Local null controllability of a cubic Ginzburg-Landau equation with dynamic boundary conditions

Nicolás Carreño[1†], Alberto Mercado[1†], Roberto Morales[1*]

[1*]Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile.

*Corresponding author(s). E-mail(s): roberto.moralesp@usm.cl;
Contributing authors: nicolas.carrenog@usm.cl; alberto.mercado@usm.cl;
[†]These authors declare no conflict of interests and the research presented herein does not rely on any specific dataset.

Abstract

This paper deals with controllability properties of a cubic Ginzburg-Landau equation with dynamic boundary conditions. More precisely, we prove a local null controllability result by using a single control supported in a small subset of the domain. In order to achieve this result, we first linearize the system around the origin and analyze it by the duality approach and an appropriate Carleman estimate. Then, using an inverse function theorem, the local null controllability of the nonlinear system is proven.

Keywords: Controllability, Ginzburg-Landau equation, Dynamic boundary conditions.

MSC Classification: 93B05 , 35Q56 , 93B07

1 Introduction and main results

1.1 Introduction

Let $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) be a bounded domain with boundary $\Gamma := \partial\Omega$ of class C^2. Given the parameters $a, b, c > 0$, $\alpha, \gamma \in \mathbb{R} \setminus \{0\}$, we consider the following cubic

Ginzburg-Landau equation with dynamic boundary conditions

$$\begin{cases} \partial_t u - a(1+\alpha i)\Delta u + c(1+\gamma i)|u|^2 u = \mathbb{1}_\omega h & \text{in } \Omega \times (0,T) \\ \partial_t u_\Gamma + a(1+\alpha i)\partial_\nu u - b(1+\alpha i)\Delta_\Gamma u_\Gamma + c(1+\gamma i)|u_\Gamma|^2 u_\Gamma = 0 & \text{on } \Gamma \times (0,T), \\ u = u_\Gamma & \text{on } \Gamma \times (0,T) \\ (u(0), u_\Gamma(0)) = (u_0, u_{\Gamma,0}) & \text{in } \Omega \times \Gamma. \end{cases} \tag{1}$$

Here, (u, u_Γ) is the state of the system, $(u_0, u_{\Gamma,0})$ the initial conditions and $h \in L^2(\omega \times (0,T); \mathbb{C})$ is a control acting on $\omega \subset \Omega$. We denote by Δ_Γ the Laplace-Beltrami operator on Γ and by $\partial_\nu y$ the normal derivative associated to the outward normal ν of Ω.

We notice that (1) can be seen as a coupled system in the variables (u, u_Γ), which is controlled by a single control h in a (small) subset ω of Ω. This means that the first equation of (1) is controlled directly by the action of the control, while the second equation is being controlled through the side condition $u = u_\Gamma$ on $\Gamma \times (0,T)$.

The main objective of this work is to obtain the local null controllability of system (1) in X (where X is an appropriate Banach space), i.e., we will prove the existence of a number $\delta > 0$ such that, for every initial state $(u_0, u_{\Gamma,0}) \in X$ which fulfills

$$\|(u_0, u_{\Gamma,0})\|_X \leq \delta,$$

we can find a control $h \in L^2(\omega \times (0,T))$ such that the associated solution (u, u_Γ) of (1) satisfies

$$y(\cdot, T) = 0 \text{ in } \Omega, \quad y_\Gamma(\cdot, T) = 0 \text{ on } \Gamma.$$

1.2 Previous results

The cubic complex Ginzburg-Landau equation is one of the most studied nonlinear equations used to model physical phenomena. This equation has been used to describe several phenomena ranging from nonlinear waves, second-order phase transitions, superconductivity, superfluidity and Bose-Einstein condensation to liquid crystals and strings in field theory. For a detailed description of relevant applications in different fields, see [1] and [2].

The existence and uniqueness of solutions of nonlinear Ginzburg-Landau equations with Dirichlet or periodic boundary conditions have been intensely investigated in several papers. For instance, we refer to [3], [4], [5], [6], [7], [8], [9] and the references therein. Concerning controllability properties of the Ginzburg-Landau equation with Dirichlet boundary conditions, only a few papers have been devoted to the study of the controllability of such problems. In [10], the stabilization of the linearized Ginzburg-Landau model with Dirichlet boundary conditions around an unstable equilibrium state is studied. Moreover, in [6], the author develop a Carleman inequality for an operator of the form

$$(a+ib)\partial_t + \mathrm{div}(A \cdot \nabla),$$

with A being a smooth, uniformly elliptic matrix, and a null controllability result for the linear PDE with a distributed control. In 2009, L. Rosier and B.-Y. Zhang in [11] proved a controllability result for the nonlinear case. In this case, the control acts on a part of the boundary, and the proof is based on a suitable Carleman estimate for the linear adjoint system. Then, combining a fixed-point argument together with the theory of sectorial operators, the authors obtained a local controllability result for a wide class of nonlinearities. In particular, controllability results for the cubic and quintic Complex Ginzburg-Landau are provided.

Recently, some results on inverse problems and controllability issues have been obtained for PDEs with dynamic boundary conditions; see, for instance, [12], [13], [14], [15], [16], and [17] for the heat equation, [18] for the wave equation, and [19] for the Schrödinger operator. In these works, the authors has been used the duality equivalence to prove the associated observability inequality by using Carleman estimates. At this level, we point out that it is not evident at all that such systems can be controlled by the action of a single control due to the tangential derivative terms. In fact, in the case of the linear wave equation with mixed boundary conditions (oscilatory boundary conditions and Dirichlet boundary conditions) [18], the authors obtain exact controllability results where the control region is on the whole boundary (and therefore on the whole system). On the other hand, in a similar setting, in [19] the authors obtained the exact controllability of the linear Schrödinger equation with dynamic boundary conditions. In this case, the control acts only on a part of the boundary. To prove the associated observability inequality, the authors used a Global Carleman estimate for the Schrödinger operator, where the weight function is adapted to the geometric properties of the domain.

Concerning the Ginzburg-Landau equation with dynamic boundary conditions, we mention [20], where well-posedness of linear/nonlinear of such models is obtained and long time behavior of solutions is characterized when Lipschitz nonlinearities are considered. However, to the best of the authors' knowledge, this is the first time that the null controllability for the cubic Ginzburg-Landau is studied.

1.3 General setting

In this section, we set up the notation and terminology used in this paper. The set $\Gamma = \partial\Omega$ can be seen as an $(d-1)$-dimensional compact Riemannian submanifold equipped by the Riemmanian metric g induced by the natural embedding $\Gamma \subset \mathbb{R}^d$. In addition, we shall denote by dS the $(d-1)$-Lebesgue measure for Γ.

Since we are considering dynamic boundary conditions, we need to define some differential operators on Γ, which can be defined in terms of the associated metric. However, for our purposes, it will be enough to use the most important properties of the underlaying operators and spaces. Details can be found, for instance, in [21]. For the sake of completeness, we recall some of those properties.

The tangential gradient ∇_Γ of y_Γ at each point $x \in \Gamma$ can be seen as the projection of the standard Euclidean gradient ∇y onto the tangent space of Γ at $x \in \Gamma$, where y_Γ is the trace of y on Γ, i.e., we have the identity

$$\nabla_\Gamma y_\Gamma = \nabla y - \nu \partial_\nu y$$

where $y = y_\Gamma$ on Γ and $\partial_\nu y$ is the normal derivative associated to the outward normal ν. In particular, notice that $\nabla_\Gamma y_\Gamma$ is well defined for every $y \in H^2(\Omega)$ such that $y = y_\Gamma$ on Γ. In this way, the tangential divergence div_Γ in Γ is defined by

$$\mathrm{div}_\Gamma(F_\Gamma) : H^1(\Gamma;\mathbb{R}) \to \mathbb{R}, \quad y_\Gamma \mapsto -\int_\Gamma F_\Gamma \cdot \nabla_\Gamma y_\Gamma dS.$$

The Laplace-Beltrami operator is given by $\Delta_\Gamma y_\Gamma := \mathrm{div}(\nabla_\Gamma y_\Gamma)$, for all $y_\Gamma \in H^2(\Gamma;\mathbb{R})$. In particular, the surface divergence theorem holds:

$$\int_\Gamma \Delta_\Gamma y_\Gamma z_\Gamma dS = -\int_\Gamma \nabla_\Gamma y_\Gamma \cdot \nabla_\Gamma z_\Gamma dS \quad \forall y_\Gamma \in H^2(\Gamma;\mathbb{R}), \quad \forall z_\Gamma \in H^1(\Gamma;\mathbb{R}).$$

In order to simplify the notation, here and subsequently, the function spaces refer to complex-valued functions unless otherwise stated.

For $1 \le p \le +\infty$, we consider the Banach space $\mathbb{L}^p := L^p(\Omega) \times L^p(\Gamma)$, endowed by the norm given by the relation

$$\|(u, u_\Gamma)\|_{\mathbb{L}^p}^2 := \|u\|_{L^p(\Omega)}^2 + \|u_\Gamma\|_{L^p(\Gamma)}^2.$$

In particular, for $p = 2$, the space $\mathbb{L}^2 := L^2(\Omega) \times L^2(\Gamma)$ is a Hilbert space equipped with the scalar product

$$\langle (u, u_\Gamma), (v, v_\Gamma) \rangle_{\mathbb{L}^2} := \Re \int_\Omega u\overline{v} dx + \Re \int_\Gamma u_\Gamma \overline{v_\Gamma} dS.$$

For $k \ge 1$, we also introduce the space

$$\mathbb{H}^k := \{(y, y_\Gamma) \in H^k(\Omega) \times H^k(\Gamma)\,;\, y\big|_\Gamma = y_\Gamma\},$$

where $H^k(\Omega)$ and $H^k(\Gamma)$ are the usual Sobolev spaces.

1.4 Main result

Our main result states the local null controllability on the space $\mathbb{H}^1$:

Theorem 1.1. *Suppose that $d = 2$ or $d = 3$. Let $a, b, c > 0$, $\alpha, \gamma \in \mathbb{R} \setminus \{0\}$. Then, for every $T > 0$ and $\omega \Subset \Omega$, there exists $\delta > 0$ such that, for every $(u_0, u_{\Gamma,0}) \in \mathbb{H}^1$ satisfying*

$$\|(u_0, u_{\Gamma,0})\|_{\mathbb{H}^1} \le \delta,$$

there exists a control $h \in L^2(\omega \times (0,T))$ *such that the unique corresponding solution* (u, u_Γ) *of* (1) *satisfies*

$$u(\cdot, T) = 0 \text{ in } \Omega, \quad u_\Gamma(\cdot, T) = 0 \text{ on } \Gamma,$$

To prove Theorem 1.1 we first deduce a null controllability result for a linear system associated to (1):

$$\begin{cases} \partial_t y - a(1+\alpha i)\Delta y = f + \mathbb{1}_\omega h & \text{in } \Omega \times (0,T), \\ \partial_t y_\Gamma + a(1+\alpha i)\partial_\nu y - b(1+\alpha i)\Delta_\Gamma y_\Gamma = f_\Gamma & \text{on } \Gamma \times (0,T), \\ y = y_\Gamma & \text{on } \Gamma \times (0,T), \\ (y, y_\Gamma)(0) = (y_0, y_{\Gamma,0}) & \text{in } \Omega \times \Gamma, \end{cases} \tag{2}$$

where (f, f_Γ) will be taken to decrease exponentially to zero in $t = T$. Then, we prove a new Carleman estimate for the adjoint system of (2) (see estimate (47) below). This will provide existence of a unique solution to a suitable variational problem, from which we define a solution (y, y_Γ, h) to (2) such that $y(T) = 0$ in $\overline{\Omega}$. Moreover, the solution is such that $e^{C/(T-t)}(y, y_\Gamma, h) \in \mathbb{L}^2 \times L^2(\omega \times (0,T))$, for some constant $C > 0$. Finally, by an inverse mapping theorem, we deduce the null controllability for the nonlinear system. We point out that this strategy has been widely used to prove controllability results for nonlinear PDEs. In particular, we mention that this technique has been used recently to prove local null controllability for the termistor problem [22], the null controllability a quasi-linear parabolic system [23], and to give numerical boundary controllability results for a parabolic equation with a nonlinear diffusion term [24].

The rest of the paper is organized as follows. In Section 2, we establish the existence and uniqueness of solutions of (1) and (2). In Section 3, we prove a suitable Carleman estimate for the Ginzburg-Landau operator with dynamic boundary conditions. In Section 4, we prove the observability estimate for the adjoint system and prove the null controllability of (2). In Section 5 we prove Theorem 1.1. Finally, in Section 6 we give additional comments related to our results.

2 Existence and uniqueness of solutions

In this section, we present new results concerning existence and uniqueness for the Ginzburg-Landau equations with dynamic boundary conditions.

