any $\lambda \geqslant \lambda_0$, $s \geqslant s_0(T^{2m} + T^m)$,*

$$f^{(1)} \in L^2(0,T;L^2(\Omega)), \ f^{(2)} \in L^2(0,T;L^2(\partial\Omega)),$$

and $u \in L^2(0,T;H^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$ weak solution of (2.14),

$$\iint_{(0,T)\times\Omega} s^3\lambda^4\xi_i^3 e^{-2s\alpha_i} |u|^2 \, dx\, dt + \iint_{(0,T)\times\Omega} s\lambda^2\xi_i e^{-2s\alpha_i} |\nabla u|^2 \, dx\, dt \lesssim \iint_{(0,T)\times\Omega} e^{-2s\alpha_i} |f^{(1)}|^2 \, dx\, dt + \iint_{(0,T)\times\partial\Omega} s\lambda\xi_\sharp e^{-2s\alpha_\sharp} |f^{(2)}|^2 \, d\gamma\, dt + \iint_{(0,T)\times\omega_i} s^3\lambda^4\xi_i^3 e^{-2s\alpha_i} |u|^2 \, dx\, dt.$$

Notation. In all that follows, we use s_0, λ_0 as generic positive constants that may change from one appearance to another, but always in an increasing way.

3 Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. The functions $\sigma_1, \sigma_2, \sigma_3$ are defined below by using the weights introduced in Section 2.2. More precisely, let us consider $N_1, N_2 > 0$. We write $N := N_1 + N_2$. We define

$$\sigma_1 := \begin{cases} e^{-N_2 s\alpha_\sharp} & \text{in } \left[\frac{T}{2}, T\right] \\ \left[e^{-N_2 s\alpha_\sharp}\right]\left(\frac{T}{2}\right) & \text{in } \left[0, \frac{T}{2}\right] \end{cases}, \quad \sigma_2 := \begin{cases} e^{-(N+1/2)s\alpha_\sharp} & \text{in } \left[\frac{T}{2}, T\right] \\ \left[e^{-(N+1/2)s\alpha_\sharp}\right]\left(\frac{T}{2}\right) & \text{in } \left[0, \frac{T}{2}\right] \end{cases}, \tag{3.1}$$

$$\sigma_3 := \begin{cases} s^3\lambda^4\xi_\sharp^3 e^{-(N+1)s\alpha_\sharp} & \text{in } \left[\frac{T}{2}, T\right] \\ \left[s^3\lambda^4\xi_\sharp^3 e^{-(N+1)s\alpha_\sharp}\right]\left(\frac{T}{2}\right) & \text{in } \left[0, \frac{T}{2}\right] \end{cases}. \tag{3.2}$$

Note that $\sigma_i(T) = 0$ $(i = 1, 2, 3)$.

We are going to show the existence of $s_0 > 0$ and $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $s \geqslant s_0(T^m + T^{2m})$, Theorem 1.3 with σ_i defined as above.

Then for Corollary 1.5, we assume $N_2 > 1$ and we define

$$\sigma_0 := \begin{cases} e^{-(N_2-1)s\alpha_\sharp} & \text{in } \left[\frac{T}{2}, T\right] \\ \left[e^{-(N_2-1)s\alpha_\sharp}\right]\left(\frac{T}{2}\right) & \text{in } \left[0, \frac{T}{2}\right] \end{cases}. \tag{3.3}$$

Note that

$$\frac{\sigma_2}{\sigma_0}, \ \frac{\sigma_3}{\sigma_0}, \ \frac{\sigma_0'\sigma_1}{\sigma_0^2} \in L^\infty(0,T) \tag{3.4}$$

and if

$$N_2 \geqslant N_1 + 3, \tag{3.5}$$

then, we have

$$\frac{\sigma_0^2}{\sigma_3} \in L^\infty(0,T).$$

These properties allow us to deduce Corollary 1.5 from Theorem 1.3 (see, for instance, Section 4 in [18]).

3.1 Decomposition of the adjoint system

In order to prove Theorem 1.3, we decompose the solution of (1.5) into three parts. We recall that $N_1, N_2 > 0$ and that $N = N_1 + N_2$. Then we set

$$\rho_i := e^{-N_i s\alpha_\sharp} \quad (i = 1, 2)$$

and we consider the following decomposition of the solution (φ, π) of (1.5):

$$\rho_1\rho_2\varphi = \rho_1\varphi^{(1)} + \varphi^{(2)} + \varphi^{(3)}, \tag{3.6}$$

where

$$\mathcal{L}^*(\varphi^{(1)}, \pi^{(1)}) = (\rho_2 g, 0), \tag{3.7}$$

$$\mathcal{L}^*(\varphi^{(2)}, \pi^{(2)}) = \left(N_2 s\alpha_\sharp' \rho_1\varphi^{(1)}, 0\right), \tag{3.8}$$

$$\mathcal{L}^*(\varphi^{(3)}, \pi^{(3)}) = \left(Ns\alpha_\sharp' \left(\varphi^{(2)} + \varphi^{(3)}\right), 0\right). \tag{3.9}$$

We are going to use Proposition 2.1 to estimate $\varphi^{(1)}$ and $\varphi^{(2)}$ and we show a Carleman estimate on $\varphi^{(3)}$. In order to state this estimate, we introduce the following quantities associated with $\varphi^{(3)}$:

$$\begin{aligned} I(\varphi^{(3)}) := &\iint_{(0,T)\times\Omega} e^{-2s\alpha_1} \left(s\lambda^2\xi_1 \left|\nabla^2\Delta\varphi_1^{(3)}\right|^2 + s^3\lambda^4\xi_1^3 \left|\nabla\Delta\varphi_1^{(3)}\right|^2\right) dx\ dt \\ &+ \iint_{(0,T)\times\Omega} e^{-2s\alpha_1} \left(s^5\lambda^6\xi_1^5 \left|\Delta\varphi_1^{(3)}\right|^2 + s^6\lambda^8\xi_1^6 \left|\nabla\varphi_1^{(3)}\right|^2 + s^8\lambda^{10}\xi_1^8 \left|\varphi_1^{(3)}\right|^2\right) dx\ dt \\ &+ \iint_{(0,T)\times\Omega} e^{-2s\alpha_\sharp} s^6\lambda^8\xi_\sharp^6 \left|\varphi^{(3)}\right|^2\ dx\ dt \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} J(\varphi^{(3)}) := &\left\|\theta_1\varphi^{(3)}\right\|^2_{L^2(0,T;H^2(\Omega))} + \left\|\theta_1\partial_t\varphi^{(3)}\right\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ \left\|\theta_2\varphi^{(3)}\right\|^2_{L^2(0,T;H^4(\Omega))} + \left\|\theta_2\partial_t\varphi^{(3)}\right\|^2_{L^2(0,T;H^2(\Omega))} + \left\|\theta_2\partial_t^2\varphi^{(3)}\right\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ \left\|\theta_3\varphi^{(3)}\right\|^2_{L^2(0,T;H^6(\Omega))} + \left\|\theta_3\partial_t\varphi^{(3)}\right\|^2_{L^2(0,T;H^4(\Omega))} + \left\|\theta_3\partial_t^2\varphi^{(3)}\right\|^2_{L^2(0,T;H^2(\Omega))} + \left\|\theta_3\partial_t^3\varphi^{(3)}\right\|^2_{L^2(0,T;L^2(\Omega))}, \end{aligned} \tag{3.11}$$

where

$$\theta_1 := s^{2-1/m}\lambda^4 \left(\xi_\sharp\right)^{2-1/m} e^{-s\alpha_\sharp}, \quad \theta_2 := s^{1-2/m}\lambda^4 \left(\xi_\sharp\right)^{1-2/m} e^{-s\alpha_\sharp}, \quad \theta_3 := s^{-3/m}\lambda^4 \left(\xi_\sharp\right)^{-3/m} e^{-s\alpha_\sharp}. \tag{3.12}$$

With the above notation, we can state the Carleman estimates we obtain for $\varphi^{(3)}$:

Proposition 3.1. *Let us consider $T > 0$ and assume (1.7), (1.9) and (1.10). There exist $s_0 > 0$ and $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $s \geqslant s_0(T^m + T^{2m})$, any solution φ of (1.5) can be decomposed as in (3.6), (3.7), (3.8) and (3.9) and the functions $\varphi^{(1)}$, $\varphi^{(2)}$ and $\varphi^{(3)}$ satisfy*

$$\left\|\varphi^{(1)}\right\|_{X_1} + \left\|\varphi^{(2)}\right\|_{X_2} \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}, \tag{3.13}$$

and

$$I(\varphi^{(3)}) + J(\varphi^{(3)}) \lesssim \|\rho_2 g\|^2_{L^2((0,T)\times\Omega)} + \iint_{(0,T)\times\omega} e^{-s\alpha_\sharp} \left(\varphi_1^{(3)}\right)^2\ dx\ dt. \tag{3.14}$$

In particular, φ verifies the following estimate:

$$\begin{aligned} &\iint_{(0,T)\times\Omega} s^6\lambda^8\xi_\sharp^6 e^{-2(N+1)s\alpha_\sharp} |\varphi|^2\ dx\ dt \\ &\qquad\lesssim \iint_{(0,T)\times\Omega} e^{-2N_2 s\alpha_\sharp} |g|^2\ dx\ dt + \iint_{(0,T)\times\omega} e^{-s(2N+1)\alpha_\sharp} |\varphi_1|^2\ dx\ dt. \end{aligned} \tag{3.15}$$

We deduce Theorem 1.3 from Proposition 3.1 in a standard way by using the well-posedness of (1.5) (see Proposition 2.1). In the remainder of this section, we prove the above proposition.

First, let us show (3.13). Applying Proposition 2.1, we have

$$\left\|\varphi^{(1)}\right\|_{X_1} \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}.$$

Using (2.11), we deduce from the above relation that for $s \geqslant T^m$,

$$\left\|N_2 s\alpha_\sharp'\rho_1\varphi^{(1)}\right\|_{X_1} \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}$$

and combining this with Proposition 2.1, we deduce

$$\left\|\varphi^{(2)}\right\|_{X_2} \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)} \tag{3.16}$$

and we have proved (3.13). In order to prove (3.14), we proceed in several steps. We start by showing the following lemma:

Lemma 3.2. *With the hypotheses of Proposition 3.1, we have the following estimate*

$$J(\varphi^{(3)}) \lesssim \|\rho_2 g\|^2_{L^2((0,T)\times\Omega)} + \left\|s^3\lambda^4\left(\xi_\sharp\right)^3 e^{-s\alpha_\sharp}\varphi^{(3)}\right\|^2_{L^2((0,T)\times\Omega)}. \tag{3.17}$$

Proof. From (3.9), we deduce

$$\mathcal{L}^*(\theta_1\varphi^{(3)}, \theta_1\pi^{(3)}) = \left(Ns\theta_1\alpha_\sharp'\left(\varphi^{(2)}+\varphi^{(3)}\right) - \theta_1'\varphi^{(3)}, 0\right), \tag{3.18}$$

$$\mathcal{L}^*(\theta_2\varphi^{(3)}, \theta_2\pi^{(3)}) = \left(Ns\theta_2\alpha_\sharp'\left(\varphi^{(2)}+\varphi^{(3)}\right) - \theta_2'\varphi^{(3)}, 0\right), \tag{3.19}$$

$$\mathcal{L}^*(\theta_3\varphi^{(3)}, \theta_3\pi^{(3)}) = \left(Ns\theta_3\alpha_\sharp'\left(\varphi^{(2)}+\varphi^{(3)}\right) - \theta_3'\varphi^{(3)}, 0\right). \tag{3.20}$$

From (3.12), (2.11) and (2.13), for $s \geqslant T^m + T^{2m}$,

$$\left|Ns\theta_1\alpha_\sharp'\right| + |\theta_1'| \lesssim s^3\lambda^4\xi_\sharp^3 e^{-s\alpha_\sharp}. \tag{3.21}$$

Using the above estimate and (3.16), and applying Proposition 2.1 to (3.18), we deduce

$$\left\|\theta_1\varphi^{(3)}\right\|^2_{X_1} \lesssim \|\rho_2 g\|^2_{L^2((0,T)\times\Omega)} + \left\|s^3\lambda^4\left(\xi_\sharp\right)^3 e^{-s\alpha_\sharp}\varphi^{(3)}\right\|^2_{L^2((0,T)\times\Omega)}.$$

Combining this estimate with (3.21), we obtain

$$\left\|\theta_1\varphi^{(3)}\right\|^2_{L^2(0,T;H^2(\Omega))} + \left\|\theta_1\partial_t\varphi^{(3)}\right\|^2_{L^2(0,T;L^2(\Omega))} \lesssim \|\rho_2 g\|^2_{L^2((0,T)\times\Omega)} + \left\|s^3\lambda^4\left(\xi_\sharp\right)^3 e^{-s\alpha_\sharp}\varphi^{(3)}\right\|^2_{L^2((0,T)\times\Omega)}. \tag{3.22}$$

From (3.12), (2.11) and (2.13), we have for $s \geqslant T^m + T^{2m}$,

$$\left|Ns\alpha_\sharp'\theta_2\right| + |\theta_2'| \lesssim \theta_1, \quad \left|Ns\left(\alpha_\sharp'\theta_2\right)'\right| + |\theta_2''| \lesssim s^3\lambda^4\xi_\sharp^3 e^{-s\alpha_\sharp}.$$

Thus, from (3.22)

$$\left\|Ns\theta_2\alpha_\sharp'\left(\varphi^{(2)}+\varphi^{(3)}\right) - \theta_2'\varphi^{(3)}\right\|^2_{X_1} \lesssim \|\rho_2 g\|^2_{L^2((0,T)\times\Omega)} + \left\|s^3\lambda^4\left(\xi_\sharp\right)^3 e^{-s\alpha_\sharp}\varphi^{(3)}\right\|^2_{L^2((0,T)\times\Omega)}.$$

Applying Proposition 2.1 to (3.19) and using the above estimate yield

$$\left\|\theta_2\varphi^{(3)}\right\|^2_{X_2} \lesssim \|\rho_2 g\|^2_{L^2((0,T)\times\Omega)} + \left\|s^3\lambda^4\left(\xi_\sharp\right)^3 e^{-s\alpha_\sharp}\varphi^{(3)}\right\|^2_{L^2((0,T)\times\Omega)}.$$

Using (3.21) and (3.22), we deduce

$$\left\|\theta_2\varphi^{(3)}\right\|^2_{L^2(0,T;H^4(\Omega))} + \left\|\theta_2\partial_t\varphi^{(3)}\right\|^2_{L^2(0,T;H^2(\Omega))} + \left\|\theta_2\partial_t^2\varphi^{(3)}\right\|^2_{L^2(0,T;L^2(\Omega))} \lesssim \|\rho_2 g\|^2_{L^2((0,T)\times\Omega)} + \left\|s^3\lambda^4\left(\xi_\sharp\right)^3 e^{-s\alpha_\sharp}\varphi^{(3)}\right\|^2_{L^2((0,T)\times\Omega)}. \tag{3.23}$$

From (3.12), (2.11) and (2.13), we have for $s \geqslant T^{2m}$,

$$\left|Ns\alpha_\sharp'\theta_3\right| + |\theta_3'| \lesssim \theta_2, \quad \left|Ns\left(\alpha_\sharp'\theta_3\right)'\right| + |\theta_3''| \lesssim \theta_1, \quad \left|Ns\left(\alpha_\sharp'\theta_3\right)''\right| + |\theta_3'''| \lesssim s^3\lambda^4\left(\xi_\sharp\right)^3 e^{-s\alpha_\sharp}. \tag{3.24}$$

Thus, using (3.22) and (3.23), we deduce

$$\left\|Ns\theta_3\alpha_\sharp'\left(\varphi^{(2)}+\varphi^{(3)}\right) - \theta_3'\varphi^{(3)}\right\|_{X_2}^2 \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}^2 + \left\|s^3\lambda^4\left(\xi_\sharp\right)^3 e^{-s\alpha_\sharp}\varphi^{(3)}\right\|_{L^2((0,T)\times\Omega)}^2.$$

Applying Proposition 2.1 to (3.20), we obtain

$$\left\|\theta_3\varphi^{(3)}\right\|_{X_3}^2 \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}^2 + \left\|s^3\lambda^4\left(\xi_\sharp\right)^3 e^{-s\alpha_\sharp}\varphi^{(3)}\right\|_{L^2((0,T)\times\Omega)}^2.$$

Using (3.24), (3.23), (3.22), we deduce

$$\begin{aligned}\left\|\theta_3\varphi^{(3)}\right\|_{L^2(0,T;H^6(\Omega))}^2 + \left\|\theta_3\partial_t\varphi^{(3)}\right\|_{L^2(0,T;H^4(\Omega))}^2 + \left\|\theta_3\partial_t^2\varphi^{(3)}\right\|_{L^2(0,T;H^2(\Omega))}^2 + \left\|\theta_3\partial_t^3\varphi^{(3)}\right\|_{L^2(0,T;L^2(\Omega))}^2 \\ \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}^2 + \left\|s^3\lambda^4\left(\xi_\sharp\right)^3 e^{-s\alpha_\sharp}\varphi^{(3)}\right\|_{L^2((0,T)\times\Omega)}^2.\end{aligned} \tag{3.25}$$

Gathering (3.25), (3.23) and (3.22) and recalling (3.11) yield (3.17). □

3.2 Carleman estimates

We are now going to use Carleman estimates (see Section 2.2) and Lemma 3.2 to show (3.14) First we have the following result:

Lemma 3.3. *With the hypotheses of Proposition 3.1, there exist $s_0 > 0$ and $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $s \geqslant s_0(T^m + T^{2m})$, we have the following estimate*

$$\begin{aligned}I(\varphi^{(3)}) + J(\varphi^{(3)}) \lesssim \|\rho_2 g\|_{L^2((0,T)\times\Omega)}^2 + \sum_{i=1}^n \iint_{(0,T)\times\Omega} e^{-2s\alpha_1}\left(\int_\Omega a^{(i)}\cdot\varphi^{(3)}\ dx\right)^2\ dx\ dt \\ + \iint_{(0,T)\times\omega_1} e^{-2s\alpha_1}\left(s^3\lambda^4\xi_1^3\left|\nabla\Delta\varphi_1^{(3)}\right|^2 + s^5\lambda^6\xi_1^5\left|\Delta\varphi_1^{(3)}\right|^2 + s^8\lambda^{10}\xi_1^8\left|\varphi_1^{(3)}\right|^2\right)\ dx\ dt.\end{aligned} \tag{3.26}$$

Proof. Taking the divergence of the first equation of (3.9), we obtain the following relation for $\pi^{(3)}$:

$$\Delta\pi^{(3)} = -\sum_{i=1}^n\left(\int_\Omega a^{(i)}\cdot\varphi^{(3)}\ dx\right)\operatorname{div} b^{(i)}. \tag{3.27}$$

In particular, recalling that $\breve{b}^{(i)}$ is given by (1.8), we can get rid of the pressure in the first equation of (3.9) by applying the operator $\nabla\Delta$ on its first component:

$$-\partial_t\nabla\Delta\varphi_1^{(3)} - \Delta\nabla\Delta\varphi_1^{(3)} + \sum_{i=1}^n\left(\int_\Omega a^{(i)}\cdot\varphi^{(3)}\ dx\right)\nabla\breve{b}^{(i)} = Ns\alpha_\sharp'\left(\nabla\Delta\varphi_1^{(2)} + \nabla\Delta\varphi_1^{(3)}\right) \quad \text{in } (0,T)\times\Omega.$$

Then we apply Lemma 2.5 to the above system with $i = 1$: there exist $s_0 > 0$ and $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$

and $s \geqslant s_0(T^m + T^{2m})$,

$$\iint_{(0,T)\times\Omega} s^3\lambda^4\xi_1^3 e^{-2s\alpha_1} \left|\nabla\Delta\varphi_1^{(3)}\right|^2 \,dx\,dt + \iint_{(0,T)\times\Omega} s\lambda^2\xi_1 e^{-2s\alpha_1} \left|\nabla^2\Delta\varphi_1^{(3)}\right|^2 \,dx\,dt$$
$$\lesssim \iint_{(0,T)\times\Omega} s^2 \left(\alpha_\sharp'\right)^2 e^{-2s\alpha_1} \left|\nabla\Delta\varphi_1^{(3)}\right|^2 \,dx\,dt + \iint_{(0,T)\times\Omega} s^2 \left(\alpha_\sharp'\right)^2 e^{-2s\alpha_1} \left|\nabla\Delta\varphi_1^{(2)}\right|^2 \,dx\,dt$$
$$+ \sum_{i=1}^n \iint_{(0,T)\times\Omega} e^{-2s\alpha_1} \left(\int_\Omega a^{(i)}\cdot\varphi^{(3)}\,dx\right)^2 \,dx\,dt + \iint_{(0,T)\times\partial\Omega} s\lambda\xi_\sharp e^{-2s\alpha_\sharp} \left|\frac{\partial}{\partial n}\nabla\Delta\varphi_1^{(3)}\right|^2 \,d\gamma\,dt$$
$$+ \iint_{(0,T)\times\omega_1} s^3\lambda^4\xi_1^3 e^{-2s\alpha_1} \left|\nabla\Delta\varphi_1^{(3)}\right|^2 \,dx\,dt.$$

Note that, from (2.11) and (2.9), for any $\varepsilon > 0$, there exists $s_0 > 0$ such that for $s \geqslant s_0 T^m$,

$$\left|s\alpha_\sharp' e^{-s\alpha_1}\right| \leqslant \varepsilon \left(s\xi_1\right)^{1+1/m} e^{-s\alpha_1}.$$

In particular, using (2.13) and (3.16), there exist $s_0 > 0$ and $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $s \geqslant s_0(T^m + T^{2m})$,

$$\iint_{(0,T)\times\Omega} s^3\lambda^4\xi_1^3 e^{-2s\alpha_1} \left|\nabla\Delta\varphi_1^{(3)}\right|^2 \,dx\,dt + \iint_{(0,T)\times\Omega} s\lambda^2\xi_1 e^{-2s\alpha_1} \left|\nabla^2\Delta\varphi_1^{(3)}\right|^2 \,dx\,dt$$
$$\lesssim \|\rho_2 g\|^2_{L^2((0,T)\times\Omega)} + \sum_{i=1}^n \iint_{(0,T)\times\Omega} e^{-2s\alpha_1} \left(\int_\Omega a^{(i)}\cdot\varphi^{(3)}\,dx\right)^2 \,dx\,dt$$
$$+ \iint_{(0,T)\times\partial\Omega} s\lambda\xi_\sharp e^{-2s\alpha_\sharp} \left|\frac{\partial}{\partial n}\nabla\Delta\varphi_1^{(3)}\right|^2 \,d\gamma\,dt + \iint_{(0,T)\times\omega_1} s^3\lambda^4\xi_1^3 e^{-2s\alpha_1} \left|\nabla\Delta\varphi_1^{(3)}\right|^2 \,dx\,dt. \quad (3.28)$$

Then, we apply Lemma 2.3 and Lemma 2.4 (with $i = 1$) to deduce

$$\iint_{(0,T)\times\Omega} s^8\lambda^{10}\xi_1^8 e^{-2s\alpha_1} \left|\varphi_1^{(3)}\right|^2 \,dx\,dt + \iint_{(0,T)\times\Omega} s^6\lambda^8\xi_1^6 e^{-2s\alpha_1} \left|\nabla\varphi_1^{(3)}\right|^2 \,dx\,dt$$
$$+ \iint_{(0,T)\times\Omega} s^5\lambda^6\xi_1^5 e^{-2s\alpha_1} \left|\Delta\varphi_1^{(3)}\right|^2 \,dx\,dt \lesssim \iint_{(0,T)\times\Omega} s^3\lambda^4\xi_1^3 e^{-2s\alpha_1} \left|\nabla\Delta\varphi_1^{(3)}\right|^2 \,dx\,dt$$
$$+ \iint_{(0,T)\times\omega_1} s^5\lambda^6\xi_1^5 e^{-2s\alpha_1} \left|\Delta\varphi_1^{(3)}\right|^2 \,dx\,dt + \iint_{(0,T)\times\omega_1} s^8\lambda^{10}\xi_1^8 e^{-2s\alpha_1} \left|\varphi_1^{(3)}\right|^2 \,dx\,dt. \quad (3.29)$$

Moreover, using the Poincaré inequality and the fact that $\varphi_2^{(3)} = 0$ on $\partial\Omega$, we have for a.e. $t \in (0,T)$,

$$\int_\Omega \left|\varphi_2^{(3)}\right|^2 \,dx \lesssim \int_\Omega \left|\partial_{x_2}\varphi_2^{(3)}\right|^2 \,dx$$

and with the divergence condition, we obtain

$$\iint_{(0,T)\times\Omega} s^6\lambda^8\xi_\sharp^6 e^{-2s\alpha_\sharp} \left|\varphi_2^{(3)}\right|^2 \,dx\,dt \lesssim \iint_{(0,T)\times\Omega} s^6\lambda^8\xi_1^6 e^{-2s\alpha_1} \left|\nabla\varphi_1^{(3)}\right|^2 \,dx\,dt.$$

Combining the above estimate with (3.28) and (3.29) and recalling (3.10), we deduce that

$$I(\varphi^{(3)}) \lesssim \|\rho_2 g\|^2_{L^2((0,T)\times\Omega)} + \sum_{i=1}^n \iint_{(0,T)\times\Omega} e^{-2s\alpha_1} \left(\int_\Omega a^{(i)}\cdot\varphi^{(3)}\,dx\right)^2 \,dx\,dt$$
$$+ \iint_{(0,T)\times\omega_1} e^{-2s\alpha_1} \left(s^3\lambda^4\xi_1^3 \left|\nabla\Delta\varphi_1^{(3)}\right|^2 + s^5\lambda^6\xi_1^5 \left|\Delta\varphi_1^{(3)}\right|^2 + s^8\lambda^{10}\xi_1^8 \left|\varphi_1^{(3)}\right|^2\right) \,dx\,dt$$
$$+ \iint_{(0,T)\times\partial\Omega} s\lambda\xi_\sharp e^{-2s\alpha_\sharp} \left|\frac{\partial}{\partial n}\nabla\Delta\varphi_1^{(3)}\right|^2 \,d\gamma\,dt.$$

The above estimate and (3.17) yield

$$I(\varphi^{(3)}) + J(\varphi^{(3)}) \lesssim \|\rho_2 g\|^2_{L^2((0,T)\times\Omega)} + \sum_{i=1}^{n} \iint_{(0,T)\times\Omega} e^{-2s\alpha_1} \left(\int_\Omega a^{(i)} \cdot \varphi^{(3)} \, dx \right)^2 \, dx \, dt$$
$$+ \iint_{(0,T)\times\omega_1} e^{-2s\alpha_1} \left(s^3\lambda^4\xi_1^3 \left|\nabla\Delta\varphi_1^{(3)}\right|^2 + s^5\lambda^6\xi_1^5 \left|\Delta\varphi_1^{(3)}\right|^2 + s^8\lambda^{10}\xi_1^8 \left|\varphi_1^{(3)}\right|^2 \right) \, dx \, dt$$
$$+ \iint_{(0,T)\times\partial\Omega} s\lambda\xi_\sharp e^{-2s\alpha_\sharp} \left| \frac{\partial}{\partial n} \nabla\Delta\varphi_1^{(3)} \right|^2 \, d\gamma \, dt. \quad (3.30)$$

From the trace theorem, interpolation results, (2.12) and the fact that $m \geqslant 9$, we also have

$$\iint_{(0,T)\times\partial\Omega} e^{-2s\alpha_\sharp} s\lambda\xi_\sharp \left| \frac{\partial}{\partial n} \nabla\Delta\varphi_1^{(3)} \right|^2 \, d\gamma \, dt \lesssim \lambda^{-7} \left\| \theta_2 \varphi^{(3)} \right\|^{3/2}_{L^2(0,T;H^4(\Omega))} \left\| \theta_3 \varphi^{(3)} \right\|^{1/2}_{L^2(0,T;H^6(\Omega))}.$$

Therefore, the above estimate and (3.30) imply the existence of $\lambda_0, s_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $s \geqslant s_0 \left(T^{2m} + T^m\right)$, (3.26) holds. □

3.3 Proof of Proposition 3.1

We are now in a position to prove Proposition 3.1 (and thus to prove Theorem 1.3). We have already (3.13) and to obtain (3.14), with respect to Lemma 3.3, it remains to remove the nonlocal spatial terms at the right-hand side of the inequality and then the local terms involving derivatives of $\varphi_1^{(3)}$.