2.1 Linear problem

We consider the Cauchy problem

$$\begin{cases} Lu = f & \text{in } \Omega \times (0,T), \\ L_\Gamma(u, u_\Gamma) = f_\Gamma & \text{on } \Gamma \times (0,T), \\ u = u_\Gamma & \text{on } \Gamma \times (0,T), \\ (u(0), u_\Gamma(0)) = (u_0, u_{\Gamma,0}) & \text{in } \Omega \times \Gamma. \end{cases} \tag{3}$$

where

$$Lu := \partial_t u - a(1+\alpha i)\Delta u, \quad L_\Gamma(u,u_\Gamma) := \partial_t u_\Gamma + a(1+\alpha i)\partial_\nu u - b(1+\alpha i)\Delta_\Gamma u_\Gamma, \tag{4}$$

respectively. Notice that the problem (3) can be seen in the abstract form

$$\begin{cases} U'(t) = \mathcal{A}_{GL}U(t) + F(t) & t \in (0,T), \\ U(0) = U_0, \end{cases} \tag{5}$$

where $\mathcal{A}_{GL} : D(\mathcal{A}_{GL}) \subset \mathbb{L}^2 \to \mathbb{L}^2$ is the operator defined by

$$\mathcal{A}_{GL}(U) := \begin{bmatrix} a(1+\alpha i)\Delta u \\ -a(1+\alpha i)\partial_\nu u + b(1+\alpha i)\Delta_\Gamma u_\Gamma \end{bmatrix} \quad \forall U := \begin{bmatrix} u \\ u_\Gamma \end{bmatrix} \in D(\mathcal{A}_{GL}), \tag{6}$$

with domain

$$D(\mathcal{A}_{GL}) := \left\{ U = \begin{bmatrix} u \\ u_\Gamma \end{bmatrix} \in \mathbb{H}^1 \, : \, (\Delta u, \Delta_\Gamma u_\Gamma) \in \mathbb{L}^2 \right\} = \mathbb{H}^2,$$

where the last equivalence is justified in [25], and

$$F(t) = \begin{bmatrix} f(t) \\ f_\Gamma(t) \end{bmatrix} \in L^2(0,T;\mathbb{L}^2).$$

If A_W is the Wentzell-Laplacian operator introduced in [12], then it is easy to see that $\mathcal{A}_{GL}$ can be written as

$$\mathcal{A}_{GL} = (1+\alpha i)A_W, \quad D(A_W) = D(\mathcal{A}_{GL}) = \mathbb{H}^2.$$

Then, arguing as in [11, Section 2] we have the following result:

Proposition 2.1. *The operator $\mathcal{A}_{GL}$ defined in (6) is densely defined and generates an analytic semigroup $(e^{t\mathcal{A}_{GL}})_{t\geq 0}$ in $\mathbb{L}^2$.*

According to Proposition 2.1, the existence and uniqueness of strong solutions of (5) in the usual sense are guaranteed. In the next subsection, we provide existence and uniqueness of solutions in appropriate spaces by energy estimates and density arguments.

Proposition 2.2. *Suppose that $(u_0, u_{\Gamma,0}) \in \mathbb{L}^2$ and $(f, f_\Gamma) \in L^2(0,T;\mathbb{L}^2)$. Then, the weak solution (u,u_Γ) of (3) belongs to $C^0([0,T];\mathbb{L}^2) \cap L^2(0,T;\mathbb{H}^1)$. Moreover, there exists a constant $C_1 > 0$ such that the associated solution (u,u_Γ) of (3) satisfies*

$$\|(u,u_\Gamma)\|_{C^0([0,T];\mathbb{L}^2)} + \|(u,u_\Gamma)\|_{L^2(0,T;\mathbb{H}^1)} \leq C_1\|(f,f_\Gamma)\|_{L^2(0,T;\mathbb{L}^2)} + C_1\|(u_0,u_{\Gamma,0})\|_{\mathbb{L}^2}. \tag{7}$$

Proof. Firstly, we multiply by $\overline{u}$ the first equation of (3) and integrate in Ω. Secondly, we multiply the second equation of (3) by $\overline{u}_\Gamma$ and integrate on Γ. Next, we add these identities and take the real part on the obtained equation. After integration by parts, this yields

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |u(t)|^2dx + \int_\Gamma |u_\Gamma(t)|^2dS\right) + a\int_\Omega |\nabla u(t)|^2dx + b\int_\Gamma |\nabla_\Gamma u_\Gamma(t)|^2dS$$
$$=\Re\int_\Omega f(t)\overline{u}(t)dx + \Re\int_\Gamma f_\Gamma(t)\overline{u}_\Gamma(t)dS.$$

By Young's inequality, it is easy to check that

$$\|(u,u_\Gamma)\|^2_{C^0([0,T];\mathbb{L}^2)} + \|(u,u_\Gamma)\|^2_{L^2(0,T;\mathbb{H}^1)} \le C\|(f,f_\Gamma)\|^2_{L^2(0,T;\mathbb{L}^2)} + C\|(u_0,u_{\Gamma,0})\|^2_{\mathbb{L}^2},$$

which clearly implies (7). ☐

Proposition 2.3. *Let $(u,u_{\Gamma,0})\in\mathbb{H}^1$ and $(f,f_\Gamma)\in L^2(0,T;\mathbb{L}^2)$. Then, the associated weak solution (u,u_Γ) of (3) belongs to $H^1(0,T;\mathbb{L}^2)\cap C^0([0,T];\mathbb{H}^1)\cap L^2(0,T;\mathbb{H}^2)$. Moreover, there exists a constant $C_2>0$ such that (u,u_Γ) satisfies*

$$\begin{aligned}&\|(u,u_\Gamma)\|_{H^1(0,T;\mathbb{L}^2)} + \|(u,u_\Gamma)\|_{C^0([0,T];\mathbb{H}^1)} + \|(u,u_\Gamma)\|_{L^2(0,T;\mathbb{H}^2)}\\ \le& C_2\|(f,f_\Gamma)\|_{L^2(0,T;\mathbb{L}^2)} + C_2\|(u_0,u_{\Gamma,0})\|_{\mathbb{H}^1}.\end{aligned} \tag{8}$$

Proof. The proof is divided into three steps.

• Step 1: Our first task is to obtain estimates $L^2(0,T;\mathbb{L}^2)$. In order to do that, we multiply the first equation of (3) by $(1-\alpha i)\partial_t\overline{u}$ and integrate in Ω. In addition, we multiply the second equation of (3) by $(1-\alpha i)\partial_t\overline{u}_\Gamma$ and integrate on Γ. Then, we sum up these identities and take the real part. This yields

$$\int_\Omega |\partial_t u(t)|^2dx + \int_\Gamma |\partial_t u_\Gamma(t)|^2dS + \frac{1}{2}a(1+\alpha^2)\frac{d}{dt}\int_\Omega |\nabla u|^2dx$$
$$+\frac{1}{2}b(1+\alpha^2)\frac{d}{dt}\int_\Gamma |\nabla_\Gamma u_\Gamma|^2dS = \Re\int_\Omega (1-\alpha i)f\partial_t\overline{u}dx + \Re\int_\Gamma (1-\alpha i)f_\Gamma\partial_t\overline{u}_\Gamma dS.$$

Integrating in time the above inequality, we easily get

$$\|(u,u_\Gamma)\|^2_{H^1(0,T;\mathbb{L}^2)} + \|(u,u_\Gamma)\|^2_{C^0([0,T];\mathbb{H}^1)} \le C\|(f,f_\Gamma)\|^2_{L^2(0,T;\mathbb{L}^2)} + C\|(u_0,u_{\Gamma,0})\|^2_{\mathbb{H}^1}. \tag{9}$$

• Step 2: We shall derive estimates in $L^2(0,T;\mathbb{H}^2)$. To do this, we firstly point out that the estimate of $\partial_t u$ implies that

$$\|\Delta u\|_{L^2(0,T;L^2(\Omega))} \le C\|(f,f_\Gamma)\|_{L^2(0,T;\mathbb{L}^2)} + C\|(u_0,u_{\Gamma,0})\|_{\mathbb{H}^1}.$$

By elliptic regularity applied to the first equation of (3), we have

$$\|u(t)\|_{H^2(\Omega)} \le C\|f(t)\|_{L^2(\Omega)} + C\|\partial_t u(t)\|_{L^2(\Omega)} + C\|u_\Gamma(t)\|_{H^{3/2}(\Gamma)},$$

a.e. in $(0,T)$. Integrating on $t \in [0,T]$, we deduce that

$$\|u\|_{L^2(0,T;H^2(\Omega))} \leq C_* \|(f, f_\Gamma)\|_{L^2(0,T;\mathbb{L}^2)} + C_* \|(u_0, u_{\Gamma,0})\|_{\mathbb{H}^1} + C_* \|u_\Gamma\|_{L^2(0,T;H^{3/2}(\Gamma))}, \tag{10}$$

for some constant $C_* > 0$. Now, from the second equation of (3), we deduce that

$$\begin{aligned}\|u_\Gamma\|_{L^2(0,T;H^2(\Gamma))} \leq & C\|f_\Gamma\|_{L^2(0,T;L^2(\Gamma))} + C\|\partial_\nu u\|_{L^2(0,T;L^2(\Gamma))} + C\|\partial_t u_\Gamma\|_{L^2(0,T;L^2(\Gamma))} \\ \leq & C\|(f, f_\Gamma)\|_{L^2(0,T;\mathbb{L}^2)} + C\|(u_0, u_{\Gamma,0})\|_{\mathbb{H}^1} + C\|\partial_\nu u\|_{L^2(0,T;L^2(\Gamma))}.\end{aligned} \tag{11}$$

Moreover, by interpolation inequalities, we have that for every $0 < s < 1/2$ and $\varepsilon > 0$, there are positive constants C_s and C_ε such that

$$\|\partial_\nu u\|_{L^2(0,T;L^2(\Gamma))} \leq C_s \|u\|_{L^2(0,T;H^{3/2+s}(\Omega))} \leq C_\varepsilon \|u\|_{L^2(0,T;H^1(\Omega))} + \varepsilon \|u\|_{L^2(0,T;H^2(\Omega))}. \tag{12}$$

Combining (10), (11) and (12) together with estimate (7), we get

$$\|u\|_{L^2(0,T;H^2(\Omega))} \leq C\|(f, f_\Gamma)\|_{L^2(0,T;L^2(\Omega))} + C\|(u_0, u_{\Gamma,0})\|_{\mathbb{H}^1}, \tag{13}$$

where we have chosen $\varepsilon > 0$ small enough. With the estimate (13) at hand, by (12) we can assert that

$$\|\partial_\nu u\|_{L^2(0,T;L^2(\Gamma))} \leq C\|(f, f_\Gamma)\|_{L^2(0,T;\mathbb{L}^2)} + C\|(u_0, u_{\Gamma,0})\|_{\mathbb{H}^1}. \tag{14}$$

Thus, substituting (14) into (11), we conclude that

$$\|u_\Gamma\|_{L^2(0,T;H^2(\Gamma))} \leq C\|(f, f_\Gamma)\|_{L^2(0,T;\mathbb{L}^2)} + C\|(u_0, u_{\Gamma,0})\|_{\mathbb{H}^1}. \tag{15}$$

Finally, combining (10), (15) and (9), we deduce (8). $\square$

Now, we study local solutions of the nonlinear problem

$$\begin{cases} Lu + c(1+\gamma i)|u|^2 u = f & \text{in } \Omega \times (0,T), \\ L_\Gamma(u, u_\Gamma) + c(1+\gamma i)|u_\Gamma|^2 u_\Gamma = f_\Gamma & \text{in } \Gamma \times (0,T), \\ u = u_\Gamma & \text{on } \Gamma \times (0,T), \\ (u(0), u_\Gamma(0)) = (u_0, u_{\Gamma,0}) & \text{in } \Omega \times \Gamma, \end{cases} \tag{16}$$

where L and L_Γ are given by (4), with $a, b, c > 0$ and $\alpha, \gamma \neq 0$.

Proposition 2.4. *Let $d = 2$ or $d = 3$. There exist $\varepsilon > 0$ and $C > 0$ such that, for every $(f, f_\Gamma) \in L^2(0,T;\mathbb{L}^2)$, $(u_0, u_{\Gamma,0}) \in \mathbb{H}^1$ such that*

$$\|(f, f_\Gamma)\|_{L^2(0,T;\mathbb{L}^2)} + \|(u_0, u_{\Gamma,0})\|_{\mathbb{H}^1} \leq \varepsilon, \tag{17}$$

there exists a unique solution (u, u_Γ) *of* (16) *which satisfies*

$$\|(u, u_\Gamma)\|_{C^0([0,T];\mathbb{H}^1)} + \|(u, u_\Gamma)\|_{L^2(0,T;\mathbb{H}^2)} \leq C \left(\|(f, f_\Gamma)\|_{L^2(0,T;\mathbb{L}^2)} + \|(u_0, u_{\Gamma,0})\|_{\mathbb{H}^1}\right).$$

Proof. We denote $\mathcal{B} = C^0([0,T];\mathbb{H}^1) \cap L^2(0,T;\mathbb{H}^2)$. Given $(f, f_\Gamma) \in L^2(0,T;\mathbb{L}^2)$ and $(u_0, u_{\Gamma,0}) \in \mathbb{H}^1$, for each $(z, z_\Gamma) \in \mathcal{B}$, we consider the system

$$\begin{cases} Lu = f - c(1+\gamma i)|z|^2 z & \text{in } \Omega \times (0,T), \\ L_\Gamma(u, u_\Gamma) = f_\Gamma - c(1+\gamma i)|z_\Gamma|^2 z_\Gamma & \text{on } \Gamma \times (0,T), \\ u = u_\Gamma & \text{on } \Gamma \times (0,T), \\ (u(0), u_\Gamma(0)) = (u_0, u_{\Gamma,0}) & \text{in } \Omega \times \Gamma. \end{cases} \tag{18}$$

Notice that since $d \leq 3$, we have $\mathcal{B} \hookrightarrow L^\infty(0,T;\mathbb{L}^6) \cap L^2(0,T;\mathbb{L}^\infty)$. Furthermore, we can see that this space can be continuously embedded in $L^6(0,T;\mathbb{L}^9)$. Indeed, let $v \in L^\infty(0,T;\mathbb{L}^6) \cap L^2(0,T;\mathbb{L}^\infty)$. By the interpolation of $L^9(\Omega)$ between $L^6(\Omega)$ and $L^\infty(\Omega)$ (see for instance [26]), we have

$$\begin{aligned} \|v\|^6_{L^6(0,T;L^9(\Omega))} &= \int_0^T \|v(t)\|^6_{L^9(\Omega)}\, dt \\ &\leq \int_0^T \|v(\cdot,t)\|^4_{L^6(\Omega)} \|v(\cdot,t)\|^2_{L^\infty(\Omega)}\, dt \\ &\leq \|v\|^4_{L^\infty(0,T;L^6(\Omega))} \|v\|^2_{L^2(0,T;L^\infty(\Omega))}. \end{aligned}$$

The same argument holds for Γ instead of Ω, so the embedding follows.