Proof of Proposition 3.1. We set

$$\delta := \frac{1}{4^5(n+1)}.$$

Using (2.10) and Lemma 2.3, there exist $s_0 > 0$ and $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $s \geqslant s_0(T^m + T^{2m})$,

$$\iint_{(0,T)\times\Omega} e^{-2s\alpha_1} \left(\int_\Omega a^{(n)} \cdot \varphi^{(3)} \, dx \right)^2 \, dx \, dt \leqslant \iint_{(0,T)\times\Omega} e^{-2(1-\delta)s\alpha_n} \left(\int_\Omega a^{(n)} \cdot \varphi^{(3)} \, dx \right)^2 \, dx \, dt$$
$$\lesssim \iint_{(0,T)\times\omega_n} e^{-2(1-\delta)s\alpha_n} \left(\int_\Omega a^{(n)} \cdot \varphi^{(3)} \, dx \right)^2 \, dx \, dt. \quad (3.31)$$

Using (3.27) and applying the operator Δ on the first component of the first equation of (3.9) we deduce

$$-\partial_t \Delta\varphi_1^{(3)} - \Delta^2\varphi_1^{(3)} + \sum_{i=1}^{n} \left(\int_\Omega a^{(i)} \cdot \varphi^{(3)} \, dx \right) \breve{b}^{(i)} = N s\alpha'_\sharp \left(\Delta\varphi_1^{(2)} + \Delta\varphi_1^{(3)} \right) \quad \text{in } (0,T)\times\Omega,$$

where we recall that $\breve{b}^{(i)}$ is given by (1.8). Using (1.10) for $i = n$, we deduce from (3.31) and the above equation that

$$\iint_{(0,T)\times\Omega} e^{-2s\alpha_1} \left(\int_\Omega a^{(n)} \cdot \varphi^{(3)} \, dx \right)^2 \, dx \, dt \lesssim \iint_{(0,T)\times\omega_n} e^{-2(1-\delta)s\alpha_n} \left(\partial_t \Delta\varphi_1^{(3)} \right)^2 \, dx \, dt$$
$$+ \iint_{(0,T)\times\omega_n} e^{-2(1-\delta)s\alpha_n} \left(\Delta^2\varphi_1^{(3)} \right)^2 \, dx \, dt + \iint_{(0,T)\times\omega_n} e^{-2(1-\delta)s\alpha_n} \left(s\alpha'_\sharp \Delta\varphi_1^{(3)} \right)^2 \, dx \, dt$$
$$+ \|\rho_2 g\|^2_{L^2((0,T)\times\Omega)} + \sum_{i=1}^{n-1} \iint_{(0,T)\times\Omega} e^{-2(1-\delta)s\alpha_n} \left(\int_\Omega a^{(i)} \cdot \varphi^{(3)} \, dx \right)^2 \, dx \, dt.$$

Then we can proceed by induction, using (1.10), and we deduce from (3.26) that

$$
\begin{aligned}
I(\varphi^{(3)}) + J(\varphi^{(3)}) &\lesssim \|\rho_2 g\|^2_{L^2((0,T)\times\Omega)} \\
&+ \iint_{(0,T)\times\omega_1} e^{-2s\alpha_1} \left(s^3\lambda^4\xi_1^3 \left|\nabla\Delta\varphi_1^{(3)}\right|^2 + s^5\lambda^6\xi_1^5 \left|\Delta\varphi_1^{(3)}\right|^2 + s^8\lambda^{10}\xi_1^8 \left|\varphi_1^{(3)}\right|^2 \right) \, dx \, dt. \\
&+ \sum_{i=1}^{n} \iint_{(0,T)\times\omega_i} e^{-2(1-(n-i+1)\delta)s\alpha_i} \left(\left(\partial_t\Delta\varphi_1^{(3)}\right)^2 + \left(\Delta^2\varphi_1^{(3)}\right)^2 + \left(s\alpha_\sharp'\Delta\varphi_1^{(3)}\right)^2 \right) \, dx \, dt. \quad (3.32)
\end{aligned}
$$

Using (1.9), let us consider open subsets $\widetilde{\omega}^{(i)}$ $i = 1,\dots,5$, such that

$$
\bigcup_{i=1}^{n} \omega_i \Subset \widetilde{\omega}^{(1)} \Subset \ldots \Subset \widetilde{\omega}^{(5)} \Subset \omega. \quad (3.33)
$$

We also consider smooth functions

$$
\chi^{(i)} \in C^\infty(\mathbb{R}^2;[0,1]), \quad \chi^{(i)} \equiv 1 \text{ in } \widetilde{\omega}^{(i)}, \quad \operatorname{supp}\chi^{(i)} \subset \widetilde{\omega}^{(i+1)} \quad (i=1,\dots,4).
$$

We deduce from (3.32) and (2.10) that

$$
\begin{aligned}
I(\varphi^{(3)}) + J(\varphi^{(3)}) &\lesssim \|\rho_2 g\|^2_{L^2((0,T)\times\Omega)} \\
&+ \iint_{(0,T)\times\widetilde{\omega}^{(1)}} e^{-2s\left(1-\frac{1}{4^5}\right)\alpha_\sharp} \left(\left|\nabla\Delta\varphi_1^{(3)}\right|^2 + \left|\Delta\varphi_1^{(3)}\right|^2 + \left|\varphi_1^{(3)}\right|^2 + \left(\partial_t\Delta\varphi_1^{(3)}\right)^2 + \left(\Delta^2\varphi_1^{(3)}\right)^2 \right) \, dx \, dt. \quad (3.34)
\end{aligned}
$$

We have

$$
\begin{aligned}
&\iint_{(0,T)\times\widetilde{\omega}^{(1)}} e^{-2s(1-\frac{1}{4^5})\alpha_\sharp} \left(\Delta^2\varphi_1^{(3)}\right)^2 \, dx \, dt \leqslant \iint_{(0,T)\times\widetilde{\omega}^{(2)}} \chi^{(1)} e^{-2s(1-\frac{1}{4^5})\alpha_\sharp} \left(\Delta^2\varphi_1^{(3)}\right)^2 \, dx \, dt \\
&= -\iint_{(0,T)\times\widetilde{\omega}^{(2)}} \chi^{(1)} e^{-2s(1-\frac{1}{4^5})\alpha_\sharp} \nabla\Delta^2\varphi_1^{(3)} \cdot \nabla\Delta\varphi_1^{(3)} \, dx \, dt - \iint_{(0,T)\times\widetilde{\omega}^{(2)}} e^{-2s(1-\frac{1}{4^5})\alpha_\sharp} \Delta^2\varphi_1^{(3)} \nabla\chi^{(1)} \cdot \nabla\Delta\varphi_1^{(3)} \, dx \, dt \\
&\lesssim \left\|\theta_3\varphi^{(3)}\right\|_{L^2(0,T;H^5(\Omega))} \left(\iint_{(0,T)\times\widetilde{\omega}^{(2)}} e^{-2s(1-\frac{1}{4^4})\alpha_\sharp} \left|\nabla\Delta\varphi_1^{(3)}\right|^2 \, dx \, dt \right)^{1/2}.
\end{aligned}
$$

We can proceed similarly to estimate successively

$$
\iint_{(0,T)\times\widetilde{\omega}^{(2)}} e^{-2s(1-\frac{1}{4^4})\alpha_\sharp} \left|\nabla\Delta\varphi_1^{(3)}\right|^2 \, dx \, dt, \quad \iint_{(0,T)\times\widetilde{\omega}^{(3)}} e^{-2s(1-\frac{1}{4^3})\alpha_\sharp} \left|\Delta\varphi_1^{(3)}\right|^2 \, dx \, dt,
$$
$$
\iint_{(0,T)\times\widetilde{\omega}^{(4)}} e^{-2s(1-\frac{1}{4^2})\alpha_\sharp} \left|\nabla\varphi_1^{(3)}\right|^2 \, dx \, dt.
$$

For the last term, we integrate by parts in time:

$$
\begin{aligned}
\iint_{(0,T)\times\widetilde{\omega}^{(1)}} e^{-2s\left(1-\frac{1}{4^5}\right)\alpha_\sharp} \left(\partial_t\Delta\varphi_1^{(3)}\right)^2 \, dx \, dt &= -\iint_{(0,T)\times\widetilde{\omega}^{(1)}} e^{-2s\left(1-\frac{1}{4^5}\right)\alpha_\sharp} \left(\partial_t^2\Delta\varphi_1^{(3)}\right)\left(\Delta\varphi_1^{(3)}\right) \, dx \, dt \\
&+ \frac{1}{2} \iint_{(0,T)\times\widetilde{\omega}^{(1)}} \left(e^{-2s\left(1-\frac{1}{4^5}\right)\alpha_\sharp}\right)'' \left(\Delta\varphi_1^{(3)}\right)^2 \, dx \, dt \\
&\lesssim \left\|\theta_3\partial_t^2\varphi^{(3)}\right\|_{L^2(0,T;H^2(\Omega))} \left(\iint_{(0,T)\times\widetilde{\omega}^{(3)}} e^{-2s(1-\frac{1}{4^3})\alpha_\sharp} \left|\Delta\varphi_1^{(3)}\right|^2 \, dx \, dt \right)^{1/2} \\
&+ \iint_{(0,T)\times\widetilde{\omega}^{(3)}} e^{-2s(1-\frac{1}{4^3})\alpha_\sharp} \left|\Delta\varphi_1^{(3)}\right|^2 \, dx \, dt.
\end{aligned}
$$

Using the above computations, we deduce (3.14) from (3.34). We obtain (3.15) by combining (3.14) with (3.6) and (3.13). This ends the proof of Proposition 3.1. □

4 Proof of Theorem 1.7

This section is devoted to the proof of Theorem 1.7. First we need to define the functions $\widehat{\sigma}_i$, $i=1,2,3$. As in Section 2.2, there exist (see [17] or [26, Theorem 9.4.3, p.299]) for any $i=1,\dots,n$, $\widehat{\eta}_i \in C^2(\overline{\Omega})$ satisfying

$$\widehat{\eta}_i > 0 \text{ in } \Omega, \quad \widehat{\eta}_i = 0 \text{ on } \partial\Omega, \quad \max_{\Omega} \widehat{\eta}_i = 1, \quad \nabla \widehat{\eta}_i \neq 0 \text{ in } \overline{\Omega \setminus \widehat{\omega}_i}. \tag{4.1}$$

We use $\widehat{\eta}_i$ to define the following standard functions:

$$\widehat{\alpha}_i(t,x) = \frac{\exp\{\lambda(2m+2)\} - \exp\{\lambda(2m+\widehat{\eta}_i(x))\}}{t^m(T-t)^m}, \quad \widehat{\xi}_i(t,x) = \frac{\exp\{\lambda(2m+\widehat{\eta}_i(x))\}}{t^m(T-t)^m},$$

where

$$m \geqslant 11, \quad \lambda > 1.$$

Recalling the definitions (2.7) and (2.8), we have

$$\xi_\sharp \leqslant \widehat{\xi}_i \leqslant \xi_\flat, \quad e^{-s\alpha_\sharp} \leqslant e^{-s\widehat{\alpha}_i} \leqslant e^{-s\alpha_\flat} \quad (i=1,\dots,n).$$

Let us consider $M_1, M_2 > 0$ and let us write $M := M_1 + M_2$. We recall that N_1, N_2 and N are defined at the beginning of Section 3. We assume

$$N_2 \geqslant M+1 \tag{4.2}$$

and we define

$$\widehat{\sigma}_1 := \begin{cases} e^{-M_2 s\alpha_\sharp} & \text{in } \left[0, \frac{T}{2}\right] \\ \left[e^{-M_2 s\alpha_\sharp}\right]\left(\frac{T}{2}\right) & \text{in } \left[\frac{T}{2}, T\right] \end{cases}, \quad \widehat{\sigma}_2 := \begin{cases} e^{-M s\alpha_\sharp} & \text{in } \left[0, \frac{T}{2}\right] \\ \left[e^{-M s\alpha_\sharp}\right]\left(\frac{T}{2}\right) & \text{in } \left[\frac{T}{2}, T\right] \end{cases}, \tag{4.3}$$

$$\widehat{\sigma}_3 := \begin{cases} e^{-(N+1) s\alpha_\sharp} & \text{in } \left[0, \frac{T}{2}\right] \\ \left[e^{-(N+1) s\alpha_\sharp}\right]\left(\frac{T}{2}\right) & \text{in } \left[\frac{T}{2}, T\right] \end{cases}. \tag{4.4}$$

Note that $\widehat{\sigma}_i(0) = 0$ $i=1,2,3$.

We are going to show the existence of $s_0 > 0$ and $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $s \geqslant s_0(T^m + T^{2m})$, Theorem 1.7 holds with the weights defined as above.

For Corollary 1.9, we assume $M_2 > 1$ and we define

$$\widehat{\sigma}_0 := \begin{cases} e^{-(M_2-1) s\alpha_\sharp} & \text{in } \left[0, \frac{T}{2}\right] \\ \left[e^{-(M_2-1) s\alpha_\sharp}\right]\left(\frac{T}{2}\right) & \text{in } \left[\frac{T}{2}, T\right] \end{cases}. \tag{4.5}$$

Note that

$$\frac{\widehat{\sigma}_2}{\widehat{\sigma}_0}, \ \frac{\widehat{\sigma}_3}{\widehat{\sigma}_0}, \ \frac{\widehat{\sigma}_0'\widehat{\sigma}_1}{\widehat{\sigma}_0^2} \in L^\infty(0,T)$$

and if

$$2M_2 \geqslant N+3. \tag{4.6}$$

then, we have

$$\frac{\widehat{\sigma}_0^2}{\widehat{\sigma}_3} \in L^\infty(0,T). \tag{4.7}$$

We are going to show Corollary 1.9 with the above choices and for for $\lambda \geqslant \lambda_0$ and $s \geqslant s_0(T^m + T^{2m})$, where $s_0 > 0$ and $\lambda_0 > 0$ are large enough.

Remark 4.1. There exist N_1, N_2, M_1 and M_2 satisfying (3.5), (4.2) and (4.6). For instance one can take

$$N_1 = 2, \ N_2 = 13, \ M_1 = 2, \ M_2 = 9.$$

4.1 Decomposition of the adjoint system

As for the proof of Theorem 1.7, we first start by decomposing the solution of (1.16) into three parts. We only need to focus on ψ since we have already obtained a Carleman estimate for φ in Proposition 3.1. We recall that $M_1, M_2 > 0$ and that $M = M_1 + M_2$. We set

$$\widehat{\rho}_i := e^{-M_i s \alpha_\sharp} \quad (i = 1, 2).$$

From (1.16) and using the notation (2.1), we see that ψ is solution of

$$\mathcal{L}(\psi, \pi_\psi) = (g^{(1)}, \psi^0), \tag{4.8}$$

Let us consider the following decomposition of ψ:

$$\widehat{\rho}_1 \widehat{\rho}_2 \psi = \widehat{\rho}_1 \psi^{(1)} + \psi^{(2)} + \psi^{(3)}, \tag{4.9}$$

where

$$\mathcal{L}(\psi^{(1)}, \pi_\psi^{(1)}) = (\widehat{\rho}_2 g^{(1)}, 0), \tag{4.10}$$

$$\mathcal{L}(\psi^{(2)}, \pi_\psi^{(2)}) = \left(-M_2 s \alpha_\sharp' \widehat{\rho}_1 \psi^{(1)}, 0\right), \tag{4.11}$$

$$\mathcal{L}(\psi^{(3)}, \pi_\psi^{(3)}) = \left(-M s \alpha_\sharp' \left(\psi^{(2)} + \psi^{(3)}\right), 0\right). \tag{4.12}$$

We estimate $\psi^{(1)}$ and $\psi^{(2)}$ by using Proposition 2.2 and their regularity properties allow us to perform a Carleman estimate on $\psi^{(3)}$. Let us introduce the following quantities associated with $\psi^{(3)}$:

$$\begin{aligned}\widehat{I}(\psi^{(3)}) := &\iint_{(0,T)\times\Omega} e^{-2s\widehat{\alpha}_1} \left(s\lambda^2 \widehat{\xi}_1 \left|\nabla^3 \Delta \psi_1^{(3)}\right|^2 + s^3 \lambda^4 \widehat{\xi}_1^3 \left|\nabla^2 \Delta \psi_1^{(3)}\right|^2 \right) dx\ dt \\ &+ \iint_{(0,T)\times\Omega} e^{-2s\widehat{\alpha}_1} \left(s^5 \lambda^6 \widehat{\xi}_1^5 \left|\nabla \Delta \psi_1^{(3)}\right|^2 + s^7 \lambda^8 \widehat{\xi}_1^7 \left|\Delta \psi_1^{(3)}\right|^2 \right) dx\ dt \\ &+ \iint_{(0,T)\times\Omega} e^{-2s\alpha_\sharp} s^7 \lambda^8 \xi_\sharp^7 \left|\psi^{(3)}\right|^2\ dx\ dt\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\widehat{J}\left(\psi^{(3)}\right) := &\left\|\widehat{\theta}_1 \psi^{(3)}\right\|^2_{L^2(0,T;H^2(\Omega))} + \left\|\widehat{\theta}_1 \partial_t \psi^{(3)}\right\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ \left\|\widehat{\theta}_2 \psi^{(3)}\right\|^2_{L^2(0,T;H^4(\Omega))} + \left\|\widehat{\theta}_2 \partial_t \psi^{(3)}\right\|^2_{L^2(0,T;H^2(\Omega))} + \left\|\widehat{\theta}_2 \partial_t^2 \psi^{(3)}\right\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ \left\|\widehat{\theta}_3 \psi^{(3)}\right\|^2_{L^2(0,T;H^6(\Omega))} + \left\|\widehat{\theta}_3 \partial_t \psi^{(3)}\right\|^2_{L^2(0,T;H^4(\Omega))} + \left\|\widehat{\theta}_3 \partial_t^2 \psi^{(3)}\right\|^2_{L^2(0,T;H^2(\Omega))} + \left\|\widehat{\theta}_3 \partial_t^3 \psi^{(3)}\right\|^2_{L^2(0,T;L^2(\Omega))},\end{aligned} \tag{4.14}$$

where we have set

$$\widehat{\theta}_1 := s^{5/2-1/m} \lambda^4 \left(\xi_\sharp\right)^{5/2-1/m} e^{-s\alpha_\sharp}, \quad \widehat{\theta}_2 := s^{3/2-2/m} \lambda^4 \left(\xi_\sharp\right)^{3/2-2/m} e^{-s\alpha_\sharp}, \\ \widehat{\theta}_3 := s^{1/2-3/m} \lambda^4 \left(\xi_\sharp\right)^{1/2-3/m} e^{-s\alpha_\sharp}. \tag{4.15}$$

Using (1.19), we consider an open set $\widehat{\omega}_0$ such that

$$\bigcup_{i=1}^{n} \widehat{\omega}_i \Subset \widehat{\omega}_0 \Subset \widehat{\omega}. \tag{4.16}$$

With the above notation, we can state the following estimates:

Proposition 4.2. *Assume* (1.7), (1.9), (1.10), (1.18), (1.19), (1.20) *and* (1.21). *There exist* $s_0 > 0$ *and* $\lambda_0 > 0$ *such that for* $\lambda \geqslant \lambda_0$ *and* $s \geqslant s_0(T^m + T^{2m})$, *the solution* ψ *of* (4.8), *can be decomposed as in* (4.9), (4.10), (4.11) *and* (4.12) *and the functions* $\psi^{(1)}$, $\psi^{(2)}$ *and* $\psi^{(3)}$ *satisfy*

$$\left\|\psi^{(1)}\right\|_{X_1} + \left\|\psi^{(2)}\right\|_{X_2} \lesssim \left\|\widehat{\rho}_2 g^{(1)}\right\|_{L^2((0,T)\times\Omega)}, \tag{4.17}$$

and

$$\widehat{I}(\psi^{(3)}) + \widehat{J}(\psi^{(3)}) \lesssim \left\|\widehat{\rho}_2 g^{(1)}\right\|^2_{L^2((0,T)\times\Omega)} + \iint_{(0,T)\times\widehat{\omega}_0} e^{-2s\left(1-\frac{1}{4^3}\right)\alpha_\sharp} \left|\Delta\psi_1^{(3)}\right|^2 \, dx\, dt. \tag{4.18}$$

In particular, any solution (φ, ψ) *of* (1.16) *verifies the following estimate:*

$$\begin{aligned}\iint_{(0,T)\times\Omega} e^{-2(N+1)s\alpha_\sharp} |\varphi|^2 \, dx\, dt + \iint_{(0,T)\times\Omega} e^{-2(M+1)s\alpha_\sharp} |\psi|^2 \, dx\, dt \\ \lesssim \iint_{(0,T)\times\Omega} e^{-2Ms\alpha_\sharp} \left|g^{(0)}\right|^2 \, dx\, dt + \iint_{(0,T)\times\Omega} e^{-2M_2 s\alpha_\sharp} \left|g^{(1)}\right|^2 \, dx\, dt \\ + \iint_{(0,T)\times(\omega\cup\widehat{\omega})} e^{-2sM\alpha_\sharp} |\varphi_1|^2 \, dx\, dt.\end{aligned} \tag{4.19}$$

We now prove the above proposition. First we can show (4.17) by applying Proposition 2.2:

$$\left\|\psi^{(1)}\right\|_{X_1} \lesssim \left\|\widehat{\rho}_2 g^{(1)}\right\|_{L^2((0,T)\times\Omega)},$$

so that

$$\left\|M_2 s\alpha'_\sharp \widehat{\rho}_1 \psi^{(1)}\right\|_{X_1} \lesssim \left\|\widehat{\rho}_2 g^{(1)}\right\|_{L^2((0,T)\times\Omega)},$$

and thus

$$\left\|\psi^{(2)}\right\|_{X_2} \lesssim \left\|M_2 s\alpha'_\sharp \widehat{\rho}_1 \psi^{(1)}\right\|_{X_1} \lesssim \left\|\widehat{\rho}_2 g^{(1)}\right\|_{L^2((0,T)\times\Omega)}. \tag{4.20}$$

The above estimates yield (4.17). It remains to show (4.18). First, we have the following result:

Lemma 4.3. *With the hypotheses of Proposition 4.2, we have the following estimate*

$$\widehat{J}\left(\psi^{(3)}\right) \lesssim \left\|\widehat{\rho}_2 g^{(1)}\right\|^2_{L^2((0,T)\times\Omega)} + \left\|s^{7/2}\lambda^4 (\xi_\sharp)^{7/2} e^{-s\alpha_\sharp}\psi^{(3)}\right\|^2_{L^2((0,T)\times\Omega)}. \tag{4.21}$$

Proof. The proof is completely similar to the proof of Lemma 3.2: instead of (3.18), (3.19) and (3.20), we consider the following systems obtained from (4.12):

$$\mathcal{L}(\widehat{\theta}_1\psi^{(3)}, \widehat{\theta}_1\pi_\psi^{(3)}) = \left(-Ms\widehat{\theta}_1\alpha'_\sharp\left(\psi^{(2)} + \psi^{(3)}\right) + \widehat{\theta}'_1\psi^{(3)}, 0\right),$$

$$\mathcal{L}(\widehat{\theta}_2\psi^{(3)}, \widehat{\theta}_2\pi_\psi^{(3)}) = \left(-Ms\widehat{\theta}_2\alpha'_\sharp\left(\psi^{(2)} + \psi^{(3)}\right) + \widehat{\theta}'_2\psi^{(3)}, 0\right),$$

$$\mathcal{L}(\widehat{\theta}_3\psi^{(3)}, \widehat{\theta}_3\pi_\psi^{(3)}) = \left(-Ms\widehat{\theta}_3\alpha'_\sharp\left(\psi^{(2)} + \psi^{(3)}\right) + \widehat{\theta}'_3\psi^{(3)}, 0\right).$$

Then, we use Proposition 2.2 (instead of Proposition 2.1 in the proof of Lemma 3.2) to show that $\widehat{J}\left(\psi^{(3)}\right)$ defined by (4.14) satisfies (4.21). □

4.2 Carleman estimates

We now use Carleman estimates (see Section 2.2) and Lemma 4.3 to show (4.18). First we have the following result:

Lemma 4.4. *With the hypotheses of Proposition 4.2, there exist* $s_0 > 0$ *and* $\lambda_0 > 0$ *such that for* $\lambda \geqslant \lambda_0$ *and* $s \geqslant s_0(T^m + T^{2m})$*, we have the following estimate*

$$\widehat{I}(\psi^{(3)}) + \widehat{J}(\psi^{(3)}) \lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|^2_{L^2((0,T)\times\Omega)} + \sum_{i=1}^n \iint_{(0,T)\times\Omega} e^{-2s\widehat{\alpha}_1} \left(\int_\Omega b^{(i)} \cdot \psi^{(3)} \, dx \right)^2 \, dx \, dt$$
$$+ \iint_{(0,T)\times\widehat{\omega}_1} e^{-2s\widehat{\alpha}_1} \left(s^3\lambda^4\widehat{\xi}_1^3 \left| \nabla^2\Delta\psi_1^{(3)} \right|^2 + s^5\lambda^6\widehat{\xi}_1^5 \left| \nabla\Delta\psi_1^{(3)} \right|^2 + s^7\lambda^8\widehat{\xi}_1^7 \left| \Delta\psi_1^{(3)} \right|^2 \right) \, dx \, dt. \quad (4.22)$$

Proof. Taking the divergence of the first equation of (4.12), we obtain the following relation for $\pi_\psi^{(3)}$:

$$\Delta\pi_\psi^{(3)} = -\sum_{i=1}^n \left(\int_\Omega b^{(i)} \cdot \psi^{(3)} \, dx \right) \operatorname{div} a^{(i)}.$$