From Proposition 2.3 and the fact that $\mathcal{B} \hookrightarrow L^6(0,T;\mathbb{L}^6)$ continuously, we have that (18) has a unique solution $(u, u_\Gamma) \in \mathcal{B}$. Hence, we can define the map $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ given by $\mathcal{F}(z, z_\Gamma) = (u, u_\Gamma)$. Moreover, we also have that there exists $D > 0$ such that

$$\|\mathcal{F}(z, z_\Gamma)\|_{\mathcal{B}} \leq D \left(\|(f, f_\Gamma)\|_{L^2(0,T;\mathbb{L}^2)} + \|(u_0, u_{\Gamma,0})\|_{\mathbb{H}^1} + \|(z, z_\Gamma)\|^3_{\mathcal{B}}\right). \tag{19}$$

Clearly, $(u, u_\Gamma) \in \mathcal{B}$ is a solution of (16) if and only if it is a fixed point of the map $\mathcal{F}$. We will show that there exists $R > 0$ such that the restriction of $\mathcal{F}$ to the closed ball $B_R := \{(z, z_\Gamma) \in \mathcal{B} : \|(z, z_\Gamma)\|_{\mathcal{B}} \leq R\}$ is a contraction from B_R into B_R. Then, the proof will follow from a classic fixed-point result.

Indeed, from (19) and assumption (17), for each $(z, z_\Gamma) \in B_R$ we have

$$\|\mathcal{F}(z, z_\Gamma)\|_{\mathcal{B}} \leq D\left(\varepsilon + R^3\right). \tag{20}$$

Moreover, for each $(z, z_\Gamma), (w, w_\Gamma) \in B_R$, taking into account the equations satisfied by $\mathcal{F}(z, z_\Gamma) - \mathcal{F}(w, w_\Gamma)$, from Proposition 2.3 we have that

$$\begin{aligned} &\|\mathcal{F}(z, z_\Gamma) - \mathcal{F}(w, w_\Gamma)\|^2_{\mathcal{B}} \leq D_1 \||z|^2 z - |w|^2 w, |z_\Gamma|^2 z_\Gamma - |w_\Gamma|^2 w_\Gamma\|^2_{L^2(0,T;\mathbb{L}^2)} \\ \leq& D_1 \left\{ \|z - w\|^2_{L^\infty(0,T;L^4(\Omega))} \left(\|z^2 + zw\|^2_{L^2(0,T;L^4(\Omega))} + \|w\|^4_{L^4(0,T;L^8(\Omega))} \right) \right. \end{aligned}$$

$$+\|z_\Gamma - w_\Gamma\|^2_{L^\infty(0,T;L^4(\Gamma))}\left(\|z_\Gamma^2 + z_\Gamma w_\Gamma\|^2_{L^2(0,T;L^4(\Gamma))} + \|w_\Gamma\|^4_{L^4(0,T;L^8(\Gamma))}\right)\Big\},$$

and then, taking into account that $\mathcal{B} \hookrightarrow L^\infty(0,T;\mathbb{L}^4) \cap L^4(0,T;\mathbb{L}^8)$, we get that

$$\|\mathcal{F}(z,z_\Gamma) - \mathcal{F}(w,w_\Gamma)\|_{\mathcal{B}} \leq D_2 R^2 \|(z,z_\Gamma) - (w,w_\Gamma)\|_{\mathcal{B}}. \tag{21}$$

Therefore, in order to conclude, we choose $R > 0$ such that $R^2 < \min\left\{\dfrac{1}{2D}, \dfrac{1}{D_2}\right\}$ and $\varepsilon > 0$ such that $\varepsilon \leq \frac{R}{2D}$. From (20), (21) and the Banach Fixed point Theorem, we get the existence of a unique fixed point of $\mathcal{F}$. □

3 A new Carleman estimate for the linear Ginzburg-Landau equation with dynamic boundary conditions

In this section, we prove a Carleman estimate for the linear Ginzburg-Landau operator with dynamic boundary conditions. We start by introducing the weight functions that we shall use. For this purpose, we recall the following

Lemma 3.1 (see [27]). *Given a nonempty open set $\omega' \Subset \Omega$, there exists a function $\eta^0 \in C^2(\overline{\Omega})$ such that*

$$\eta^0 > 0 \text{ in } \Omega, \quad \eta^0 = 0 \text{ on } \Gamma, \quad |\nabla \eta^0| > 0 \text{ in } \overline{\Omega \setminus \omega'}. \tag{22}$$

Given $\omega' \Subset \Omega$, we take η^0 with respect to ω' as in Lemma 3.1. For $\lambda, m > 1$, we define

$$\varphi(x,t) := (t(T-t))^{-1}\left(e^{2\lambda m\|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}\right) \quad \forall (x,t) \in \overline{\Omega} \times (0,T), \tag{23}$$

$$\xi(x,t) := (t(T-t))^{-1} e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))} \quad \forall (x,t) \in \overline{\Omega} \times (0,T). \tag{24}$$

Theorem 3.2. *Let $\omega \Subset \Omega$. Set $\omega' \Subset \omega$ and η^0 satisfying (22). Define φ and ξ as in (23) and (24), respectively. Then, there exist constants $C, \lambda_0, s_0 > 0$ such that for all*

$\lambda \geq \lambda_0$ and $s \geq s_0$, we have

$$\begin{aligned}
&\int_0^T \int_\Omega e^{-2s\varphi} \left(s^3\lambda^4\xi^3|v|^2 + s\lambda^2\xi|\nabla v|^2 + s^{-1}\xi^{-1}|\partial_t v|^2 + s^{-1}\xi^{-1}|\Delta v|^2\right) dxdt \\
&+ \int_0^T \int_\Gamma e^{-2s\varphi}(s^3\lambda^3\xi^3|v_\Gamma|^2 + s\lambda\xi|\nabla_\Gamma v_\Gamma|^2 + s\lambda\xi|\partial_\nu v|^2)dSdt \\
&+ \int_0^T \int_\Gamma (s^{-1}\xi^{-1}|\partial_t v_\Gamma|^2 + s^{-1}\xi^{-1}|\Delta_\Gamma v_\Gamma|^2)dSdt \\
\leq & Cs^3\lambda^4 \int_0^T \int_\omega e^{-2s\varphi}\xi^3|v|^2 dxdt + C\int_0^T \int_\Omega e^{-2s\varphi}|L^*(v)|^2 dxdt \\
&+ C\int_0^T \int_\Gamma e^{-2s\varphi}|L^*_\Gamma(v, v_\Gamma)|^2 dSdt,
\end{aligned} \tag{25}$$

for all $(v, v_\Gamma) \in H^1(0,T;\mathbb{L}^2) \cap L^2(0,T;\mathbb{H}^2)$, where

$$\begin{aligned}
&L^*(v) = \partial_t v + a(1-\alpha i)\Delta v, \\
&L^*_\Gamma(v, v_\Gamma) := \partial_t v_\Gamma - a(1-\alpha i)\partial_\nu v + b(1-\alpha i)\Delta_\Gamma v_\Gamma.
\end{aligned} \tag{26}$$

Proof. For convenience, the proof has been divided into several steps:
• **Step 1.** In this step, we perform the first estimates of the conjugated variables. For simplicity, we consider $v \in C^\infty(\overline{\Omega} \times [0,T])$, $v_\Gamma = v$. This is due to the density of this space in $H^1(0,T;\mathbb{L}^2) \cap L^2(0,T;\mathbb{H}^2)$. Now, for $\lambda \geq \lambda_1 \geq 1$, and $s \geq s_0 \geq 1$, define

$$\begin{aligned}
w :=& e^{-s\varphi}v \quad \text{in } \overline{\Omega} \times (0,T), \\
f :=& e^{-s\varphi}(\partial_t v + a(1-\alpha i)\Delta v) \quad \text{in } \Omega \times (0,T), \\
f_\Gamma :=& e^{-s\varphi}(\partial_t v - a(1-\alpha i)\partial_\nu v + b(1-\alpha i)\Delta_\Gamma v) \quad \text{on } \Gamma \times (0,T).
\end{aligned}$$

Straightforward computations show that

$$\nabla\varphi = -\nabla\xi = -\lambda\xi\nabla\eta^0, \quad \Delta\varphi = -\lambda^2\xi|\nabla\eta^0|^2 - \lambda\xi\Delta\eta^0, \quad \partial_\nu\varphi = -\lambda\xi\partial_\nu\eta^0,$$

on $\Gamma \times (0,T)$ with $\partial_\nu\eta^0 \leq c < 0$ on Γ, for some constant $c > 0$ and

$$\nabla_\Gamma\varphi = \nabla_\Gamma\xi = 0, \quad \Delta_\Gamma\varphi = \Delta_\Gamma\xi = 0 \quad \text{on } \Gamma \times (0,T).$$

Then, in $\Omega \times (0,T)$ we have the following identity:

$$\begin{aligned}
f =& \partial_t w + a(1-\alpha i)\Delta w + as^2\lambda^2(1-\alpha i)|\nabla\eta^0|^2\xi^2 w - as\lambda^2(1-\alpha i)|\nabla\eta^0|^2\xi w \\
&- as\lambda(1-\alpha i)\Delta\eta^0\xi w - 2as\lambda(1-\alpha i)\xi\nabla\eta^0 \cdot \nabla w + s\partial_t\varphi w.
\end{aligned} \tag{27}$$

On the other hand, on $\Gamma \times (0,T)$ we have

$$
\begin{aligned}
f_\Gamma =& \partial_t w - a(1-\alpha i)\partial_\nu w + as\lambda(1-\alpha i)\partial_\nu \eta^0 \xi w + b(1-\alpha i)\Delta_\Gamma w \\
&+ s\partial_t \varphi w.
\end{aligned} \tag{28}
$$

Now, we define

$$
\begin{aligned}
P_1 w :=& a(s^2\lambda^2|\nabla\eta^0|^2\xi^2 w + \Delta w) + a\alpha i(2s\lambda\xi\nabla\eta^0 \cdot \nabla w + (s\lambda^2|\nabla\eta^0|^2 + s\lambda\Delta\eta^0)\xi w) \\
&+ s\partial_t\varphi w \\
P_2 w :=& -a(2s\lambda\xi\nabla\eta^0 \cdot \nabla w + (s\lambda^2|\nabla\eta^0|^2 + s\lambda\Delta\eta^0)\xi w) - a\alpha i(s^2\lambda^2|\nabla\eta^0|^2\xi^2 w + \Delta w) \\
&+ \partial_t w \\
Rw :=& f.
\end{aligned} \tag{29}
$$

and

$$
\begin{aligned}
P_{\Gamma,1} w :=& b\Delta_\Gamma w - \frac{2a^2}{b}\alpha i s\lambda\partial_\nu\eta^0\xi w + s\partial_t\varphi w, \\
P_{\Gamma,2} w :=& -\alpha b i\Delta_\Gamma w + \frac{2a^2}{b}s\lambda\partial_\nu\eta^0\xi w + \partial_t w, \\
R_\Gamma w :=& f_\Gamma - a(1-\alpha i)s\lambda\partial_\nu\eta^0\xi w + a(1-\alpha i)\partial_\nu w + \frac{2a^2}{b}(1-\alpha i)s\lambda\partial_\nu\eta^0\xi w.
\end{aligned} \tag{30}
$$

We emphasize that we have incorporated the term $\frac{2a^2}{b}\alpha i s\lambda\partial_\nu\eta^0\xi w$ in $P_{\Gamma,1}w$ and $P_{\Gamma,2}w$. This plays a crucial role since this term is used to obtain a global term of $\nabla_\Gamma w$ on the right-hand side of the Carleman estimate (see equation (37)).

Then, (27) and (28) can be written as

$$
P_1 w + P_2 w = Rw \quad \text{in } \Omega \times (0,T), \tag{31}
$$

$$
P_{\Gamma,1} w + P_{\Gamma,2} w = R_\Gamma w \quad \text{on } \Gamma \times (0,T). \tag{32}
$$

Then, taking the $L^2(\Omega \times (0,T))$-norm in (31) and the $L^2(\Gamma \times (0,T))$-norm in (32), we obtain

$$
\begin{aligned}
&\|P_1 w\|^2_{L^2(\Omega\times(0,T))} + \|P_2 w\|^2_{L^2(\Omega\times(0,T))} + \|P_{\Gamma,1} w\|^2_{L^2(\Gamma\times(0,T))} + \|P_{\Gamma,2} w\|^2_{L^2(\Gamma\times(0,T))} \\
&+ 2\Re\langle P_1 w, P_2 w\rangle_{L^2(\Omega\times(0,T))} + 2\Re\langle P_{\Gamma,1} w, P_{\Gamma,2} w\rangle_{L^2(\Gamma\times(0,T))} \\
=& \|Rw\|^2_{L^2(\Omega\times(0,T))} + \|R_\Gamma w\|^2_{L^2(\Gamma\times(0,T))}.
\end{aligned}
$$

• **Step 2.** In this step, we devote to compute the terms

$$
\langle P_1 w, P_2 w\rangle_{L^2(\Omega\times(0,T))} = \sum_{j,k=1}^{3} I_{jk},
$$

where we have used the notation I_{jk} to denote the real L^2 inner product between the j^{th}-term of $P_1 w$ and the k^{th}-term of $P_2 w$, defined in (29).