Then, applying the operators Δ and $\nabla^2\Delta$ on the first component of the first equation of (4.12), we deduce

$$\partial_t\Delta\psi_1^{(3)} - \Delta^2\psi_1^{(3)} + \sum_{i=1}^n \left(\int_\Omega b^{(i)} \cdot \psi^{(3)} \, dx \right) \breve{a}^{(i)} = -Ms\alpha_\sharp' \left(\Delta\psi_1^{(2)} + \Delta\psi_1^{(3)} \right) \quad \text{in } (0,T)\times\Omega \quad (4.23)$$

and

$$\partial_t\nabla^2\Delta\psi_1^{(3)} - \Delta\nabla^2\Delta\psi_1^{(3)} + \sum_{i=1}^n \left(\int_\Omega b^{(i)} \cdot \psi^{(3)} \, dx \right) \nabla^2\breve{a}^{(i)}$$
$$= -Ms\alpha_\sharp' \left(\nabla^2\Delta\psi_1^{(2)} + \nabla^2\Delta\psi_1^{(3)} \right) \quad \text{in } (0,T)\times\Omega \quad (4.24)$$

where $\breve{a}^{(i)}$ is defined by (1.17). We continue by following the steps in Section 3.2: first, we apply Lemma 2.5 to (4.24): there exist $s_0 > 0$ and $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $s \geqslant s_0(T^m + T^{2m})$,

$$\iint_{(0,T)\times\Omega} s^3\lambda^4\widehat{\xi}_1^3 e^{-2s\widehat{\alpha}_1} \left| \nabla^2\Delta\psi_1^{(3)} \right|^2 \, dx \, dt + \iint_{(0,T)\times\Omega} s\lambda^2\widehat{\xi}_1 e^{-2s\widehat{\alpha}_1} \left| \nabla^3\Delta\psi_1^{(3)} \right|^2 \, dx \, dt$$
$$\lesssim \iint_{(0,T)\times\Omega} s^2 \left(\alpha_\sharp'\right)^2 e^{-2s\widehat{\alpha}_1} \left| \nabla^3\Delta\psi_1^{(3)} \right|^2 \, dx \, dt + \iint_{(0,T)\times\Omega} s^2 \left(\alpha_\sharp'\right)^2 e^{-2s\widehat{\alpha}_1} \left| \nabla^3\Delta\psi_1^{(2)} \right|^2 \, dx \, dt$$
$$+ \sum_{i=1}^n \iint_{(0,T)\times\Omega} e^{-2s\widehat{\alpha}_1} \left(\int_\Omega b^{(i)} \cdot \psi^{(3)} \, dx \right)^2 \, dx \, dt + \iint_{(0,T)\times\partial\Omega} s\lambda\xi_\sharp e^{-2s\alpha_\sharp} \left| \frac{\partial}{\partial n}\nabla^2\Delta\psi_1^{(3)} \right|^2 \, d\gamma \, dt$$
$$+ \iint_{(0,T)\times\omega_1} s^3\lambda^4\widehat{\xi}_1^3 e^{-2s\widehat{\alpha}_1} \left| \nabla^2\Delta\psi_1^{(3)} \right|^2 \, dx \, dt. \quad (4.25)$$

The term

$$\iint_{(0,T)\times\Omega} s^2 \left(\alpha_\sharp'\right)^2 e^{-2s\widehat{\alpha}_1} \left| \nabla^3\Delta\psi_1^{(3)} \right|^2 \, dx \, dt,$$

can be absorbed by the right-hand side of (4.25) and by using (4.20), we have

$$\iint_{(0,T)\times\Omega} s^2 \left(\alpha_\sharp'\right)^2 e^{-2s\widehat{\alpha}_1} \left| \nabla^3\Delta\psi_1^{(2)} \right|^2 \, dx \, dt \lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|^2_{L^2((0,T)\times\Omega)}.$$

Then, we apply Lemma 2.3 to deduce

$$\iint_{(0,T)\times\Omega} s^7\lambda^8\widehat{\xi}_1^7 e^{-2s\widehat{\alpha}_1}\left|\Delta\psi_1^{(3)}\right|^2\,dx\,dt + \iint_{(0,T)\times\Omega} s^5\lambda^6\widehat{\xi}_1^5 e^{-2s\widehat{\alpha}_1}\left|\nabla\Delta\psi_1^{(3)}\right|^2\,dx\,dt$$
$$\lesssim \iint_{(0,T)\times\Omega} s^3\lambda^4\widehat{\xi}_1^3 e^{-2s\widehat{\alpha}_1}\left|\nabla^2\Delta\psi_1^{(3)}\right|^2\,dx\,dt + \iint_{(0,T)\times\widehat{\omega}_1} s^5\lambda^6\widehat{\xi}_1^5 e^{-2s\widehat{\alpha}_1}\left|\nabla\Delta\psi_1^{(3)}\right|^2\,dx\,dt$$
$$+ \iint_{(0,T)\times\widehat{\omega}_1} s^7\lambda^8\widehat{\xi}_1^7 e^{-2s\widehat{\alpha}_1}\left|\Delta\psi_1^{(3)}\right|^2\,dx\,dt. \quad (4.26)$$

Using the ellipticity of the Laplace operator with Dirichlet boundary conditions, we deduce that

$$\iint_{(0,T)\times\Omega} s^7\lambda^8\xi_\sharp^7 e^{-2s\alpha_\sharp}\left(\left|\psi_1^{(3)}\right|^2 + \left|\nabla\psi_1^{(3)}\right|^2\right)\,dx\,dt \lesssim \iint_{(0,T)\times\Omega} s^7\lambda^8\widehat{\xi}_1^7 e^{-2s\widehat{\alpha}_1}\left|\Delta\psi_1^{(3)}\right|^2\,dx\,dt. \quad (4.27)$$

Moreover, using the Poincaré inequality and the fact that $\psi_2^{(3)} = 0$ on $\partial\Omega$, we have for a.e. $t \in (0,T)$,

$$\int_{(0,T)\times\Omega}\left|\psi_2^{(3)}\right|^2\,dx \lesssim \int_{(0,T)\times\Omega}\left|\partial_{x_2}\psi_2^{(3)}\right|^2\,dx$$

and with the divergence condition, we obtain

$$\iint_{(0,T)\times\Omega} s^7\lambda^8\xi_\sharp^7 e^{-2s\alpha_\sharp}\left|\psi_2^{(3)}\right|^2\,dx\,dt \lesssim \iint_{(0,T)\times\Omega} s^7\lambda^8\xi_\sharp^7 e^{-2s\alpha_\sharp}\left|\nabla\psi_1^{(3)}\right|^2\,dx\,dt.$$

Combining the above estimate with (4.25), (4.26) and (4.27), we deduce that $\widehat{I}(\psi^{(3)})$ defined by (4.13) satisfies

$$\widehat{I}(\psi^{(3)}) \lesssim \left\|\widehat{\rho}_2 g^{(1)}\right\|^2_{L^2((0,T)\times\Omega)} + \sum_{i=1}^n \iint_{(0,T)\times\Omega} e^{-2s\widehat{\alpha}_1}\left(\int_\Omega b^{(i)}\cdot\psi^{(3)}\,dx\right)^2\,dx\,dt$$
$$+ \iint_{(0,T)\times\widehat{\omega}_1} e^{-2s\widehat{\alpha}_1}\left(s^3\lambda^4\widehat{\xi}_1^3\left|\nabla^2\Delta\psi_1^{(3)}\right|^2 + s^5\lambda^6\widehat{\xi}_1^5\left|\nabla\Delta\psi_1^{(3)}\right|^2 + s^7\lambda^8\widehat{\xi}_1^7\left|\Delta\psi_1^{(3)}\right|^2\right)\,dx\,dt$$
$$+ \iint_{(0,T)\times\partial\Omega} s\lambda\xi_\sharp e^{-2s\alpha_\sharp}\left|\frac{\partial}{\partial n}\nabla^2\Delta\psi_1^{(3)}\right|^2\,d\gamma\,dt.$$

Combining the above inequality with (4.21) we deduce

$$\widehat{I}(\psi^{(3)}) + \widehat{J}(\psi^{(3)}) \lesssim \left\|\widehat{\rho}_2 g^{(1)}\right\|^2_{L^2((0,T)\times\Omega)} + \sum_{i=1}^n \iint_{(0,T)\times\Omega} e^{-2s\widehat{\alpha}_1}\left(\int_\Omega b^{(i)}\cdot\psi^{(3)}\,dx\right)^2\,dx\,dt$$
$$+ \iint_{(0,T)\times\widehat{\omega}_1} e^{-2s\widehat{\alpha}_1}\left(s^3\lambda^4\widehat{\xi}_1^3\left|\nabla^2\Delta\psi_1^{(3)}\right|^2 + s^5\lambda^6\widehat{\xi}_1^5\left|\nabla\Delta\psi_1^{(3)}\right|^2 + s^7\lambda^8\widehat{\xi}_1^7\left|\Delta\psi_1^{(3)}\right|^2\right)\,dx\,dt$$
$$+ \iint_{(0,T)\times\partial\Omega} s\lambda\xi_\sharp e^{-2s\alpha_\sharp}\left|\frac{\partial}{\partial n}\nabla^2\Delta\psi_1^{(3)}\right|^2\,d\gamma\,dt. \quad (4.28)$$

Then, from trace theorems and interpolation inequalities

$$\int_{\partial\Omega}\left|\frac{\partial}{\partial n}\nabla^2\Delta\psi_1^{(3)}\right|^2\,d\gamma \lesssim \left\|\psi^{(3)}\right\|_{H^5(\Omega)}\left\|\psi^{(3)}\right\|_{H^6(\Omega)} \lesssim \left\|\psi^{(3)}\right\|^{1/2}_{H^4(\Omega)}\left\|\psi^{(3)}\right\|^{3/2}_{H^6(\Omega)}.$$

Thus, using (4.15) and (2.12), we have for $m \geqslant 11$, and $s \geqslant s_0(T^{2m} + T^m)$,

$$\iint_{(0,T)\times\partial\Omega} e^{-2s\alpha_\sharp} s\lambda\xi_\sharp\left|\frac{\partial}{\partial n}\nabla^2\Delta\psi_1^{(3)}\right|^2\,d\gamma\,dt \lesssim \lambda^{-7}\left\|\widehat{\theta}_2\psi^{(3)}\right\|^{1/2}_{L^2(0,T;H^4(\Omega))}\left\|\widehat{\theta}_3\psi^{(3)}\right\|^{3/2}_{L^2(0,T;H^6(\Omega))}.$$

Combining the above estimate and (4.28), we deduce the existence of $\lambda_0 > 0$ and $s_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $s \geqslant s_0(T^{2m} + T^m)$, (4.22) holds. □

4.3 Removing the nonlocal spatial terms

In this section, we are going to remove the nonlocal spatial terms in (4.22) together with several local terms in $\psi_1^{(3)}$ to obtain the following result:

Lemma 4.5. *With the hypotheses of Proposition 4.2, there exist $s_0 > 0$ and $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $s \geqslant s_0(T^m + T^{2m})$, we have the following estimate*

$$\widehat{I}(\psi^{(3)}) + \widehat{J}(\psi^{(3)}) + \left\| s^3\lambda^4\xi_\sharp^3 e^{-(N+1)s\alpha_\sharp}\varphi \right\|^2_{L^2((0,T)\times\Omega)} + \left\| s^{7/2}\lambda^4\xi_\sharp^{7/2} e^{-(M+1)s\alpha_\sharp}\psi \right\|^2_{L^2((0,T)\times\Omega)}$$
$$\lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|^2_{L^2((0,T)\times\Omega)} + \left\| \rho_2 g^{(0)} \right\|^2_{L^2((0,T)\times\Omega)}$$
$$+ \iint_{(0,T)\times\omega} e^{-s(2N+1)\alpha_\sharp} |\varphi_1|^2 \, dx\, dt + \iint_{(0,T)\times\breve{\omega}^{(3)}} e^{-2s\left(1-\frac{1}{4^3}\right)\alpha_\sharp} \left|\Delta\psi_1^{(3)}\right|^2 dx\, dt. \quad (4.29)$$

Proof. We set

$$\delta := \frac{1}{4^5(n+1)}.$$

Using (2.10) and Lemma 2.3, there exist $s_0 > 0$ and $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$ and $s \geqslant s_0(T^m + T^{2m})$,

$$\iint_{(0,T)\times\Omega} e^{-2s\widehat{\alpha}_1} \left(\int_\Omega b^{(n)}\cdot\psi^{(3)}\, dx\right)^2 dx\, dt \lesssim \iint_{(0,T)\times\Omega} e^{-2(1-\delta)s\widehat{\alpha}_n} \left(\int_\Omega b^{(n)}\cdot\psi^{(3)}\, dx\right)^2 dx\, dt$$
$$\lesssim \iint_{(0,T)\times\widehat{\omega}_n} e^{-2(1-\delta)s\widehat{\alpha}_n} \left(\int_\Omega b^{(n)}\cdot\psi^{(3)}\, dx\right)^2 dx\, dt.$$

Using (1.21) for $i = n$, (4.23) and the above equation, we deduce that

$$\iint_{(0,T)\times\Omega} e^{-2s\widehat{\alpha}_1} \left(\int_\Omega b^{(n)}\cdot\psi^{(3)}\, dx\right)^2 dx\, dt \lesssim \iint_{(0,T)\times\widehat{\omega}_n} e^{-2(1-\delta)s\widehat{\alpha}_n} \left(\partial_t\Delta\psi_1^{(3)}\right)^2 dx\, dt$$
$$+ \iint_{(0,T)\times\widehat{\omega}_n} e^{-2(1-\delta)s\widehat{\alpha}_n} \left(\Delta^2\psi_1^{(3)}\right)^2 dx\, dt + \iint_{(0,T)\times\widehat{\omega}_n} e^{-2(1-\delta)s\widehat{\alpha}_n} \left(s\alpha_\sharp'\Delta\psi_1^{(3)}\right)^2 dx\, dt$$
$$+ \left\|\widehat{\rho}_2 g^{(1)}\right\|^2_{L^2((0,T)\times\Omega)} + \sum_{i=1}^{n-1} \iint_{(0,T)\times\Omega} e^{-2(1-\delta)s\widehat{\alpha}_n} \left(\int_\Omega b^{(i)}\cdot\psi^{(3)}\, dx\right)^2 dx\, dt$$