Firstly, the term I_{11} can be written as

$$I_{11} := -a^2 \Re \int_0^T \int_\Omega z_1 z_2 dxdt = I_{11}^{(1)} + I_{11}^{(2)} + I_{11}^{(3)} + I_{11}^{(4)}.$$

where z_1 and z_2 are given by

$$z_1 = (s^2\lambda^2 |\nabla\eta^0|^2 \xi^2 w + \Delta w), \quad z_2 = (2s\lambda\xi\nabla\eta^0 \cdot \nabla\overline{w} + (s\lambda^2|\nabla\eta^0|^2 + s\lambda\Delta\eta^0)\xi\overline{w})$$

Integration by parts and some algebraic manipulations allow us to obtain

$$\begin{aligned} I_{11}^{(1)} :=& -2a^2 s^3\lambda^3 \Re \int_0^T \int_\Omega |\nabla\eta^0|^2 \xi^3 w \nabla\eta^0 \cdot \nabla\overline{w} dxdt \\ =& a^2 s^3 \lambda^3 \int_0^T \int_\Omega \left(\nabla(|\nabla\eta^0|^2)\cdot\nabla\eta^0 + |\nabla\eta^0|^2\Delta\eta^0\right)\xi^3|w|^2 dxdt \\ &+ 3a^2 s^3\lambda^4 \int_0^T \int_\Omega |\nabla\eta^0|^4 \xi^3 |w|^2 dxdt - a^2 s^3\lambda^3 \int_0^T \int_\Gamma (\partial_\nu \eta^0)^3 \xi^3 |w|^2 dSdt. \end{aligned}$$

On the other hand, we notice that

$$I_{11}^{(2)} := -a^2 \int_0^T \int_\Omega (s^3\lambda^4|\nabla\eta^0|^4 + s^3\lambda^3|\nabla\eta^0|^2\Delta\eta^0)\xi^3|w|^2 dxdt$$

Integrating by parts in space, we have

$$\begin{aligned} I_{11}^{(3)} :=& -a^2\Re\int_0^T\int_\Omega \Delta w(2s\lambda\xi\nabla\eta^0\cdot\nabla\overline{w})dxdt \\ =& 2a^2 s\lambda^2 \int_0^T \int_\Omega \xi|\nabla\eta^0\cdot\nabla w|^2 dxdt + 2a^2 s\lambda\Re\int_0^T\int_\Omega \xi\nabla w\cdot\nabla(\nabla\eta^0\cdot\nabla\overline{w})dxdt \\ &- 2a^2 s\lambda\int_0^T\int_\Gamma \xi\partial_\nu\eta^0|\partial_\nu w|^2 dSdt. \end{aligned}$$

Using the identity

$$\nabla w\cdot\nabla(\nabla\eta^0\cdot\nabla\overline{w}) = \nabla^2\eta^0(\nabla w,\nabla\overline{w}) + \frac{1}{2}\nabla\eta^0\cdot\nabla(|\nabla w|^2) \quad \text{in } \Omega\times(0,T),$$

we notice that

$$\begin{aligned} &2a^2 s\lambda\Re\int_0^T\int_\Omega \xi\nabla w\cdot\nabla(\nabla\eta^0\cdot\nabla\overline{w})dxdt \\ =& 2a^2 s\lambda\int_0^T\int_\Omega \xi\nabla^2\eta^0(\nabla w,\nabla\overline{w})dxdt - a^2 s\lambda\int_0^T\int_\Omega \Delta\eta^0\xi|\nabla w|^2 dxdt \end{aligned}$$

$$- a^2 s\lambda^2 \int_0^T \int_\Omega \xi |\nabla \eta^0|^2 |\nabla w|^2 dxdt + a^2 s\lambda \int_0^T \int_\Gamma \partial_\nu \eta^0 \xi (|\nabla_\Gamma w|^2 + |\partial_\nu w|^2) dSdt.$$

Then, $I_{11}^{(3)}$ is given by

$$\begin{aligned} I_{11}^{(3)} =& 2a^2 s\lambda^2 \int_0^T \int_\Omega \xi |\nabla \eta^0 \cdot \nabla w|^2 dxdt + 2a^2 s\lambda \int_0^T \int_\Omega \xi \nabla^2 \eta^0 (\nabla w, \nabla \overline{w}) dxdt \\ & - a^2 s\lambda \int_0^T \int_\Omega \Delta \eta^0 \xi |\nabla w|^2 dxdt - a^2 s\lambda^2 \int_0^T \int_\Omega \xi |\nabla \eta^0|^2 |\nabla w|^2 dxdt \\ & - a^2 s\lambda \int_0^T \int_\Gamma \partial_\nu \eta^0 \xi \left(|\partial_\nu w|^2 - |\nabla_\Gamma w|^2 \right) dSdt. \end{aligned}$$

After integration by parts, $I_{11}^{(4)}$ can be written as follows:

$$\begin{aligned} I_{11}^{(4)} :=& a^2 \Re \int_0^T \int_\Omega \xi \overline{w} \nabla w \cdot \nabla (s\lambda^2 |\nabla \eta^0|^2 + s\lambda \Delta \eta^0) dxdt \\ & + a^2 \Re \iint_0^T \int_\Omega \lambda \xi \left(s\lambda^3 |\nabla \eta^0|^2 + s\lambda^2 \Delta \eta^0 \right) \overline{w} \nabla \eta^0 \cdot \nabla w dxdt \\ & + a^2 \iint_0^T \int_\Omega (s\lambda^2 |\nabla \eta^0|^2 + s\lambda \Delta \eta^0) \xi |\nabla w|^2 dxdt \\ & - a^2 \Re \iint_0^T \int_\Gamma (s\lambda^2 |\partial_\nu \eta^0|^2 + s\lambda \Delta \eta^0) \xi (\partial_\nu w) \overline{w} dSdt. \end{aligned}$$

Then, by (22), I_{11} can be estimated as

$$\begin{aligned} I_{11} \geq& C \int_0^T \int_\Omega (s^3 \lambda^4 \xi^3 |w|^2 + s\lambda^2 \xi |\nabla \eta^0 \cdot \nabla w|^2) dxdt \\ & + Cs\lambda \int_0^T \int_\Gamma \xi |\partial_\nu w|^2 dSdt + a^2 s\lambda \int_0^T \int_\Gamma \partial_\nu \eta^0 \xi |\nabla_\Gamma w|^2 dSdt \\ & - Cs^3 \lambda^4 \int_0^T \int_\omega \xi^3 |w|^2 dxdt - X_{11}, \end{aligned}$$

where X_{11} satisfies the following upper bound:

$$\begin{aligned} X_{11} \leq& C \int_0^T \int_\Omega (s^3 \lambda^3 \xi^3 |w|^2 + s\lambda \xi |\nabla w|^2) dxdt \\ & + C \int_0^T \int_\Gamma \left(s^2 \lambda^3 \xi^3 |w|^2 + \lambda \xi |\partial_\nu w|^2 \right) dSdt. \end{aligned}$$

It is clear that

$$I_{12} = 0.$$

Moreover, we write I_{13} as

$$\begin{aligned} I_{13} :=& as^2\lambda^2\Re\int_0^T\int_\Omega |\nabla\eta^0|^2\xi^2 w\partial_t\overline{w}dxdt + a\Re\int_0^T\int_\Omega \Delta w\partial_t\overline{w}dxdt \\ =& I_{13}^{(1)} + I_{13}^{(2)}. \end{aligned}$$

Integrating by parts on time and using the fact that $w=0$ in $t=0$ and $t=T$, we have

$$I_{13}^{(1)} = -as^2\lambda^2\int_0^T\int_\Omega |\nabla\eta^0|^2\xi\partial_t\xi|w|^2dxdt.$$

In the same manner, $I_{13}^{(2)}$ gives

$$I_{13}^{(2)} = a\Re\int_0^T\int_\Gamma \partial_\nu w\partial_t\overline{w}dSdt.$$

Then, I_{13} is given by

$$I_{13} \geq -Cs^2\lambda^3\int_0^T\int_\Omega \xi^3|w|^2dxdt - C\int_0^T\int_\Gamma (s\lambda^{1/2}\xi|\partial_\nu w|^2 + s^{-1}\lambda^{-1/2}\xi^{-1}|\partial_t w|^2)dSdt.$$

Moreover, from the definition of I_{21}, we see that

$$I_{21} = 0$$

From the definitions in (29), we can notice that $I_{22} = \alpha^2 I_{11}$, and then

$$\begin{aligned} I_{22} \geq& C\int_0^T\int_\Omega \left(s^3\lambda^4\xi^3|w|^2 + s\lambda^2\xi|\nabla\eta^0\cdot\nabla w|^2\right)dxdt \\ &+ C\int_0^T\int_\Gamma (s^3\lambda^3\xi^3|w|^2 + s\lambda\xi|\partial_\nu w|^2)dSdt + a^2\alpha^2 s\lambda\int_0^T\int_\Gamma \partial_\nu\eta^0\xi|\nabla_\Gamma w|^2dSdt \\ &- Cs^3\lambda^4\int_0^T\int_\omega \xi^3|w|^2dSdt - X_{22}, \end{aligned}$$

where X_{22} satisfies

$$\begin{aligned} X_{22} \leq& C\int_0^T\int_\Omega (s^3\lambda^3\xi^3|w|^2 + Cs\lambda\xi|\nabla w|^2)dxdt \\ &+ C\int_0^T\int_\Gamma \left(s^2\lambda^3\xi^2|w|^2 + \lambda\xi|\partial_\nu w|^2\right)dSdt. \end{aligned}$$

To compute the term I_{23}, we follow [11]. Then, we write

$$\begin{aligned}
I_{23} =& \frac{1}{2} a\alpha i \int_0^T \int_\Omega (2s\lambda\xi\nabla\eta^0 \cdot \nabla w + (s\lambda^2|\nabla\eta^0|^2 + s\lambda\Delta\eta^0)\xi w)\partial_t \overline{w} dxdt \\
&- \frac{1}{2} a\alpha i \int_0^T \int_\Omega (2s\lambda\xi\nabla\eta^0 \cdot \nabla \overline{w} + (s\lambda^2|\nabla\eta^0|^2 + s\lambda\Delta\eta^0)\xi \overline{w})\partial_t w dxdt \\
=& I_{23}^{(1)} + I_{23}^{(2)}.
\end{aligned} \tag{33}$$

On one hand, integrating by parts in time we have

$$\begin{aligned}
& I_{23}^{(1)} \\
:=& - a\alpha i \int_0^T \int_\Omega (s\lambda\partial_t\xi\nabla\eta^0 \cdot \nabla w + s\lambda\xi\nabla\eta^0 \cdot \nabla\partial_t w)\overline{w} dxdt \\
&- \frac{1}{2} a\alpha i \int_0^T \int_\Omega \left((s\lambda^2|\nabla\eta^0|^2 + s\lambda\Delta\eta^0)\partial_t\xi|w|^2 + (s\lambda^2|\nabla\eta^0|^2 + s\lambda\Delta\eta^0)\xi\overline{w}\partial_t w\right) dxdt.
\end{aligned} \tag{34}$$

On the other hand, integration by parts in space yields

$$\begin{aligned}
I_{23}^{(2)} =& a\alpha i \int_0^T \int_\Omega \left(s\lambda^2|\nabla\eta^0|^2\xi\overline{w}\partial_t w + s\lambda\xi\overline{w}\nabla\eta^0 \cdot \nabla\partial_t w + s\lambda\Delta\eta^0\xi\overline{w}\partial_t w\right) dxdt \\
&- \frac{1}{2} a\alpha i \int_0^T \int_\Omega (s\lambda^2|\nabla\eta^0|^2 + s\lambda\Delta\eta^0)\xi\overline{w}\partial_t w dxdt \\
&- a\alpha s\lambda i \int_0^T \int_\Gamma \partial_\nu\eta^0\xi\overline{w}\partial_t w dSdt.
\end{aligned} \tag{35}$$

Then, substituting (34) and (35) into (33) and applying the Young's inequality, we show that for all $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$\begin{aligned}
I_{23} \geq& - C \int_0^T \int_\Omega \left(s^2\lambda^2\xi^3|w|^2 + \lambda\xi|\nabla w|^2\right) dxdt \\
&- \int_0^T \int_\Gamma (\epsilon s^3\lambda^3\xi^3|w|^2 + C(\epsilon)s^{-1}\lambda^{-1}\xi^{-1}|\partial_t w|^2) dSdt.
\end{aligned}$$

The term I_{31} is

$$\begin{aligned}
I_{31} =& - 2as^2\lambda\Re \int_0^T \int_\Omega \xi\partial_t\varphi w\nabla\eta^0 \cdot \nabla\overline{w} dxdt \\
&- a \int_0^T \int_\Omega (s^2\lambda^2|\nabla\eta^0|^2 + s^2\lambda\Delta\eta^0)\partial_t\varphi\xi|w|^2 dxdt
\end{aligned}$$

where the first term can be computed as

$$
\begin{aligned}
&-2as^2\lambda\Re\int_0^T\int_\Omega \xi\partial_t\varphi w\nabla\eta^0\cdot\nabla\overline{w}dxdt\\
=&as^2\lambda\int_0^T\int_\Omega (\lambda|\nabla\eta^0|^2\xi\partial_t\varphi+\xi\nabla\eta^0\cdot\nabla(\partial_t\varphi)+\Delta\eta^0\xi\partial_t\varphi)|w|^2dxdt\\
&-2as^2\lambda\int_0^T\int_\Gamma \partial_\nu\eta^0\xi\partial_t\varphi|w|^2dSdt
\end{aligned}
$$

Then, I_{31} is estimated as

$$
I_{31}\geq -Cs^2\lambda^2\int_0^T\int_\Omega \xi^3|w|^2dxdt-Cs^2\lambda^2\int_0^T\int_\Gamma \xi^3|w|^2dSdt.
$$

Now, for I_{32}, it is clear that

$$
\begin{aligned}
I_{32}=&-a\alpha s\Im\int_0^T\int_\Omega w\nabla(\partial_t\varphi)\cdot\nabla\overline{w}dxdt+a\alpha s\Im\int_0^T\int_\Gamma \partial_t\varphi w\partial_\nu\overline{w}dSdt\\
\geq&-C\int_0^T\int_\Omega \left(s^2\lambda^2\xi^3|w|^2+\xi|\nabla w|^2\right)dxdt\\
&-C\int_0^T\int_\Gamma \left(s^2\xi^3|w|^2+\xi|\partial_\nu w|^2\right)dSdt.
\end{aligned}
$$

In addition, since $w=0$ in $t=0$ and $t=T$, we have

$$
I_{33}\geq -Cs\int_0^T\int_\Omega \xi^3|w|^2dxdt.
$$

According to the above estimates, taking s_1 and $\lambda_1>0$ large enough if it is neccesary and choosing $\epsilon>0$ small enough, we deduce that

$$
\begin{aligned}
&\langle P_1w,P_2w\rangle_{L^2(\Omega\times(0,T))}\\
\geq&Cs^3\lambda^4\int_0^T\int_\Omega \xi^3|w|^2dxdt+C\int_0^T\int_\Gamma (s^3\lambda^3\xi^3|w|^2+s\lambda\xi|\partial_\nu w|^2)dSdt\\
&+a^2(1+\alpha^2)s\lambda\int_0^T\int_\Gamma \partial_\nu\eta^0\xi|\nabla_\Gamma w|^2dSdt-Cs^3\lambda^4\int_0^T\int_\omega \xi^3|w|^2dxdt-X,
\end{aligned}
\tag{36}
$$

for $s\geq s_1>0$ and $\lambda\geq\lambda_1>0$, where X satisfies

$$
X\leq Cs\lambda\int_0^T\int_\Omega \xi|\nabla w|^2dxdt+Cs^{-1}\lambda^{-1}\int_0^T\int_\Gamma \xi^{-1}|\partial_t w|^2dSdt.
$$

Before going any further, let us point out that the integral term $|\nabla_\Gamma w|^2$ is negative. However, in the next step we obtain additional terms to solve this problem.

• **Step 3.** In this step, we compute the terms

$$\int_0^T \int_\Gamma P_{\Gamma,1} w \overline{P_{\Gamma,2} w} dSdt = \sum_{j=1}^{3}\sum_{k=1}^{3} J_{jk},$$

where J_{jk} denotes the real L^2 inner product of the j$^{\text{th}}$-term of $P_{1,\Gamma}w$ with the kth-term of $P_{2,\Gamma}w$ defined in (30). Firstly, we have

$$J_{11} = b^2\alpha\Im \int_0^T \int_\Gamma |\Delta_\Gamma w|^2 dSdt = 0.$$

Secondly, by the surface divergence theorem, we get

$$\begin{aligned} J_{12} =& 2a^2 s\lambda \Re \int_0^T \int_\Gamma (\xi \overline{w} \nabla_\Gamma(\partial_\nu \eta^0)\cdot \nabla_\Gamma w) dSdt - 2a^2 s\lambda \int_0^T \int_\Gamma \partial_\nu \eta^0 \xi |\nabla_\Gamma w|^2 dSdt \\ \geq& -2a^2 s\lambda \int_0^T \int_\Gamma \partial_\nu \eta^0 \xi |\nabla_\Gamma w|^2 dSdt - C\int_0^T \int_\Gamma (s^2\lambda^2 \xi |w|^2 + \xi |\nabla_\Gamma w|^2) dSdt. \end{aligned}$$

Moreover, integrating by parts in time and space, we can assert that

$$J_{13} = 0.$$

The term J_{21} is

$$\begin{aligned} J_{21} =& -2a^2\alpha^2 s\lambda \Re \int_0^T \int_\Gamma \xi w \nabla_\Gamma(\partial_\nu \eta^0)\cdot \nabla_\Gamma \overline{w} dSdt - 2a^2\alpha s\lambda \int_0^T \int_\Gamma \partial_\nu \eta^0 \xi |\nabla_\Gamma w|^2 dSdt \\ \geq& -2a^2\alpha^2 s\lambda \int_0^T \int_\Gamma \partial_\nu \eta^0 \xi |\nabla_\Gamma w|^2 dSdt - C\int_0^T \int_\Gamma (s^2\lambda^2 \xi |w|^2 + \xi |\nabla_\Gamma w|^2) dSdt. \end{aligned}$$

Furthermore, by definition, J_{22} is

$$J_{22} = 0$$

For J_{23}, we use Young's inequality to show that for all $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$J_{23} \geq -\int_0^T \int_\Gamma \left(\epsilon s^3\lambda^3\xi^3 |w|^2 + C(\epsilon) s^{-1}\lambda^{-1}\xi^{-1} |\partial_t w|^2\right) dSdt.$$

By the surface divergence theorem and the fact that $\nabla_\Gamma \varphi = 0$ on $\Gamma \times (0,T)$, we deduce that

$$J_{31} = 0.$$

Moreover, by definition, J_{32} is given by

$$J_{32} \geq -Cs^2\lambda \int_0^T \int_\Gamma \xi^3 |w|^2 dSdt.$$

On the other hand, integration by parts yields

$$J_{33} \geq - Cs \int_0^T \int_\Gamma \xi^3 |w|^2 dSdt.$$

According to these estimates, we can assert that

$$\langle P_{\Gamma,1} w, P_{\Gamma,2} w\rangle_{L^2(\Gamma\times(0,T))} \geq -2a^2(1+\alpha^2)s\lambda \int_0^T \int_\Gamma \partial_\nu \eta^0 \xi |\nabla_\Gamma w|^2 dSdt - Y, \tag{37}$$

where Y satisfies

$$\begin{aligned} Y \leq &\int_0^T \int_\Gamma (\epsilon s^3\lambda^3\xi^3|w|^2 + C(\epsilon) s^{-1}\lambda^{-1}\xi^{-1}|\partial_t w|^2) dSdt \\ &+ C\int_0^T \int_\Gamma \xi |\nabla_\Gamma w|^2 dSdt. \end{aligned}$$

From (36) and (37), using that $\partial_\nu \eta^0 \leq -c < 0$ on $\Gamma \times (0,T)$, taking $s_1, \lambda_1 > 0$ large enough and choosing $\epsilon > 0$ small enough, we obtain

$$\begin{aligned} &\|P_1 w\|^2_{L^2(\Omega\times(0,T))} + \|P_2 w\|^2_{L^2(\Omega\times(0,T))} + \|P_{\Gamma,1} w\|^2_{L^2(\Gamma\times(0,T))} + \|P_{\Gamma,2} w\|^2_{L^2(\Gamma\times(0,T))} \\ &+ C\int_0^T \int_\Omega (s^3\lambda^4\xi^3|w|^2 + s\lambda^2\xi|\nabla\eta^0\cdot\nabla w|^2) dxdt \\ &+ C\int_0^T \int_\Gamma (s^3\lambda^3\xi^3|w|^2 + s\lambda\partial_\nu\eta^0\xi|\nabla_\Gamma w|^2) dSdt \\ \leq &\|f\|^2_{L^2(\Omega\times(0,T))} + \|f_\Gamma\|^2_{L^2(\Gamma\times(0,T))} + s^3\lambda^4 \int_0^T \int_\omega \xi^3 |w|^2 dxdt + Z, \end{aligned} \tag{38}$$

where Z is given by

$$\begin{aligned} Z \leq &C\int_0^T \int_\Gamma (s^{-1/2}\lambda^{-1/2} + s^{-1})\xi^{-1}|\partial_t w|^2 dSdt \\ &+ Cs\lambda \int_0^T \int_\Omega \xi |\nabla w|^2 dxdt. \end{aligned}$$

In order to absorb the terms of $|\nabla w|$ in $\Omega \times (0,T)$, Δw and $\partial_t w$ on $\Gamma \times (0,T)$, we shall use an indirect estimates using the definitions given in (31) and (32).

• **Step 4.** In this step, we obtain estimates for the L^2-norm of Δw, ∇w and $\partial_t w$ in $\Omega \times (0,T)$. More precisely, the purpose of this step is to prove the following inequality:

$$\begin{aligned}
&\int_0^T \int_\Omega \left(s^{-1}\xi^{-1}|\Delta w|^2 + s^{-1}\xi^{-1}|\partial_t w|^2 + s\lambda^2|\nabla w|^2\right) dxdt \\
\leq& C\int_0^T \int_\Omega \left(s^3\lambda^4\xi^3|w|^2 + s\lambda^2\xi|\nabla\eta^0 \cdot \nabla w|^2 + s^{-1}\xi^{-1}|P_1 w|^2 + s^{-1}\xi^{-1}|P_2 w|^2\right) dxdt \\
&+ C\int_0^T \int_\Gamma \left(s^3\lambda^3|w|^2 + s\lambda|\partial_\nu w|^2\right) dSdt.
\end{aligned} \tag{39}$$

To do that, we first estimate the term of Δw. From the definition of P_1 in (29), it is clear that

$$\begin{aligned}
&s^{-1}\int_0^T \int_\Omega \xi^{-1}|\Delta w|^2 dxdt \\
\leq& C\int_0^T \int_\Omega \left(s^{-1}\xi^{-1}|P_1 w|^2 + s^3\lambda^4\xi^3|w|^2 + s\lambda^2\xi|\nabla\eta^0 \cdot \nabla w|^2\right) dxdt.
\end{aligned} \tag{40}$$

Secondly, we estimate ∇w as follows:

$$\begin{aligned}
s\lambda^2\int_0^T \int_\Omega \xi|\nabla w|^2 dxdt =& - s\lambda^2\Re\int_0^T \int_\Omega w\nabla\xi \cdot \nabla\overline{w} dxdt + s\lambda^2\Re\int_0^T \int_\Gamma \xi w\partial_\nu\overline{w} dSdt \\
&- s\lambda^2\Re\int_0^T \int_\Omega \xi w\Delta\overline{w} dxdt
\end{aligned}$$

Then, by Young's inequality we obtain

$$\begin{aligned}
&s\lambda^2\int_0^T \int_\Omega \xi|\nabla w|^2 dxdt \\
\leq& C\int_0^T \int_\Omega \left(s^3\lambda^4\xi^3|w|^2 + s\lambda^2\xi|\nabla\eta^0 \cdot \nabla w|^2 + s^{-1}\xi^{-1}|\Delta w|^2\right) dxdt \\
&+ C\int_0^T \int_\Gamma (s\lambda^3\xi|w|^2 + s\lambda\xi|\partial_\nu w|^2) dSdt.
\end{aligned} \tag{41}$$

Next, we estimate the term of $\partial_t w$ directly from the definition of P_2:

$$\begin{aligned}
&s^{-1}\int_0^T \int_\Omega \xi^{-1}|\partial_t w|^2 dxdt \\
\leq& C\int_0^T \int_\Omega \left(s^3\lambda^4\xi^3|w|^2 + s\lambda^2\xi|\nabla\eta^0 \cdot \nabla w|^2 + s^{-1}\xi^{-1}|\Delta w|^2\right) dxdt.
\end{aligned} \tag{42}$$

Thus, combining (40), (41) and (42), we easily get (39).

- **Step 5** In this step, we shall prove that

$$\begin{aligned}
&s^{-1}\int_0^T\int_\Gamma\left(\xi^{-1}|\partial_t w|^2+\xi^{-1}|\Delta_\Gamma w|^2\right)dSdt\\
\leq &C\int_0^T\int_\Gamma\left(s^2\lambda^2|w|^2+\xi^{-1}|P_{\Gamma,1}w|^2+\xi^{-1}|P_{\Gamma,2}w|^2\right)dSdt.
\end{aligned}\tag{43}$$

From one hand, by definition of $P_{\Gamma,1}$ in (30), we have

$$s^{-1}\int_0^T\int_\Gamma\xi^{-1}|\Delta_\Gamma w|^2 dSdt\leq C\int_0^T\int_\Gamma\left(|P_{\Gamma,1}w|^2+s^2\lambda^2\xi^3|w|^2\right)dSdt.\tag{44}$$

On the other hand, using the definition of $P_{\Gamma,2}$ we deduce that

$$s^{-1}\int_0^T\int_\Gamma\xi^{-1}|\partial_t w|^2 dSdt\leq C\int_0^T\int_\Gamma\left(s^{-1}\xi^{-1}|\Delta_\Gamma w|^2+s\lambda\xi|w|^2+|P_{\Gamma,2}w|^2\right)dSdt.\tag{45}$$

Combining (44) and (45), we obtain (43).
Finally, combining (39), (43) and (38), and taking $s,\lambda>0$, we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega(s^3\lambda^4\xi^3|w|^2+s\lambda^2\xi|\nabla w|^2+s^{-1}\xi^{-1}|\partial_t w|^2+s^{-1}\xi^{-1}|\Delta w|^2)dxdt\\
&+\int_0^T\int_\Gamma\left(s^3\lambda^3\xi^3|w|^2+s\lambda\xi|\partial_\nu w|^2+s\lambda\xi|\nabla_\Gamma w|^2+s^{-1}\xi^{-1}|\Delta_\Gamma w|^2\right)dSdt\\
&+\int_0^T\int_\Gamma\left(s^{-1}\xi^{-1}|\partial_t w|^2\right)dSdt\\
\leq &C\|Rw\|^2_{L^2(\Omega\times(0,T))}+C\|R_\Gamma w\|^2_{L^2(\Gamma\times(0,T))}+Cs^3\lambda^4\int_0^T\int_\omega\xi^3|w|^2dxdt.
\end{aligned}$$

Finally, taking into account that $w=e^{-s\varphi}$, we come back to the original variable and conclude the inequality (25). ☐

The proof of Theorem 6.2 runs as in the proof above but in a simplified way. In fact, in this case we do not have to deal with tangential derivatives terms in the conjugation process, and therefore the computations are easy to follow.