Then we can proceed by induction by using (1.21), and we deduce from (4.22) that

$$\widehat{I}(\psi^{(3)}) + \widehat{J}(\psi^{(3)}) \lesssim \left\|\widehat{\rho}_2 g^{(1)}\right\|^2_{L^2((0,T)\times\Omega)}$$
$$+ \iint_{(0,T)\times\widehat{\omega}_1} e^{-2s\widehat{\alpha}_1} \left(s^3\lambda^4\widehat{\xi}_1^3 \left|\nabla^2\Delta\psi_1^{(3)}\right|^2 + s^5\lambda^6\widehat{\xi}_1^5 \left|\nabla\Delta\psi_1^{(3)}\right|^2 + s^7\lambda^8\widehat{\xi}_1^7 \left|\Delta\psi_1^{(3)}\right|^2 \right) dx\, dt$$
$$+ \sum_{i=1}^{n} \iint_{(0,T)\times\widehat{\omega}_i} e^{-2(1-(n-i+1)\delta)s\widehat{\alpha}_i} \left(\left(\partial_t\Delta\psi_1^{(3)}\right)^2 + \left(\Delta^2\psi_1^{(3)}\right)^2 + \left(s\alpha_\sharp'\Delta\psi_1^{(3)}\right)^2 \right) dx\, dt. \quad (4.30)$$

Using (4.16), we can consider open subsets $\breve{\omega}^{(i)}$ $i = 1, \ldots, 3$, such that

$$\bigcup_{i=1}^{n} \widehat{\omega}_i \Subset \breve{\omega}^{(1)} \Subset \breve{\omega}^{(2)} \Subset \breve{\omega}^{(3)} = \widehat{\omega}_0 \quad (4.31)$$

and smooth functions

$$\check{\chi}^{(i)} \in C^\infty(\mathbb{R}^2;[0,1]), \quad \check{\chi}^{(i)} \equiv 1 \text{ in } \breve{\omega}^{(i)}, \quad \operatorname{supp} \check{\chi}^{(i)} \subset \breve{\omega}^{(i+1)} \quad (i=1,2).$$

We deduce from (4.30) that

$$\widehat{I}(\psi^{(3)}) + \widehat{J}(\psi^{(3)}) \lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|^2_{L^2((0,T)\times\Omega)} + \iint_{(0,T)\times\breve{\omega}^{(1)}} e^{-2s\left(1-\frac{1}{4^5}\right)\alpha_\sharp} \left(\left|\nabla^2 \Delta \psi_1^{(3)}\right|^2 + \left(\partial_t \Delta \psi_1^{(3)}\right)^2 + \left|\nabla \Delta \psi_1^{(3)}\right|^2 + \left|\Delta \psi_1^{(3)}\right|^2 \right) \, dx \, dt. \quad (4.32)$$

By integrating by parts and by using (4.15), we have

$$\iint_{(0,T)\times\breve{\omega}^{(1)}} e^{-2s(1-\frac{1}{4^5})\alpha_\sharp} \left(\Delta^2 \psi_1^{(3)}\right)^2 \, dx \, dt \leqslant \iint_{(0,T)\times\breve{\omega}^{(2)}} \check{\chi}^{(1)} e^{-2s(1-\frac{1}{4^5})\alpha_\sharp} \left(\Delta^2 \psi_1^{(3)}\right)^2 \, dx \, dt$$

$$= -\iint_{(0,T)\times\breve{\omega}^{(2)}} \check{\chi}^{(1)} e^{-2s(1-\frac{1}{4^5})\alpha_\sharp} \nabla \Delta^2 \psi_1^{(3)} \cdot \nabla \Delta \psi_1^{(3)} \, dx \, dt - \iint_{(0,T)\times\breve{\omega}^{(2)}} e^{-2s(1-\frac{1}{4^5})\alpha_\sharp} \Delta^2 \psi_1^{(3)} \nabla \check{\chi}^{(1)} \cdot \nabla \Delta \psi_1^{(3)} \, dx \, dt$$

$$\lesssim \left\| \widehat{\theta}_3 \psi^{(3)} \right\|_{L^2(0,T;H^5(\Omega))} \left(\iint_{(0,T)\times\widetilde{\omega}^{(2)}} e^{-2s(1-\frac{1}{4^4})\alpha_\sharp} \left|\nabla \Delta \psi_1^{(3)}\right|^2 \, dx \, dt \right)^{1/2}.$$

We can proceed similarly to estimate

$$\iint_{(0,T)\times\breve{\omega}^{(2)}} e^{-2s(1-\frac{1}{4^4})\alpha_\sharp} \left|\nabla \Delta \psi_1^{(3)}\right|^2 \, dx \, dt, \quad \iint_{(0,T)\times\breve{\omega}^{(1)}} e^{-2s(1-\frac{1}{4^5})\alpha_\sharp} \left|\nabla^2 \Delta \psi_1^{(3)}\right|^2 \, dx \, dt.$$

For the last term, we integrate by parts in time:

$$\iint_{(0,T)\times\breve{\omega}^{(1)}} e^{-2s\left(1-\frac{1}{4^5}\right)\alpha_\sharp} \left(\partial_t \Delta \psi_1^{(3)}\right)^2 \, dx \, dt = -\iint_{(0,T)\times\breve{\omega}^{(1)}} e^{-2s\left(1-\frac{1}{4^5}\right)\alpha_\sharp} \left(\partial_t^2 \Delta \psi_1^{(3)}\right) \left(\Delta \psi_1^{(3)}\right) \, dx \, dt$$

$$+ \frac{1}{2} \iint_{(0,T)\times\breve{\omega}^{(1)}} \left(e^{-2s\left(1-\frac{1}{4^5}\right)\alpha_\sharp}\right)'' \left(\Delta \psi_1^{(3)}\right)^2 \, dx \, dt$$

$$\lesssim \left\| \widehat{\theta}_3 \partial_t^2 \psi^{(3)} \right\|_{L^2(0,T;H^2(\Omega))} \left(\iint_{(0,T)\times\breve{\omega}^{(3)}} e^{-2s(1-\frac{1}{4^3})\alpha_\sharp} \left|\Delta \psi_1^{(3)}\right|^2 \, dx \, dt \right)^{1/2} + \iint_{(0,T)\times\breve{\omega}^{(3)}} e^{-2s(1-\frac{1}{4^3})\alpha_\sharp} \left|\Delta \psi_1^{(3)}\right|^2 \, dx \, dt.$$

Using the above computations, (4.32) yields (4.18). Recalling (4.13), (4.9), and (4.17), we have

$$\iint_{(0,T)\times\Omega} e^{-2s\alpha_\sharp} s^7 \lambda^8 \xi_\sharp^7 \left|\widehat{\rho}_1 \widehat{\rho}_2 \psi\right|^2 \, dx \, dt \lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|^2_{L^2((0,T)\times\Omega)} + \widehat{I}(\psi^{(3)}).$$

On the other hand, since φ satisfies (1.16), it is solution of (1.5) with

$$g := \psi 1_{\mathcal{O}} + g^{(0)}.$$

Applying (3.15) yields

$$\left\| s^3 \lambda^4 \xi_\sharp^3 e^{-(N+1)s\alpha_\sharp} \varphi \right\|^2_{L^2((0,T)\times\Omega)} \lesssim \left\| e^{-N_2 s\alpha_\sharp} \psi \right\|^2_{L^2((0,T)\times\Omega)} + \left\| e^{-N_2 s\alpha_\sharp} g^{(0)} \right\|^2_{L^2((0,T)\times\Omega)} + \iint_{(0,T)\times\omega} e^{-s(2N+1)\alpha_\sharp} \left|\varphi_1\right|^2 \, dx \, dt$$

and since $N_2 \geqslant M+1$, we can combine this relation with (4.18) and deduce (4.29). □

4.4 Proof of Proposition 4.2

We are now in a position to prove Proposition 4.2. With respect to Lemma 4.5, we only need to remove the local term in ψ in (4.29).

Proof of Proposition 4.2. Using (4.16), there exists $\check{\chi} \in C^\infty(\mathbb{R}^2; [0,1])$, such that

$$\check{\chi} \equiv 1 \text{ in } \breve{\omega}^{(3)} = \widehat{\omega}_0, \quad \operatorname{supp} \check{\chi} \subset \widehat{\omega}$$

and we can write

$$\iint_{(0,T)\times\breve{\omega}^{(3)}} e^{-2s\left(1-\frac{1}{4^3}\right)\alpha_\sharp} \left|\Delta\psi_1^{(3)}\right|^2 \,dx\,dt \leqslant \iint_{(0,T)\times\widehat{\omega}} \check{\chi} e^{-2s\left(1-\frac{1}{4^3}\right)\alpha_\sharp} \left|\Delta\psi_1^{(3)}\right|^2 \,dx\,dt$$
$$= \iint_{(0,T)\times\widehat{\omega}} \check{\chi} e^{-2s\left(1-\frac{1}{4^3}\right)\alpha_\sharp} \left(\Delta\psi_1^{(3)}\right)\left(\widehat{\rho}_1\widehat{\rho}_2\Delta\psi_1 - \widehat{\rho}_1\Delta\psi_1^{(1)} - \Delta\psi_1^{(2)}\right) dx\,dt. \quad (4.33)$$

Applying the operator Δ on the first component of the first equation of (1.16) and using (1.20), we deduce

$$-\partial_t\Delta\varphi_1 - \Delta^2\varphi_1 = \Delta\psi_1 + \Delta g_1^{(0)} - \partial_{x_1}\operatorname{div} g^{(0)} \quad \text{in } (0,T)\times\widehat{\omega}.$$

Combining the above relation with (4.33), we find

$$\iint_{(0,T)\times\breve{\omega}^{(3)}} e^{-2s\left(1-\frac{1}{4^3}\right)\alpha_\sharp} \left|\Delta\psi_1^{(3)}\right|^2 \,dx\,dt$$
$$\leqslant -\iint_{(0,T)\times\widehat{\omega}} \check{\chi} e^{-2s\left(1-\frac{1}{4^3}\right)\alpha_\sharp} \left(\Delta\psi_1^{(3)}\right)\Big(\widehat{\rho}_1\widehat{\rho}_2\partial_t\Delta\varphi_1$$
$$+ \widehat{\rho}_1\widehat{\rho}_2\Delta^2\varphi_1 + \widehat{\rho}_1\widehat{\rho}_2\Delta g_1^{(0)} - \widehat{\rho}_1\widehat{\rho}_2\partial_{x_1}\operatorname{div} g^{(0)} + \widehat{\rho}_1\Delta\psi_1^{(1)} + \Delta\psi_1^{(2)}\Big) dx\,dt. \quad (4.34)$$

Now, we estimate each term of right-hand side of the above relation. First,

$$-\iint_{(0,T)\times\widehat{\omega}} \check{\chi} e^{-2s\left(1-\frac{1}{4^3}\right)\alpha_\sharp} \left(\Delta\psi_1^{(3)}\right)\left(\widehat{\rho}_1\widehat{\rho}_2\partial_t\Delta\varphi_1\right) dx\,dt$$
$$= \iint_{(0,T)\times\widehat{\omega}} \left(e^{-s\left(2+M-\frac{2}{4^3}\right)\alpha_\sharp}\right)' \left(\Delta\check{\chi}\Delta\psi_1^{(3)} + 2\nabla\check{\chi}\cdot\nabla\Delta\psi_1^{(3)} + \check{\chi}\Delta^2\psi_1^{(3)}\right)\varphi_1 \,dx\,dt$$
$$+ \iint_{(0,T)\times\widehat{\omega}} e^{-s\left(2+M-\frac{2}{4^3}\right)\alpha_\sharp} \left(\Delta\check{\chi}\partial_t\Delta\psi_1^{(3)} + 2\nabla\check{\chi}\cdot\partial_t\nabla\Delta\psi_1^{(3)} + \check{\chi}\partial_t\Delta^2\psi_1^{(3)}\right)\varphi_1 \,dx\,dt$$

so that, using (4.15),

$$-\iint_{(0,T)\times\widehat{\omega}} \check{\chi} e^{-2s\left(1-\frac{1}{4^3}\right)\alpha_\sharp} \left(\Delta\psi_1^{(3)}\right)\left(\widehat{\rho}_1\widehat{\rho}_2\partial_t\Delta\varphi_1\right) dx\,dt$$
$$\lesssim \left(\left\|\widehat{\theta}_3\partial_t\psi^{(3)}\right\|_{L^2(0,T;H^4(\Omega))} + \left\|\widehat{\theta}_2\psi^{(3)}\right\|_{L^2(0,T;H^4(\Omega))}\right)\left(\iint_{(0,T)\times\widehat{\omega}} e^{-2s\left(1+M-\frac{1}{4^2}\right)\alpha_\sharp}|\varphi_1|^2 \,dx\,dt\right)^{1/2}. \quad (4.35)$$

Similarly,

$$-\iint_{(0,T)\times\widehat{\omega}} \check{\chi} e^{-2s\left(1-\frac{1}{4^3}\right)\alpha_\sharp} \left(\Delta\psi_1^{(3)}\right)\left(\widehat{\rho}_1\widehat{\rho}_2\Delta^2\varphi_1\right) dx\,dt$$
$$\lesssim \left\|\widehat{\theta}_3\psi^{(3)}\right\|_{L^2(0,T;H^6(\Omega))}\left(\iint_{(0,T)\times\widehat{\omega}} e^{-2s\left(1+M-\frac{1}{4^2}\right)\alpha_\sharp}|\varphi_1|^2 \,dx\,dt\right)^{1/2},$$

$$
-\iint_{(0,T)\times\widehat{\omega}} \check{\chi} e^{-2s\left(1-\frac{1}{4^3}\right)\alpha_\sharp} \left(\Delta\psi_1^{(3)}\right) \widehat{\rho}_1\widehat{\rho}_2 \left(\Delta g_1^{(0)} - \partial_{x_1} \operatorname{div} g^{(0)}\right) \, dx\, dt
$$
$$
\lesssim \left\| \widehat{\theta}_2 \psi^{(3)} \right\|_{L^2(0,T;H^4(\Omega))} \left(\iint_{(0,T)\times\tilde{\omega}} e^{-2s\left(1+M-\frac{1}{4^2}\right)\alpha_\sharp} \left| g_1^{(0)} \right|^2 \, dx\, dt \right)^{1/2}
$$
$$
\lesssim \left\| \widehat{\theta}_2 \psi^{(3)} \right\|_{L^2(0,T;H^4(\Omega))} \left\| \widehat{\rho}_1\widehat{\rho}_2 g^{(0)} \right\|_{L^2(0,T;L^2(\Omega))},
$$

and

$$
-\iint_{(0,T)\times\widehat{\omega}} \check{\chi} e^{-2s\left(1-\frac{1}{4^3}\right)\alpha_\sharp} \left(\Delta\psi_1^{(3)}\right) \left(\widehat{\rho}_1 \Delta\psi_1^{(1)} + \Delta\psi_1^{(2)}\right) \, dx\, dt.
$$
$$
\lesssim \left(\iint_{(0,T)\times\widehat{\omega}} e^{-4s\left(1-\frac{1}{4^3}\right)\alpha_\sharp} \left(\Delta\psi_1^{(3)}\right)^2 \, dx\, dt \right)^{1/2} \left(\left\|\psi^{(1)}\right\|_{L^2(0,T;H^2(\Omega))} + \left\|\psi^{(2)}\right\|_{L^2(0,T;H^2(\Omega))} \right)
$$
$$
\lesssim \left\| \widehat{\theta}_1 \psi^{(3)} \right\|_{L^2(0,T;H^2(\Omega))} \left\| \widehat{\rho}_2 g^{(1)} \right\|_{L^2((0,T)\times\Omega)}. \tag{4.36}
$$

Gathering (4.29), (4.34), (4.35)–(4.36) and using $N_2 \geqslant M+1$ imply

$$
\widehat{I}(\psi^{(3)}) + \widehat{J}(\psi^{(3)}) + \left\| s^3\lambda^4\xi_\sharp^3 e^{-(N+1)s\alpha_\sharp}\varphi \right\|^2_{L^2((0,T)\times\Omega)} + \left\| s^{7/2}\lambda^4\xi_\sharp^{7/2} e^{-(M+1)s\alpha_\sharp}\psi \right\|^2_{L^2((0,T)\times\Omega)}
$$
$$
\lesssim \left\| \widehat{\rho}_2 g^{(1)} \right\|^2_{L^2((0,T)\times\Omega)} + \left\| \widehat{\rho}_1\widehat{\rho}_2 g^{(0)} \right\|^2_{L^2((0,T)\times\Omega)}
$$
$$
+ \iint_{(0,T)\times\omega} e^{-s(2N+1)\alpha_\sharp} |\varphi_1|^2 \, dx\, dt + \iint_{(0,T)\times\widehat{\omega}} e^{-2s\left(1+M-\frac{1}{4^2}\right)\alpha_\sharp} |\varphi_1|^2 \, dx\, dt. \tag{4.37}
$$

The above relation implies (4.19) and this concludes the proof of Proposition 4.2. □

4.5 Controllability results

First let us deduce Theorem 1.7 from Proposition 4.2:

Proof of Theorem 1.7. We consider $s_0 > 0$ and $\lambda_0 > 0$ from Proposition 4.2 and in what follows, we fix $\lambda = \lambda_0$ and $s = s_0(T^m + T^{2m})$.

Since $N_2 \geqslant M+1$,

$$
N+1 > M > M_2
$$

and there exists $P > 2$ such that

$$
\left(\frac{1}{4}\frac{P^2}{(P-1)}\right)^m < \frac{N+1}{M} < \frac{N+1}{M_2}.
$$

In particular, from (2.7)

$$
-(N+1)s\alpha_\sharp(T/2) + M_2 s\alpha_\sharp(T/P) < 0, \quad -(N+1)s\alpha_\sharp(T/2) + Ms\alpha_\sharp(T/P) < 0. \tag{4.38}
$$

We consider $\chi_0 \in C^\infty([0,T])$ such that

$$
\chi_0 \equiv 0 \quad \text{in } [0,T/P], \quad \chi_0 \equiv 1 \quad \text{in } [T/2,T], \quad |\chi_0'| \lesssim 1/T.
$$

Then we deduce from (1.16) that

$$
\begin{cases}
\mathcal{L}^*\left(\chi_0\varphi, \chi_0\pi_\varphi\right) = \left(\chi_0\psi 1_{\mathcal{O}} + \chi_0 g^{(0)} - \chi_0'\varphi, 0\right), \\
\mathcal{L}(\chi_0\psi, \chi_0\pi_\psi) = \left(\chi_0 g^{(1)} + \chi_0'\psi, 0\right).
\end{cases} \tag{4.39}
$$

Then, combining Proposition 2.1 and Proposition 2.2, we deduce

$$\|\chi_0\varphi\|_{L^2(0,T;L^2(\Omega))} + \|\chi_0\psi\|_{L^2(0,T;L^2(\Omega))} \lesssim \left\|\chi_0 g^{(0)}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\chi_0 g^{(1)}\right\|_{L^2(0,T;L^2(\Omega))} + \|\chi_0'\varphi\|_{L^2(0,T;L^2(\Omega))} + \|\chi_0'\psi\|_{L^2(0,T;L^2(\Omega))}.$$

In particular, from (4.3), (4.4) and (4.38),

$$\begin{aligned}
&\|\widehat{\sigma}_3\varphi\|_{L^2(T/2,T;L^2(\Omega))} + \|\widehat{\sigma}_3\psi\|_{L^2(T/2,T;L^2(\Omega))} \\
&\qquad \lesssim e^{-(N+1)s\alpha_\sharp(T/2)+M_2 s\alpha_\sharp(T/P)} \left(\left\|\widehat{\sigma}_1 g^{(0)}\right\|_{L^2(T/P,T;L^2(\Omega))} + \left\|\widehat{\sigma}_1 g^{(1)}\right\|_{L^2(T/P,T;L^2(\Omega))}\right) \\
&\qquad\quad + \frac{1}{T} e^{-(N+1)s\alpha_\sharp(T/2)+Ms\alpha_\sharp(T/P)} \left(\|\widehat{\sigma}_2\varphi\|_{L^2(T/P,T/2;L^2(\Omega))} + \|\widehat{\sigma}_2\psi\|_{L^2(T/P,T/2;L^2(\Omega))}\right) \\
&\lesssim \left\|\widehat{\sigma}_1 g^{(0)}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\widehat{\sigma}_1 g^{(1)}\right\|_{L^2(0,T;L^2(\Omega))} + \|\widehat{\sigma}_2\varphi\|_{L^2(T/P,T/2;L^2(\Omega))} + \|\widehat{\sigma}_2\psi\|_{L^2(T/P,T/2;L^2(\Omega))}.
\end{aligned} \tag{4.40}$$

Then, we deduce from (4.19) that

$$\|\widehat{\sigma}_3\varphi\|_{L^2(0,T/2;L^2(\Omega))} + \|\widehat{\sigma}_3\psi\|_{L^2(0,T/2;L^2(\Omega))} \lesssim \left\|\widehat{\sigma}_1 g^{(0)}\right\|_{L^2(0,T;L^2(\Omega))} + \left\|\widehat{\sigma}_1 g^{(1)}\right\|_{L^2(0,T;L^2(\Omega))} + \|\widehat{\sigma}_2\varphi_1\|_{L^2(0,T;L^2(\omega\cup\widehat{\omega}))}.$$

Combining the above relation and (4.40) yields the result. □

The proof of Corollary 1.9 is completely standard and we only present the main ideas to prove it from Theorem 1.7.

Proof of Corollary 1.9. First, we define the space

$$\mathcal{X}_0 := \left\{(\varphi,\pi_\varphi,\psi,\pi_\psi) \in C^\infty([0,T]\times\overline{\Omega}) \ : \ \operatorname{div}\varphi = \operatorname{div}\psi = 0, \quad \varphi=\psi=0 \text{ on } (0,T)\times\partial\Omega, \ \int_\Omega \pi_\varphi \ dx = \int_\Omega \pi_\psi \ dx = 0\right\},$$

the operators

$$L^*\varphi := -\partial_t\varphi - \Delta\varphi + \sum_{i=1}^n \left(\int_\Omega a^{(i)}\cdot\varphi \ dx\right) b^{(i)}$$

$$L\psi := \partial_t\psi - \Delta\psi + \sum_{i=1}^n \left(\int_\Omega b^{(i)}\cdot\psi \ dx\right) a^{(i)},$$

and the bilinear symmetric form

$$\begin{aligned}
\langle(\varphi,\pi_\varphi,\psi,\pi_\psi),(\check\varphi,\check\pi_\varphi,\check\psi,\check\pi_\psi)\rangle_{\mathcal{X}} := &\iint_{(0,T)\times\Omega} \widehat{\sigma}_1^2(L^*\varphi+\nabla\pi_\varphi-\psi 1_{\mathcal{O}})\cdot(L^*\check\varphi+\nabla\check\pi_\varphi-\check\psi 1_{\mathcal{O}}) \ dx \ dt \\
&+ \iint_{(0,T)\times\Omega} \widehat{\sigma}_1^2(L\psi+\nabla\pi_\psi)\cdot(L^*\check\varphi+\nabla\check\pi_\psi) \ dx \ dt + \iint_{(0,T)\times(\omega\cup\widehat{\omega})} \widehat{\sigma}_2^2\varphi_1\check\varphi_1 \ dx \ dt.
\end{aligned}$$

From (1.22), we deduce that

$$\|\widehat{\sigma}_3\varphi\|_{L^2(0,T;L^2(\Omega))} + \|\widehat{\sigma}_3\psi\|_{L^2(0,T;L^2(\Omega))} \lesssim \|(\varphi,\pi_\varphi,\psi,\pi_\psi)\|_{\mathcal{X}} := \langle(\varphi,\pi_\varphi,\psi,\pi_\psi),(\varphi,\pi_\varphi,\psi,\pi_\psi)\rangle_{\mathcal{X}}^{1/2}. \tag{4.41}$$

and thus $\|\cdot\|_{\mathcal{X}}$ is a norm and we can define the completion $\mathcal{X}$ of $\mathcal{X}_0$ for this norm.

We also define

$$\ell((\check{\varphi}, \check{\pi}_\varphi, \check{\psi}, \check{\pi}_\psi)) := \iint_{(0,T)\times\Omega} \left(f^{(0)} \cdot \check{\varphi} + f^{(1)} \cdot \check{\psi} \right) \, dx \, dt,$$

From (4.41), we deduce that ℓ is a linear continuous form of $\mathcal{X}$ and

$$\|\ell\|_{\mathcal{X}'} \lesssim \left\| \frac{f^{(0)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T)\times\Omega)} + \left\| \frac{f^{(1)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T)\times\Omega)}.$$

Thus from the Riesz theorem, there exists a unique $(\varphi, \pi_\varphi, \psi, \pi_\psi) \in \mathcal{X}$ such that

$$\forall (\check{\varphi}, \check{\pi}_\varphi, \check{\psi}, \check{\pi}_\psi) \in \mathcal{X}, \quad \langle (\varphi, \pi_\varphi, \psi, \pi_\psi), (\check{\varphi}, \check{\pi}_\varphi, \check{\psi}, \check{\pi}_\psi) \rangle_{\mathcal{X}} = \ell((\check{\varphi}, \check{\pi}_\varphi, \check{\psi}, \check{\pi}_\psi)). \tag{4.42}$$

We set

$$z := \widehat{\sigma}_1^2 (L^* \varphi + \nabla \pi_\varphi - \psi 1_{\mathcal{O}}), \quad w := \widehat{\sigma}_1^2 (L\psi + \nabla \pi_\psi), \quad u := -\widehat{\sigma}_2^2 \varphi_1 e_1, \tag{4.43}$$

and from (4.42), we deduce that

$$\left\| \frac{z}{\widehat{\sigma}_1} \right\|_{L^2((0,T)\times\Omega)} + \left\| \frac{w}{\widehat{\sigma}_1} \right\|_{L^2((0,T)\times\Omega)} + \left\| \frac{u}{\widehat{\sigma}_2} \right\|_{L^2((0,T)\times(\omega\cup\widehat{\omega}))} \lesssim \left\| \frac{f^{(0)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T)\times\Omega)} + \left\| \frac{f^{(1)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T)\times\Omega)}$$

and that

$$\iint_{(0,T)\times\Omega} z \cdot (L^* \check{\varphi} + \nabla \check{\pi}_\varphi - 1_{\mathcal{O}} \check{\psi}) \, dx \, dt + \iint_{(0,T)\times\Omega} w \cdot (L\check{\psi} + \nabla \check{\pi}_\psi) \, dx \, dt$$
$$= \iint_{(0,T)\times\omega_\natural} u \cdot \check{\varphi} \, dx \, dt + \iint_{(0,T)\times\Omega} f^{(0)} \cdot \check{\varphi} \, dx \, dt + \iint_{(0,T)\times\Omega} f^{(1)} \cdot \check{\psi} \, dx \, dt. \tag{4.44}$$

The last relation yields that (z, w) is a weak solution of (1.15). We recall that $\widehat{\sigma}_0$ is defined by (4.5). We can check that

$$\begin{cases} \mathcal{L}\left(\dfrac{z}{\widehat{\sigma}_0}, \dfrac{p}{\widehat{\sigma}_0} \right) = \left(\dfrac{f^{(0)}}{\widehat{\sigma}_0} - \dfrac{\widehat{\sigma}_0' z}{\widehat{\sigma}_0^2}, 0 \right), \\ \mathcal{L}^*\left(\dfrac{w}{\widehat{\sigma}_0}, \dfrac{q}{\widehat{\sigma}_0} \right) = \left(\dfrac{f^{(1)}}{\widehat{\sigma}_0} + \dfrac{z}{\widehat{\sigma}_0} 1_{\mathcal{O}} + \dfrac{\widehat{\sigma}_0' w}{\widehat{\sigma}_0^2}, 0 \right), \end{cases}$$

and thus using (3.4), Proposition 2.1 and Proposition 2.2, we deduce that

$$\left\| \frac{z}{\widehat{\sigma}_0} \right\|_{X_1} + \left\| \frac{w}{\widehat{\sigma}_0} \right\|_{X_1} \lesssim \left\| \frac{f^{(0)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T)\times\Omega)} + \left\| \frac{f^{(1)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T)\times\Omega)}. \tag{4.45}$$

This implies in particular that $w(0, \cdot) = 0$.