4 Observability and null controllability of the linear system

The goal of this section is to prove a null controllability result for the linear Ginzburg-Landau

$$\begin{cases} L(y) = f + \mathbb{1}_\omega h & \text{in } \Omega \times (0,T), \\ L_\Gamma(y, y_\Gamma) = f_\Gamma & \text{in } \Gamma \times (0,T), \\ y = y_\Gamma & \text{on } \Gamma \times (0,T), \\ (y(0), y_\Gamma(0)) = (y_0, y_{\Gamma,0}) & \text{in } \Omega \times \Gamma. \end{cases} \tag{46}$$

where the operators L and L_Γ were defined in (4). Naturally, the null controllability for (46) can be expressed in the following terms:

Definition 1. *We say that* (46) *is null controllable in* $\mathbb{L}^2$ *if for all* $T > 0$, $(y_0, y_{\Gamma,0}) \in \mathbb{L}^2$, *there exists a control* $h \in L^2(\omega \times (0,T))$ *such that the associated solution* (y, y_Γ) *of* (46) *satisfies*

$$y(\cdot, T) = 0 \ \textit{in } \Omega, \quad y_\Gamma(\cdot, T) = 0 \ \textit{on } \Gamma.$$

Following the classical duality between controllability and observability in the context of parabolic equations (see, e.g., [27]), the null controllability of (46) is equivalent to prove a suitable observability inequality for its adjoint system. Then, thanks to this inequality and the Lax-Milgram lemma, we allow us to deduce the null controllability of (46), which is crucial for the proof of Theorem 1.1 and an interesting result by itself.

4.1 Observability inequality

As we explained above, we introduce the adjoint system

$$\begin{cases} L^* z = g & \text{in } \Omega \times (0,T), \\ L_\Gamma^*(z, z_\Gamma) z = g_\Gamma & \text{on } \Gamma \times (0,T), \\ z = z_\Gamma & \text{on } \Gamma \times (0,T), \\ (z(T), z_\Gamma(T)) = (z_T, z_{\Gamma,T}) & \text{in } \Omega \times \Gamma. \end{cases} \tag{47}$$

where L^* and L_Γ^* are given by (26). We point out that if $(z_T, z_{\Gamma,T}) \in \mathbb{L}^2$, then the associated weak solution (z, z_Γ) belongs to $C^0([0,T]; \mathbb{L}^2) \cap L^2(0,T; \mathbb{H}^1)$. This is done by results of Section 2.

To formulate the result of this subsection, we shall define some additional functions. For $t \in (0,T)$, we define

$$\mu(t) := \begin{cases} \dfrac{4}{T^2} & \text{if } t \in (0, T/2], \\ \dfrac{1}{t(T-t)} & \text{if } t \in (T/2, T), \end{cases}$$

$$\begin{aligned}
\check{\varphi}(t) :=& \mu(t) \min_{x\in\overline{\Omega}} \left(e^{2s\lambda m\|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))} \right) \quad t \in (0,T), \\
\hat{\varphi}(t) :=& \mu(t) \max_{x\in\overline{\Omega}} \left(e^{2s\lambda m\|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))} \right) \quad t \in (0,T), \\
\check{\xi}(t) :=& \mu(t) \min_{x\in\overline{\Omega}} e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))} \quad t \in (0,T), \\
\hat{\xi}(t) :=& \mu(t) \max_{x\in\overline{\Omega}} e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))} \quad t \in (0,T).
\end{aligned}$$

Thus, we have the following.

Proposition 4.1 (Observability inequality)**.** *Let $d \geq 2$. Suppose that $(z_T, z_{\Gamma,T}) \in \mathbb{L}^2$ and $(g, g_\Gamma) \in L^2(0,T;\mathbb{L}^2)$. Then, there exists a constant $C > 0$ such that the associated weak solution $(z, z_\Gamma) \in C^0([0,T];\mathbb{L}^2) \cap L^2(0,T;\mathbb{H}^1)$ of* (47) *satisfies*

$$\begin{aligned}
&\int_\Omega |z(0)|^2 dx + \int_\Gamma |z_\Gamma(0)|^2 dS + \int_0^T \int_\Omega e^{-2s\hat{\varphi}} (\check{\xi}^3 |z|^2 + \check{\xi} |\nabla z|^2) dx dt \\
&+ \int_0^T \int_\Gamma e^{-2s\hat{\varphi}} (\check{\xi}^3 |z_\Gamma|^2 + \check{\xi} |\nabla_\Gamma z_\Gamma|^2) dS dt \\
\leq& C \int_0^T \int_\Omega e^{-2s\check{\varphi}} |g|^2 dx dt + C \int_0^T \int_\Gamma e^{-2s\check{\varphi}} |g_\Gamma|^2 dS dt + C \int_0^T \int_\omega e^{-2s\check{\varphi}} \hat{\xi}^3 |z|^2 dx dt.
\end{aligned} \tag{48}$$

We skip the proof of Proposition 4.1 since it relies on well-known facts used to prove observability inequalities for parabolic equations (see, for instance, [28]).

4.2 Proof of the null controllability for the linear system

From the observability inequality (48), we can deduce the null controllability of the linear system (46). In the following, for $r \in [1, +\infty]$, a linear space $\mathcal{H}$ and a measurable function $\rho : (0,T) \longrightarrow \mathbb{R}$, we shall consider the notation

$$L^r(\rho(0,T);\mathcal{H}) = \{y \in L^r(0,T;\mathcal{H}); \rho y \in L^r(0,T;\mathcal{H})\}.$$

Consider the Banach space $\mathcal{V}$ defined by

$$\begin{aligned}
\mathcal{V} := \{(y, y_\Gamma, h) :& (y, y_\Gamma) \in L^2(e^{s\check{\varphi}}(0,T);\mathbb{L}^2),\ h\mathbb{1}_\omega \in L^2(e^{s\check{\varphi}}\hat{\xi}^{-3/2}(0,T);L^2(\Omega)), \\
& (L(y) - \mathbb{1}_\omega h, L_\Gamma(y, y_\Gamma)) \in L^2(e^{s\hat{\varphi}}\check{\xi}^{-3/2}(0,T);\mathbb{L}^2), \\
& (y, y_\Gamma) \in L^2(e^{\frac{1}{3}s\hat{\varphi}}(0,T);\mathbb{H}^2) \cap L^\infty(e^{\frac{1}{3}s\hat{\varphi}}(0,T);\mathbb{H}^1)\},
\end{aligned} \tag{49}$$

endowed by its natural norm:

$$\begin{aligned}
\|(y, y_\Gamma, h)\|_{\mathcal{V}}^2 :=& \|e^{s\check{\varphi}}(y, y_\Gamma)\|^2_{L^2(0,T;\mathbb{L}^2)} + \|e^{s\check{\varphi}}\hat{\xi}^3 h\mathbb{1}_\omega\|^2_{L^2(\Omega\times(0,T))} \\
&+ \|e^{s\hat{\varphi}}\check{\xi}^{-3/2}(L(y) - \mathbb{1}_\omega h, N(y, y_\Gamma))\|^2_{L^2(0,T;\mathbb{L}^2)} \\
&+ \|e^{\frac{1}{3}s\hat{\varphi}}(y, y_\Gamma)\|^2_{L^2(0,T;\mathbb{H}^2)} + \|e^{\frac{1}{3}s\hat{\varphi}}(y, y_\Gamma)\|^2_{L^\infty(0,T;\mathbb{H}^1)}.
\end{aligned}$$

Proposition 4.2. *Let $(y_0, y_{\Gamma,0}) \in \mathbb{L}^2$ and assume that*

$$(f, f_\Gamma) \in L^2(e^{s\hat{\varphi}}\check{\xi}^{-3/2}(0,T);\mathbb{L}^2). \tag{50}$$

Then, we can find a control h such that the associated solution (y, y_Γ) of (46) satisfies $(y, y_\Gamma, h) \in \mathcal{V}$. In particular, we have

$$y(\cdot, T) = 0 \text{ in } \Omega, \quad y_\Gamma(\cdot, T) = 0 \text{ on } \Gamma.$$

Proof. For our purposes, we define

$$P_0 := \left\{(z, z_\Gamma) \in C^\infty(\overline{\Omega} \times [0,T]) \times C^\infty(\Gamma \times [0,T]) \,:\, z\big|_\Gamma = z_\Gamma \text{ on } \Gamma \times [0,T]\right\}.$$

Let $\mathbb{a} : P_0 \times P_0 \to \mathbb{R}$ be the bilinear form

$$\begin{aligned}&\mathbb{a}((z, z_\Gamma), (w, w_\Gamma))\\ :=&\Re \int_0^T \int_\Omega e^{-2s\check{\varphi}} L^* z \overline{L^*(w)} dx dt + \Re \int_0^T \int_\Gamma e^{-2s\check{\varphi}} L_\Gamma^*(z, z_\Gamma) \overline{L_\Gamma^*(w, w_\Gamma)} dS dt\\ &+ \Re \int_0^T \int_\omega e^{-2s\check{\varphi}} \hat{\xi}^3 z \overline{w} dx dt \quad \forall (z, z_\Gamma), (w, w_\Gamma) \in P_0 \times P_0.\end{aligned}$$

We also define the linear form $\ell : P_0 \to \mathbb{R}$ as

$$\begin{aligned}\ell(w, w_\Gamma) :=&\Re \int_\Omega y_0 \overline{w(0)} dx + \Re \int_\Gamma y_{\Gamma,0} \overline{w_\Gamma(0)} dS + \Re \int_0^T \int_\Omega f \overline{w} dx dt\\ &+ \Re \int_0^T \int_\Gamma f_\Gamma \overline{w_\Gamma} dS dt \quad \forall (w, w_\Gamma) \in P_0.\end{aligned}$$

In view of the observability inequality (48), it is clear that

$$\|(z, z_\Gamma)\|_{\mathbb{a}} := \sqrt{\mathbb{a}((z, z_\Gamma), (z, z_\Gamma))} \quad \forall (z, z_\Gamma) \in P_0,$$

defines a norm in P_0. Let P be the completion of $(P_0, \|\cdot\|_{\mathbb{a}})$. Then, $\mathbb{a}(\cdot,\cdot)$ is well-defined, continuous and again definite positive on P. On the other hand, by (50) and the observability inequality (48), it is easy to see that

$$|\ell(w, w_\Gamma)| \leq C\|(w, w_\Gamma)\|_{\mathbb{a}} \quad \forall (w, w_\Gamma) \in P_0,$$

i.e. ℓ is continuous in P_0 and, by density, in P. Thus, by Lax-Milgram Theorem, the variational problem

$$\begin{cases}\text{Find } (z, z_\Gamma) \in P \text{ such that}\\ \mathbb{a}((z, z_\Gamma), (w, w_\Gamma)) = \ell(w, w_\Gamma) \quad \forall (w, w_\Gamma) \in P,\end{cases}$$

possesses exactly one solution (z^*, z^*_Γ). Let (y^*, y^*_Γ, h^*) defined by

$$\begin{cases} y^* := e^{-2s\check{\varphi}} L^*(z^*) & \text{in } \Omega \times (0,T), \\ y^*_\Gamma := e^{-2s\check{\varphi}} L^*_\Gamma(z^*, z^*_\Gamma) & \text{on } \Gamma \times (0,T), \\ h^* := e^{-2s\check{\varphi}} \hat{\xi}^3 z^* & \text{in } \omega \times (0,T). \end{cases}$$

We point out that (y^*, y^*_Γ, h^*) satisfies

$$\begin{aligned} &\int_0^T \int_\Omega e^{2s\check{\varphi}} |y^*|^2 dxdt + \int_0^T \int_\Gamma e^{2s\check{\varphi}} |y^*_\Gamma|^2 dSdt + \int_0^T \int_\omega e^{2s\check{\varphi}} \hat{\xi}^{-3} |h^*|^2 dxdt \\ =&\mathrm{a}((z^*, z^*_\Gamma), (z^*, z^*_\Gamma)) < \infty, \end{aligned}$$

and also that

$$\begin{aligned} &\Re \int_0^T \int_\Omega y^* \overline{L^*(w)} dxdt + \Re \int_0^T \int_\Gamma y^*_\Gamma \overline{L^*_\Gamma(w, w_\Gamma)} dSdt + \Re \int_0^T \int_\omega h^* \overline{w} dxdt \\ =&\Re \int_\Omega y_0 \overline{w(0)} dx + \Re \int_\Gamma y_{\Gamma,0} \overline{w_\Gamma(0)} dS + \Re \int_0^T \int_\Omega f \overline{w} dxdt \\ &+ \Re \int_0^T \int_\Gamma f_\Gamma \overline{w_\Gamma} dSdt, \end{aligned} \tag{51}$$

i.e., from (51) we deduce that $(y^*, y^*_\Gamma) \in C^0([0,T]; \mathbb{L}^2) \cap L^2(0,T; \mathbb{H}^1)$ is a distributional solution with control h^* and initial datum $(y_0, y_{\Gamma,0})$ such that $y^*(\cdot, T) = 0$ in Ω and $y_\Gamma(\cdot, T) = 0$ on Γ.

It remains to check that

$$(y^*, y^*_\Gamma) \in L^2(e^{\frac{1}{3}s\hat{\varphi}}(0,T); \mathbb{H}^2) \cap L^\infty(e^{\frac{1}{3}s\hat{\varphi}}(0,T); \mathbb{H}^1). \tag{52}$$

In order to do that, we notice that the new variables

$$(y^\star, y^\star_\Gamma) := (e^{\frac{1}{3}s\hat{\varphi}} y^*, e^{\frac{1}{3}s\hat{\varphi}} y^*_\Gamma)$$

solves the problem

$$\begin{cases} L(y^\star) = e^{\frac{1}{3}s\hat{\varphi}}(f + h\mathbb{1}_\omega) + (e^{\frac{1}{3}s\hat{\varphi}})_t y^*, & \text{in } \Omega \times (0,T), \\ L_\Gamma(y^\star, y^\star_\Gamma) = e^{\frac{1}{3}s\hat{\varphi}} f_\Gamma + (e^{\frac{1}{3}s\hat{\varphi}})_t y^*_\Gamma, & \text{in } \Gamma \times (0,T), \\ y^\star = y^\star_\Gamma, & \text{on } \Gamma \times (0,T), \\ (y(0), y_\Gamma(0)) = e^{\frac{1}{3}s\hat{\varphi}(0)}(y_0, y_{\Gamma,0}), & \text{in } \Omega \times \Gamma. \end{cases} \tag{53}$$

Since

$$\left| \left(e^{\frac{1}{3}s\hat{\varphi}} \right)_t y^* \right| \le C\hat{\xi}^2 e^{\frac{1}{3}s\hat{\varphi}} |y^*| \le C e^{s\check{\varphi}} |y^*| \in L^2(0,T; L^2(\Omega)),$$

$$\left|\left(e^{\frac{1}{3}s\hat{\varphi}}\right)_t y_\Gamma^*\right| \leq e^{\frac{1}{3}s\check{\varphi}y_\Gamma^*} \in L^2(0,T;L^2(\Gamma)).$$

and

$$\left(e^{\frac{1}{3}s\hat{\varphi}}(f+h\mathbb{1}_\omega), e^{\frac{1}{3}s\hat{\varphi}}f_\Gamma\right) \in L^2(0,T;\mathbb{L}^2),$$

and since $e^{\frac{1}{3}s\hat{\varphi}(0)}(y_0, y_{\Gamma,0}) \in \mathbb{H}^1$, it follows from Proposition 2.3 that the solution $(y^\star, y_\Gamma^\star)$ of (53) satisfies

$$(y^\star, y_\Gamma^\star) \in L^2(0,T;\mathbb{H}^2 \cap C^0([0,T];\mathbb{H}^1)),$$

which is equivalently to (52). We conclude that $(y^*, y_\Gamma^*, h^*) \in \mathcal{V}$. This ends the proof of Proposition 4.2. □

5 Proof of the Theorem 1.1

To prove the main theorem of this article, we shall use a local inversion argument. This will be done by using the following local inversion mapping theorem in Banach spaces (see, for instance, [29], page 107).

Theorem 5.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces and let $\mathcal{A} : \mathcal{B}_1 \to \mathcal{B}_2$ be a $C^1(\mathcal{B}_1;\mathcal{B}_2)$ function. Suppose that $b_1 \in \mathcal{B}_1$, $b_2 = \mathcal{A}(b_1)$ and that $\mathcal{A}'(b_1) : \mathcal{B}_1 \to \mathcal{B}_2$ is surjective. Then, there exists $\delta > 0$ such that, for every $b' \in \mathcal{B}_2$ satisfying $\|b' - b_2\|_{\mathcal{B}_2} < \delta$, there exists a solution of the equation*

$$\mathcal{A}(b) = b', \quad b \in \mathcal{B}_1.$$

Moreover, there exists a constant $C > 0$ such that

$$\|b_1 - b\|_{\mathcal{B}_1} \leq C\|b_2 - b'\|_{\mathcal{B}_2}.$$

Now, we have all the ingredients to prove Theorem 1.1.

Proof of the Theorem 1.1. Consider the spaces

$$\mathcal{B}_1 := \mathcal{V}, \quad \mathcal{B}_2 := L^2(e^{s\hat{\varphi}}\check{\xi}^{-3/2}(0,T);\mathbb{L}^2) \times \mathbb{L}^2,$$

where $\mathcal{V}$ is the linear space defined in (49).

Consider $\mathcal{A} : \mathcal{B}_1 \to \mathcal{B}_2$ be the operator defined by

$$\mathcal{A}(u, u_\Gamma, h) = \left(Lu + c(1+\gamma i)|u|^2 u - \mathbb{1}_\omega h, L_\Gamma(u, u_\Gamma) + c(1+\gamma i)|u_\Gamma|^2 u_\Gamma, u(0), u_\Gamma(0)\right).$$

We choose, $b_1 = (0,0,0)$ and $b_2 = \mathcal{A}(0,0,0) = (0,0,u_0,u_{\Gamma,0})$. In order to use Theorem 5.1, we shall check that the following two assertions:

(a) $\mathcal{A}'(0,0,0) : \mathcal{B}_1 \to \mathcal{B}_2$ is surjective.
(b) $\mathcal{A}$ is an operator of class C^1 from $\mathcal{B}_1$ to $\mathcal{B}_2$.

A simple computation shows that for all $(u^*, u_\Gamma^*, h^*) \in \mathcal{B}_1$,

$$\begin{aligned}\mathcal{A}'(u, u_\Gamma, h)(u^*, u_\Gamma^*, h^*) =& (L(u^*) + 3c(1+\gamma i)|u|^2 u^* - \mathbb{1}_\omega h^*, L_\Gamma(u^*, u_\Gamma^*) \\ & + 3c(1+\gamma i)|u_\Gamma|^2 u_\Gamma^*, u^*(0), u_\Gamma^*(0)).\end{aligned} \tag{54}$$

In particular, taking $(u, u_\Gamma, h) = (0, 0, 0)$ in (54), we have

$$\mathcal{A}'(0,0,0)(u^*, u_\Gamma^*, h^*) = (L(u^*) - \mathbb{1}_\omega h^*, N(u^*, u_\Gamma^*), u^*(0), u_\Gamma^*(0)).$$

Now, it is clear that, in view of Proposition 4.2, the operator $\mathcal{A}'(0,0,0)$ is surjective.

On the other hand, to prove that $\mathcal{A} \in C^1(\mathcal{B}_1, \mathcal{B}_2)$, it is sufficient to check that the map

$$(u_1, u_{\Gamma,1}, h_1), (u_2, u_{\Gamma,2}, h_2), (u_3, u_{\Gamma,3}, h_3) \mapsto (u_1 u_2 u_3, u_{\Gamma,1} u_{\Gamma,2} u_{\Gamma,3}),$$

is continuous from $\mathcal{V} \times \mathcal{V} \times \mathcal{V}$ to $L^2(e^{s\hat{\varphi}} \check{\xi}^{-3/2}(0,T); \mathbb{L}^2)$. Now, observe that the classical embeddings for Sobolev spaces (see [26] and [30] for such embeddings for Ω and Γ, respectively) and Hölder's inequality implies that

$$L^2(0,T;\mathbb{H}^2) \cap L^\infty(0,T;\mathbb{H}^1) \hookrightarrow L^6(0,T;\mathbb{L}^9) \hookrightarrow L^6(0,T;\mathbb{L}^6),$$

This implies that

$$\begin{aligned} & \|e^{s\hat{\varphi}} \check{\xi}^{-3/2}(u_1 u_2 u_3, u_{\Gamma,1} u_{\Gamma,2} u_{\Gamma,3})\|_{L^2(0,T;\mathbb{L}^2)} \\ \leq & C \prod_{j=1}^{3} \|e^{\frac{1}{3}s\hat{\varphi}}(u_j, u_{\Gamma,j})\|_{L^6(0,T;\mathbb{L}^6)} \\ \leq & C \prod_{j=1}^{3} \|e^{\frac{1}{3}s\hat{\varphi}}(u_j, u_{\Gamma,j})\|_{L^2(0,T;\mathbb{H}^2)\cap L^\infty(0,T;\mathbb{H}^1)} \\ \leq & C \prod_{j=1}^{3} \|(u_j, u_{\Gamma,j}, h_j)\|_{\mathcal{V}}. \end{aligned} \tag{55}$$

Therefore, we have $\mathcal{A} \in C^1(\mathcal{B}_1, \mathcal{B}_2)$, and we can apply Theorem 5.1 to guarantee the existence of $\delta > 0$ such that $\|(u_0, u_{\Gamma,0})\|_{\mathbb{H}^1} \leq \delta$, one can find $(y, y_\Gamma, h) \in \mathcal{B}_1 := \mathcal{V}$ such that the associated solution (u, u_Γ) (with initial datum $(u_0, u_{\Gamma,0})$) satisfies

$$y(\cdot, T) = 0, \text{ in } \Omega, \quad y_\Gamma(\cdot, T) = 0, \text{ on } \Gamma.$$

This ends the proof of Theorem 1.1. □

6 Further comments

In this paper, we proved a local null controllability result for the cubic Ginzburg-Landau equation with dynamic boundary conditions in dimension 2 or 3. Some comments are in order.

6.1 General nonlinearities

Consider the Ginzburg-Landau equation with with dynamic boundary conditions and a general potential:

$$\begin{cases} \partial_t u - a(1+\alpha i)\Delta u + c(u) = \mathbb{1}_\omega h & \text{in } \Omega \times (0,T) \\ \partial_t u_\Gamma + a(1+\alpha i)\partial_\nu u - b(1+\alpha i)\Delta_\Gamma u_\Gamma + c_\Gamma(u_\Gamma) = 0 & \text{on } \Gamma \times (0,T), \\ u = u_\Gamma & \text{on } \Gamma \times (0,T) \\ (u(0), u_\Gamma(0)) = (u_0, u_{\Gamma,0}) & \text{in } \Omega \times \Gamma. \end{cases} \tag{56}$$

We suppose that the potentials c and c_Γ verify the following assumptions:

$$c, c_\Gamma \in C^2(\mathbb{R}, \mathbb{C}), \;\; \text{with } c(0) = c_\Gamma(0) = 0.$$

Under these assumptions on the potentials, Theorem 1.1 can also be applied to equation (57). The linearized equation is

$$\begin{cases} \partial_t u - a(1+\alpha i)\Delta u + c'(0)u = f + \mathbb{1}_\omega h & \text{in } \Omega \times (0,T) \\ \partial_t u_\Gamma + a(1+\alpha i)\partial_\nu u - b(1+\alpha i)\Delta_\Gamma u_\Gamma + c_\Gamma'(0)u_\Gamma = f_\Gamma & \text{on } \Gamma \times (0,T), \\ u = u_\Gamma & \text{on } \Gamma \times (0,T) \\ (u(0), u_\Gamma(0)) = (u_0, u_{\Gamma,0}) & \text{in } \Omega \times \Gamma. \end{cases} \tag{57}$$

Notice that the Carleman estimate (3.2) remains valid in this case. Indeed, it suffices to consider

$$\begin{aligned} &L^*(v) := \partial_t v + a(1-\alpha i)\Delta v - c'(0)v, \\ &L^*_\Gamma(v, v_\Gamma) := \partial_t v_\Gamma - a(1-\alpha i)\partial_\nu v + b(1-\alpha i)\Delta_\Gamma v_\Gamma - c_\Gamma'(0)v_\Gamma \end{aligned}$$

in Theorem 3.2. The extra terms can be easily absorbed. The rest of the proof can be achieved following the arguments of [31], where this kind of potentials are considered. The crucial point is to choose the correct spaces to use properly a Local Inverse Mapping Theorem. Of course, these functional spaces must depend on the Carleman weight functions defined in (23) and (24).