In order to prove the existence of insensitizing controls for (1.1) and (1.14), we define

$$\mathcal{F}_3 := \left\{ \left(f^{(0)}, f^{(1)} \right) \; ; \; \frac{f^{(0)}}{\widehat{\sigma}_3}, \frac{f^{(1)}}{\widehat{\sigma}_3} \in L^2((0,T)\times\Omega) \right\}$$

and the mapping

$$\mathcal{N} : \mathcal{F}_3 \to \mathcal{F}_3, \quad \left(f^{(0)}, f^{(1)} \right) \mapsto \left(f - (z \cdot \nabla) z, -(\nabla z)^* w + (z \cdot \nabla) w \right)$$

where (z, w) is the above solution, that is given by (4.43) and that satisfies (4.45) and (4.44).

Using (4.7), one can check that the map $\mathcal{N}$ is well-defined and from (4.45), we can also show that if

$$\left\| \frac{f}{\widehat{\sigma}_3} \right\|_{L^2((0,T)\times\Omega)} \leqslant r$$

and if r is small enough, the closed ball

$$\mathcal{B}_3 := \left\{ \left(f^{(0)}, f^{(1)}\right) \in \mathcal{F}_3 \ ; \ \left\| \frac{f^{(0)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T)\times\Omega)} + \left\| \frac{f^{(1)}}{\widehat{\sigma}_3} \right\|_{L^2((0,T)\times\Omega)} \leqslant 2r \right\}$$

is invariant by $\mathcal{N}$ and is a strict contraction on this set. This yields the existence of a fixed point for $\mathcal{N}$. The corresponding solution (z, w) satisfies (1.1) and (1.14), and (1.23). This concludes the proof of Corollary 1.9. □

5 Conclusion

In this article, we have obtained the local controllability of the system (1.1) that corresponds to the Navier-Stokes with a nonlocal spatial term. In order to do this, we have shown a weighted observability type inequality in Theorem 1.3 for the adjoint system. One of the main ingredients to prove such an inequality is Carleman estimates that are a standard tool in the study of the controllability of parabolic systems. Here, due to the nonlocal spatial terms, we needed to use several weight functions and, to overcome regularity issues, we introduce a new decomposition (see Section 3.1). Such a decomposition can be used in other problems for the controllability of parabolic systems. We have also managed to show the existence of insensitizing controls for (1.1), by using a similar method.

The results obtained here could be extended to other systems such as the heat equation and the Boussinesq system with similar nonlocal spatial terms.

There are several open questions that remain to be solved in the future: is it possible to replace the condition (1.9) by a weaker condition? Is it possible to show similar results in the case where the nonlocal spatial term has a general form such as (1.2)? At the moment, there are few results in this general case, one can quote for instance [16] or [23] where there are strong hypotheses on the regularity of the kernel k.

A Proof of Proposition 2.1

We give here a sketch of the proof of Proposition 2.1 that is quite standard. The proof of Proposition 2.2 can be obtained with a similar method.

Proof of Proposition 2.1. The proof of Proposition 2.1 can be obtained by using the standard Galerkin method and by following for instance the proof of [25, Proposition 1.2, pp. 267–268] for the Stokes system. The main idea is to obtain uniform a priori estimates for the approximate solutions. Here, to simplify, we only show a priori estimates on the solutions of (2.2), but one can recover similar estimates for the Galerkin approximation.

First, by multiplying the first equation of (2.3) by φ and integrating in $(t,T)\times\Omega$, we obtain that for $t \in (0,T)$

$$\int_\Omega |\varphi(t,x)|^2 \ dx + 2\int_t^T \int_\Omega |\nabla\varphi|^2 \ dx\,ds \leqslant \int_\Omega |\varphi_T|^2 \ dx + \int_0^T \int_\Omega |g|^2 \ dx\,ds \\ + \left(1 + \sum_{i=1}^n \left\|a^{(i)}\right\|_{L^\infty(0,T;L^2(\Omega))} \left\|b^{(i)}\right\|_{L^\infty(0,T;L^2(\Omega))}\right) \int_t^T \int_\Omega |\varphi|^2 \ dx\,ds$$

and with the Grönwall lemma, this yields that for $t \in (0,T)$

$$\int_\Omega |\varphi(t,x)|^2 \ dx + 2\int_t^T \int_\Omega |\nabla\varphi|^2 \ dx\,ds \lesssim \int_\Omega |\varphi_T|^2 \ dx + \int_0^T \int_\Omega |g|^2 \ dx\,ds,$$

where the constant of the above inequality depends on T and on the norms of $a^{(i)}$ and $b^{(i)}$ in $H^2(0,T;L^2(\Omega))$. With the Poincaré inequality, we thus deduce that,

$$\|\varphi\|_{L^\infty(0,T;L^2(\Omega))} + \|\varphi\|_{L^2(0,T;H^1(\Omega))} \lesssim \|\varphi_T\|_{L^2(\Omega)} + \|g\|_{L^2(0,T;L^2(\Omega))}. \tag{A.1}$$

Second, by multiplying the first equation of (2.3) by $-\partial_t\varphi$ and integrating in $(t,T)\times\Omega$, we obtain that for $t\in(0,T)$

$$\int_t^T\int_\Omega |\partial_t\varphi|^2\ dx\,ds + \frac{1}{2}\int_\Omega |\nabla\varphi(t,x)|^2\ dx \leqslant \frac{1}{2}\int_\Omega |\nabla\varphi_T|^2\ dx + \int_0^T\int_\Omega |g|^2\ dx\,ds$$
$$+\int_t^T \sum_{i=1}^n \left\|a^{(i)}\right\|^2_{L^\infty(0,T;L^2(\Omega))} \left\|b^{(i)}\right\|^2_{L^\infty(0,T;L^2(\Omega))} \|\varphi(s,\cdot)\|^2_{L^2(\Omega)}\ ds$$

and combining the above relation with (A.1) implies

$$\|\varphi\|_{H^1(0,T;L^2(\Omega))} + \|\varphi\|_{L^\infty(0,T;H^1(\Omega))} \lesssim \|\varphi_T\|_{H^1(\Omega)} + \|g\|_{L^2(0,T;L^2(\Omega))}. \tag{A.2}$$

Then, we can see (2.3) as a stationary Stokes system and apply the elliptic regularity of such a system (see, for instance, [25, Proposition 2.2, p.33]):

$$\|\varphi\|_{L^2(0,T;H^2(\Omega))} + \|\pi\|_{L^2(0,T;H^1(\Omega)/\mathbb{R})} \lesssim \|\partial_t\varphi\|_{L^2(0,T;L^2(\Omega))}$$
$$+\sum_{i=1}^n \left\|a^{(i)}\right\|_{L^\infty(0,T;L^2(\Omega))} \left\|b^{(i)}\right\|_{L^\infty(0,T;L^2(\Omega))} \|\varphi\|_{L^2(0,T;L^2(\Omega))}. \tag{A.3}$$

Combining (A.1), (A.2) and (A.3) we deduce (2.4).

Then to obtain (2.5), we differentiate (2.3) in time to obtain

$$\begin{cases} -\partial_t(\partial_t\varphi) - \Delta(\partial_t\varphi) + \displaystyle\sum_{i=1}^n \left(\int_\Omega a^{(i)}\cdot\partial_t\varphi\ dx\right) b^{(i)} + \nabla(\partial_t\pi) = g^{(1)} & \text{in } (0,T)\times\Omega, \\ \operatorname{div}(\partial_t\varphi) = 0 & \text{in } (0,T)\times\Omega, \\ \partial_t\varphi = 0 & \text{on } (0,T)\times\partial\Omega, \\ (\partial_t\varphi)(T,\cdot) = 0 & \text{in } \Omega, \end{cases} \tag{A.4}$$

with

$$g^{(1)} := \partial_t g - \sum_{i=1}^n \left(\int_\Omega \partial_t a^{(i)}\cdot\varphi\ dx\right) b^{(i)} - \sum_{i=1}^n \left(\int_\Omega a^{(i)}\cdot\varphi\ dx\right)\partial_t b^{(i)}.$$

From the first part of the proof, we have

$$\|\partial_t\varphi\|_{H^1(0,T;L^2(\Omega))} + \|\partial_t\varphi\|_{L^2(0,T;H^2(\Omega))} \lesssim \left\|g^{(1)}\right\|_{L^2(0,T;L^2(\Omega))}. \tag{A.5}$$

We can check that

$$\left\|g^{(1)}\right\|_{L^2(0,T;L^2(\Omega))} \lesssim \|g\|_{H^1(0,T;L^2(\Omega))} + \sum_{i=1}^n \left\|a^{(i)}\right\|_{H^1(0,T;L^2(\Omega))} \left\|b^{(i)}\right\|_{L^\infty(0,T;L^2(\Omega))} \|\varphi\|_{L^\infty(0,T;L^2(\Omega))}$$
$$+\sum_{i=1}^n \left\|b^{(i)}\right\|_{H^1(0,T;L^2(\Omega))} \left\|a^{(i)}\right\|_{L^\infty(0,T;L^2(\Omega))} \|\varphi\|_{L^\infty(0,T;L^2(\Omega))}.$$

Combining the above estimate with (A.5) and (2.4) implies

$$\|\partial_t\varphi\|_{H^1(0,T;L^2(\Omega))} + \|\partial_t\varphi\|_{L^2(0,T;H^2(\Omega))} \lesssim \|g\|_{H^1(0,T;L^2(\Omega))}. \tag{A.6}$$

Then, using the elliptic regularity of the Stokes system (see, for instance, [25, Proposition 2.2, p.33]), we deduce

$$\|\varphi\|_{L^2(0,T;H^4(\Omega))} + \|\pi\|_{L^2(0,T;H^3(\Omega)/\mathbb{R})} \lesssim \|\partial_t\varphi\|_{L^2(0,T;H^2(\Omega))} + \sum_{i=1}^n \left\|a^{(i)}\right\|_{L^\infty(0,T;L^2(\Omega))} \left\|b^{(i)}\right\|_{L^2(0,T;H^2(\Omega))} \|\varphi\|_{L^\infty(0,T;L^2(\Omega))}.$$

Gathering (A.6), (2.4) and the above estimate yields (2.5).

Finally, to obtain (2.6), we differentiate (A.4) in time to obtain

$$\begin{cases} -\partial_t\left(\partial_t^2\varphi\right) - \Delta\left(\partial_t^2\varphi\right) + \displaystyle\sum_{i=1}^n \left(\int_\Omega a^{(i)}\cdot\partial_t^2\varphi\, dx\right) b^{(i)} + \nabla\left(\partial_t^2\pi\right) = g^{(2)} & \text{in } (0,T)\times\Omega, \\ \operatorname{div}\left(\partial_t^2\varphi\right) = 0 & \text{in } (0,T)\times\Omega, \\ \partial_t^2\varphi = 0 & \text{on } (0,T)\times\partial\Omega, \\ \left(\partial_t^2\varphi\right)(T,\cdot) = 0 & \text{in } \Omega, \end{cases}$$

with

$$g^{(2)} := \partial_t^2 g - \sum_{i=1}^n \left[\left(\int_\Omega \partial_t^2 a^{(i)}\cdot\varphi\, dx\right) b^{(i)} + \left(\int_\Omega a^{(i)}\cdot\varphi\, dx\right)\partial_t^2 b^{(i)} + 2\left(\int_\Omega \partial_t a^{(i)}\cdot\partial_t\varphi\, dx\right) b^{(i)} + 2\left(\int_\Omega \partial_t a^{(i)}\cdot\varphi\, dx\right)\partial_t b^{(i)} + 2\left(\int_\Omega a^{(i)}\cdot\partial_t\varphi\, dx\right)\partial_t b^{(i)}\right].$$

We can check that

$$\left\|g^{(2)}\right\|_{L^2(0,T;L^2(\Omega))} \lesssim \|g\|_{H^2(0,T;L^2(\Omega))} + \sum_{i=1}^n \left\|a^{(i)}\right\|_{H^2(0,T;L^2(\Omega))} \left\|b^{(i)}\right\|_{H^2(0,T;L^2(\Omega))} \|\varphi\|_{L^\infty(0,T;L^2(\Omega))}.$$

From the first part of the proof and the above estimate, we have

$$\left\|\partial_t^2\varphi\right\|_{H^1(0,T;L^2(\Omega))} + \left\|\partial_t^2\varphi\right\|_{L^2(0,T;H^2(\Omega))} \lesssim \|g\|_{H^2(0,T;L^2(\Omega))}. \tag{A.7}$$

Using the elliptic regularity of the Stokes system (see, for instance, [25, Proposition 2.2, p.33]) on (A.4), we deduce from the above estimate that

$$\|\partial_t\varphi\|_{L^2(0,T;H^4(\Omega))} \lesssim \|g\|_{H^2(0,T;L^2(\Omega))} + \|g\|_{H^1(0,T;H^2(\Omega))}.$$

Then, using the elliptic regularity of the Stokes system (see, for instance, [25, Proposition 2.2, p.33]) on (2.3) and the above estimate, we obtain

$$\|\varphi\|_{L^2(0,T;H^6(\Omega))} \lesssim \|g\|_{H^2(0,T;L^2(\Omega))} + \|g\|_{H^1(0,T;H^2(\Omega))} + \|g\|_{L^2(0,T;H^4(\Omega))}.$$

Combining the above relation and (A.7) gives (2.6). □

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Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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