6.2 The one-dimensional case

The approach of this paper can also be used to prove a local controllability result for the 1-D version of our problem. To fix some ideas, let $\Omega = (0,1)$ and $\omega = (a_\star, b_\star) \subseteq \Omega$ and $T > 0$. Then, consider the following Ginzburg-Landau equation with dynamic boundary conditions:

$$\begin{cases} \partial_t u - a(1+\alpha i)\partial_x^2 u + c(1+\gamma i)|u|^2 u = \mathbb{1}_\omega h & (x,t) \in \Omega \times (0,T), \\ \dot{u}_{\Gamma_0}(t) - a(1+\alpha i)\partial_x u(0,t) + c(1+\gamma i)|u_{\Gamma_0}(t)|^2 u_{\Gamma_0}(t) = 0 & t \in (0,T), \\ \dot{u}_{\Gamma_1}(t) + a(1+\alpha i)\partial_x u(1,t) + c(1+\gamma i)|u_{\Gamma_1}(t)|^2 u_{\Gamma_1}(t) = 0 & t \in (0,T), \\ u_{\Gamma_0}(t) = u(0,t),\ u_{\Gamma_1}(t) = u(1,t) & t \in (0,T), \\ u(x,0) = u^0(x),\ u_{\Gamma_0}(0) = u^0_{\Gamma_0},\ u_{\Gamma_1}(0) = u^0_{\Gamma_1} & x \in \Omega. \end{cases} \tag{58}$$

Here, $a > 0$, and $c, \alpha, \gamma \in \mathbb{R}$ with $c \neq 0$. Then, given a initial condition $(u^0, u^0_{\Gamma_0}, u^0_{\Gamma_1}) \in L^2(\Omega) \times \mathbb{R} \times \mathbb{R}$, we are interested in finding a control $h \in L^2(\omega \times (0,T))$ such that the state $(u, u_{\Gamma_0}, u_{\Gamma_1})$ of (58) vanishes at final time $T > 0$. For $k \in \mathbb{N}$, let us define the space

$$\mathbb{X}^k := \{(u, u_{\Gamma_0}, u_{\Gamma_1}) \in L^2(\Omega) \times \mathbb{R} \times \mathbb{R} \,:\, u \in H^k(\Omega),\ u(0) = u_{\Gamma_0},\ u(1) = u_{\Gamma_1}\},$$

and set $\mathbb{X}^0 := L^2(\Omega) \times \mathbb{R} \times \mathbb{R}$. Then, we have the following result:

Theorem 6.1. *Suppose that $\omega \Subset \Omega$ and $T > 0$. Then, there exists $\delta > 0$ such that, for every $(u^0, u^0_{\Gamma_0}, u^0_{\Gamma_1}) \in \mathbb{X}^1$ satisfying*

$$\|(u^0, u^0_{\Gamma_0}, u^0_{\Gamma_1})\|_{\mathbb{X}^1} \leq \delta,$$

there exists a control $h \in L^2(\omega \times (0,T))$ such that the unique corresponding solution $(u, u_{\Gamma_0}, u_{\Gamma_1})$ of (58) satisfies

$$u(\cdot, T) = 0 \text{ in } \Omega, \text{ and } u_{\Gamma_0}(T) = u_{\Gamma_1}(T) = 0.$$

The proof of Theorem 6.1 relies on the same approach used for the general case, i.e., we consider the linearized system of (58) and we prove a null controllability result. This problem is equivalent to the observability inequality for the corresponding adjoint system. To prove such inequality, it is enough to apply the following Carleman estimate for the 1-D Ginzburg-Landau operator with dynamic boundary conditions:

Theorem 6.2. *Let $\omega \Subset \Omega$. Let us choose $\omega' \Subset \omega$ and η^0 as in Lemma 3.1. We also define φ and ξ given by (23) and (24), respectively. Then, there exist three positive* *constants C, λ_0, s_0 such that for all $\lambda \geq \lambda_0$ and $s \geq s_0$, the following identity holds*

$$\begin{aligned}
&\int_0^T \int_\Omega e^{-2s\varphi} \left(s^3\lambda^4\xi^3|v|^2 + s\lambda^2\xi|\partial_x v|^2 + s^{-1}\xi^{-1}|\partial_t v|^2 + s^{-1}\xi^{-1}|\partial_x^2 v|^2\right) dxdt \\
&+ \int_0^T e^{-2s\varphi(0,\cdot)} \left(s^3\lambda^3\xi^3(0,\cdot)|v_{\Gamma_0}|^2 + s\lambda\xi(0,\cdot)|\partial_x v(0,\cdot)|^2 + s^{-1}\xi^{-1}(0,\cdot)|\dot{v}_{\Gamma_0}|^2\right) dt \\
&+ \int_0^T e^{-2s\varphi(1,\cdot)} \left(s^3\lambda^3\xi^3(0,\cdot)|v_{\Gamma_1}|^2 + s\lambda\xi(1,\cdot)|\partial_x v(1,\cdot)|^2 + s^{-1}\xi^{-1}(1,\cdot)|\dot{v}_{\Gamma_1}|^2\right) dt \\
\leq& C\int_0^T \int_\Omega e^{-2s\varphi}|\ell(v)|^2\, dxdt + C\int_0^T e^{-2s\varphi(0,\cdot)}|\ell_{\Gamma_0}(v, v_{\Gamma_0})|^2\, dt \\
&+ C\int_0^T e^{-2s\varphi(1,\cdot)}|\ell_{\Gamma_1}(v, v_{\Gamma_1})|^2\, dt + s^3\lambda^4 \int_0^T \int_\omega e^{-2s\varphi}\xi^3|v|^2\, dxdt,
\end{aligned}$$

for all $(v, v_{\Gamma_0}, v_{\Gamma_1}) \in H^1(0,T;\mathbb{X}^0) \cap L^2(0,T;\mathbb{X}^2)$. Here,

$$\ell(v) := \partial_t v + a(1-\alpha i)\partial_x^2 v, \quad \ell_{\Gamma_0}(v, v_{\Gamma_0}) := \dot{v}_{\Gamma_0}(t) + a(1-\alpha i)\partial_x v(0,t),$$
$$\ell_{\Gamma_1}(v, v_{\Gamma_1}) := \dot{v}_{\Gamma_1}(t) - a(1-\alpha i)\partial_x v(1,t).$$

The proof of Theorem 6.2 follows the same structure as the Carleman estimate for the case $d \geq 2$, and is actually simplified by the fact that no tangential derivatives exist in this case.

Acknowledgments

We express our gratitude to the anonymous referees for carefully reviewing the manuscript and providing valuable comments and suggestions to further improve the quality and impact of the article. Nicolás Carreño has been funded by ANID FONDECYT 1211292. Alberto Mercado has been funded by ANID FONDECYT 1211292 and ANID Millennium Science Initiative Program, Code NCN19-161. Roberto Morales has been funded by ANID FONDECYT 3200830.

References

[1] Aranson, I.S., Kramer, L.: The world of the complex ginzburg-landau equation. Reviews of modern physics **74**(1), 99 (2002)

[2] García-Morales, V., Krischer, K.: The complex ginzburg–landau equation: an introduction. Contemporary Physics **53**(2), 79–95 (2012)

[3] Levermore, C.D., Oliver, M.: The complex Ginzburg-Landau equation as a model problem. Lectures in Appl. Math., vol. 31. Amer. Math. Soc., Providence, RI (1996). https://doi.org/10.1080/03605309708821254

[4] Gao, H., Bu, C.: Dirichlet inhomogeneous boundary value problem for the $n+1$ complex Ginzburg-Landau equation. J. Differential Equations **198**(1), 176–195 (2004) https://doi.org/10.1016/j.jde.2003.09.006

[5] Gao, H., Gu, X., Bu, C.: A Dirichlet inhomogeneous boundary value problem for a generalized Ginzburg-Landau equation. J. Math. Anal. Appl. **330**(1), 699–714 (2007) https://doi.org/10.1016/j.jmaa.2006.07.023

[6] Fu, X.: A weighted identity for partial differential operators of second order and its applications. C. R. Math. Acad. Sci. Paris **342**(8), 579–584 (2006) https://doi.org/10.1016/j.crma.2006.02.023

[7] Chen, Z., Hoffmann, K.-H.: Numerical solutions of an optimal control problem governed by a Ginzburg-Landau model in superconductivity. Numer. Funct. Anal. Optim. **19**(7-8), 737–757 (1998) https://doi.org/10.1080/01630569808816856

[8] Bu, C.: An initial-boundary value problem for the Ginzburg-Landau equation. Appl. Math. Lett. **5**(5), 31–34 (1992) https://doi.org/10.1016/0893-9659(92)90058-H

[9] Bartuccelli, M., Constantin, P., Doering, C.R., Gibbon, J.D., Gisselfält, M.: On the possibility of soft and hard turbulence in the complex Ginzburg-Landau equation. Phys. D **44**(3), 421–444 (1990) https://doi.org/10.1016/0167-2789(90)90156-J

[10] Aamo, O.M., Smyshlyaev, A., Krstić, M.: Boundary control of the linearized Ginzburg-Landau model of vortex shedding. SIAM J. Control Optim. **43**(6), 1953–1971 (2005) https://doi.org/10.1137/S036301290342601X

[11] Rosier, L., Zhang, B.-Y.: Null controllability of the complex Ginzburg-Landau equation. Ann. Inst. H. Poincaré C Anal. Non Linéaire **26**(2), 649–673 (2009) https://doi.org/10.1016/j.anihpc.2008.03.003

[12] Maniar, L., Meyries, M., Schnaubelt, R.: Null controllability for parabolic equations with dynamic boundary conditions. Evol. Equ. Control Theory **6**(3), 381–407 (2017) https://doi.org/10.3934/eect.2017020

[13] Khoutaibi, A., Maniar, L., Oukdach, O.: Null controllability for semilinear heat equation with dynamic boundary conditions. Discrete Contin. Dyn. Syst. Ser. S **15**(6), 1525–1546 (2022) https://doi.org/10.3934/dcdss.2022087

[14] Ait Ben Hassi, E.M., Chorfi, S.-E., Maniar, L., Oukdach, O.: Lipschitz stability for an inverse source problem in anisotropic parabolic equations with dynamic boundary conditions. Evol. Equ. Control Theory **10**(4), 837–859 (2021) https://doi.org/10.3934/eect.2020094

[15] Ait Ben Hassi, E.M., Chorfi, S.-E., Maniar, L.: An inverse problem of radiative

potentials and initial temperatures in parabolic equations with dynamic boundary conditions. J. Inverse Ill-Posed Probl. **30**(3), 363–378 (2022) https://doi.org/10.1515/jiip-2020-0067

[16] Ait Ben Hassi, E.M., Chorfi, S.-E., Maniar, L.: Stable determination of coefficients in semilinear parabolic system with dynamic boundary conditions. Inverse Problems **38**(11), 115007–28 (2022) https://doi.org/10.1088/1361-6420/ac91ed

[17] Lecaros, R., Morales, R., Pérez, A., Zamorano, S.: Discrete Carleman estimates and application to controllability for a fully-discrete parabolic operator with dynamic boundary conditions. J. Differential Equations **365**, 832–881 (2023) https://doi.org/10.1016/j.jde.2023.05.014

[18] Gal, C.G., Tebou, L.: Carleman inequalities for wave equations with oscillatory boundary conditions and application. SIAM J. Control Optim. **55**(1), 324–364 (2017) https://doi.org/10.1137/15M1032211

[19] Mercado, A., Morales, R.: Exact controllability for a Schrödinger equation with dynamic boundary conditions. SIAM J. Control Optim. **61**(6), 3501–3525 (2023) https://doi.org/10.1137/21M1439407

[20] Corrêa, W.J., Özsarı, T.: Complex Ginzburg-Landau equations with dynamic boundary conditions. Nonlinear Anal. Real World Appl. **41**, 607–641 (2018) https://doi.org/10.1016/j.nonrwa.2017.12.001

[21] Taylor, M.E.: Partial Differential Equations I. Basic Theory. Springer, Applied Mathematical Sciences (2023). https://doi.org/10.1007/978-3-031-33859-5

[22] Huaman, D.N., Nuñez-Chávez, M.R., Límaco, J., Carvalho, P.P.: Local null controllability for the thermistor problem. Nonlinear Anal. **236**, 113330–34 (2023) https://doi.org/10.1016/j.na.2023.113330

[23] Fernández-Cara, E., Límaco, J., Thamsten, Y., Menezes, D.: Local null controllability of a quasi-linear system and related numerical experiments. ESAIM Control Optim. Calc. Var. **29**, 27–34 (2023) https://doi.org/10.1051/cocv/2023009

[24] Fernández-Cara, E., Nina-Huamán, D., Nuñez-Chávez, M.R., Vieira, F.B.: On the theoretical and numerical control of a one-dimensional nonlinear parabolic partial differential equation. J. Optim. Theory Appl. **175**(3), 652–682 (2017) https://doi.org/10.1007/s10957-017-1190-4

[25] Gal, C.G.: The role of surface diffusion in dynamic boundary conditions: Where do we stand? Milan J. Math. **83**(2), 237–278 (2015) https://doi.org/10.1007/s00032-015-0242-1

[26] Adams, R.A., Fournier, J.J.F.: Sobolev Spaces vol. 140, 2nd edn. Elsevier/Academic Press, Amsterdam, Pure and Applied Mathematics (Amsterdam) (2003)

[27] Fursikov, A.V., Imanuvilov, O.Y.: Controllability of Evolution Equations. Lecture Notes Series, vol. 34, p. 163. Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul (1996)

[28] Fernández-Cara, E., Guerrero, S.: Global carleman inequalities for parabolic systems and applications to controllability. SIAM journal on control and optimization **45**(4), 1395–1446 (2006)

[29] Alekseev, V.M., Tikhomirov, V.M., Fomin, S.V.: Optimal Control. Consultants Bureau, New York, Contemporary Soviet Mathematics (1987). https://doi.org/10.1007/978-1-4615-7551-1

[30] Lee, J.M., Parker, T.H.: The Yamabe problem. Bull. Amer. Math. Soc. (N.S.) **17**(1), 37–91 (1987) https://doi.org/10.1090/S0273-0979-1987-15514-5

[31] Et-tahri, F., Chorfi, S.-E., Maniar, L., Boutaayamou, I.: Null controllability of a volume-surface reaction-diffusion equation with dynamic boundary conditions. J. Math. Anal. Appl. **542**(2), 128793–24 (2025) https://doi.org/10.1016/j.jmaa.2024.128793