Stackelberg-Nash exact controllability for the Kuramoto-Sivashinsky equation with boundary and distributed controls

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Abstract

This paper deals with a multi-objective control problem for the Kuramoto-Sivashinsky equation by following a Stackelberg-Nash strategy. We have a distributed control called Leader, and two boundary controls called Followers, each of them has to act over the equation to influence the behavior of the state in a particular way, by reaching or approaching to many targets at once. To be more precise, the Leader wants to drive the solution to a prescribed target at a final time, and the followers have to minimize some given cost functionals, adapting themselves to what the Leader wants. The main difficulty here is that, since the Followers are in the boundary, the problem turns to be equivalent to prove a partial null controllability result for a system of nonlinear fourth-order equations with boundary coupling terms.

Keywords: Kuramoto-Sivashinsky equation, Null controllability, Carleman estimates, Hierarchical controls, Stackelberg-Nash Strategy.

Subject Classification: 35K41, 35K55, 93C10, 93B05

1 Introduction

The Kuramoto-Sivashinsky (KS) equation is a nonlinear fourth-order parabolic equation with a great variety of applications. Its deduction is due to Kuramoto et al. in [23,24], for modeling phase turbulence in reaction-diffusion systems, and independently by Sivashinsky, in [34], for the description of the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front. This nonlinear partial differential equation describes incipient instabilities in a variety of physical and chemical systems (see [11,22,25]).

Due to the great variety of applications, the KS equation has gained the attention of the mathematical community. Many references with dynamical properties and well-posedness are present in the literature, as we can see in [18,29,30].

For controllability properties, many results are known. By utilizing the moment method, E. Cerpa proved in [7] a boundary null controllability result for a linear fourth-order equation with controls positioned over the normal derivate of the state. For this result to hold, some conditions on the "anti-diffusion" parameter are necessary. By deriving a suitable Carleman estimate, the authors in [9] proved a local exact controllability for the KS equation with controls acting over the state and normal derivative of the state on the left boundary. In what concerns distributed controls, the papers [5, 10] deals with the controllability of a type of KS system with only one distributed control. For a linearized version of the KS equation, the authors in [8,20] deduced the null controllability with boundary and distributed controls for both Dirichlet and Neumann boundary conditions. All these references consider control problems for fourth-order equations with the null controllability as the only objective.

It is not difficult to realize that in problems of real life, one may be interested in controlling many aspects, or variables, simultaneously. For instance, in a heat conduction problem, one may want to drive

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the temperature to a given value maintaining the temperature at some reasonable levels in given regions. In fluid dynamics, one may want to control the velocity of the fluid, but in some zones, the fluid must behave satisfactorily. In a polluted river, we may want to apply a chemical product to clean it, but in some regions, we cannot change some properties of the water and not extinguish a population living there. In all these examples, the concept of single objective control problem is insufficient, being necessary the development of methods for solving multi-objective control problems.

In the present paper, we want to control the KS equation to zero in a finite time and also control it in other aspects simultaneously. Let us be more precise about the problem under view: we consider a distributed system governed by the KS equation and a control v positioned inside the domain or in the boundary, or a combination of both. We assume that there are two (or more) goals we would like to achieve, the main one being of controllability type, and the others to ensure that the state does not move too far from given functions. We split the control v into two (or more) parts, say f, $v^1, v^2, \dots$ where f is the principal control, usually called the leader, and v^i are the secondary controls, denominated the followers. We assume the leader is responsible for the objective of controllability type, and the followers must deal with the other targets. This way of splitting the controls and giving roles to them stating degrees of importance is what we call a hierarchical control problem.

The application of hierarchical control for PDEs is due to J.-L. Lions in [26, 27], where we may find many techniques. These pioneering works motivated studies on this subject, and a lot of other results appeared after that (see [13, 14, 21, 32, 33]). We remark that all these classical works deal with hierarchical control problems whose leader's objective is approximate controllability. It is an interesting question to know the equations and the circumstances that allow us to solve these problems at the exact or null controllability level. In this framework, we can mention the work of Araruna et al., see [2], pioneering in considering the null controllability as the leader objective, but with some geometric assumptions. Then, in [3], the authors improved the results of [2] by taking less restrictive geometric assumptions. It is relevant to mention that the works in [2, 3] assume both leader and followers as distributed controls. An interesting question is whether we can extend the analysis to the case of controls acting on the boundary. For some positive results in this direction, we can cite [4] for the heat equation with nonlinearities that are uniformly bounded.

Concerning the KS equation, what is known so far is the case where both leader and followers are distributed at the interior of the domain (see [6]). In this way, what we propose here is to extend the result in [6] to the cases where the leader or the followers are assumed to be on the boundary. This particular setting brings new difficulties that are more complicated to solve than [6]. Indeed, as we are going to see further, it turns to be equivalent to a partial null controllability problem for a fourth-order nonlinear system, where some coupling terms are located on the boundary, thus some careful computations are needed to deal with them.

1.1 Statement of the problem

Let L and T some positive real numbers, define $Q = (0, L) \times (0, T)$ and, for $i = 1, 2$, let l_i and r_i be $L^2(0, T)$ functions. Consider the KS equation

$$\begin{cases} y_t + y_{xxxx} + \nu y_{xx} + y y_x = f \mathbb{1}_{\mathcal{O}} & (x,t) \in Q, \\ y(0,t) = l_1(t), \quad y(L,t) = r_1(t) & t \in (0,T), \\ y_x(0,t) = l_2(t), \quad y_x(L,t) = r_2(t) & t \in (0,T), \\ y(x,0) = y^0(x) & x \in (0,L), \end{cases} \tag{1.1}$$

where $y = y(x, t)$ is the state and y^0 is a prescribed initial condition. We write $\mathcal{O} = (a, b)$ and $\mathbb{1}_{\mathcal{O}}$ is the characteristic functions of $\mathcal{O}$. The distributed control f is called the *leader* and, for $i = 1, 2$, the boundary controls l^i and r^i are called the *followers*.

Let $\mathcal{O}_{1,d}$, $\mathcal{O}_{2,d} \subset (0, L)$ be open sets of the form $O_{i,d} = (a_{i,d}, b_{i,d})$, representing observation domains for the followers. For $l = (l_1, l_2)$ and $r = (r_1, r_2)$, we consider the functionals

$$\mathcal{L}(f; l, r) := \frac{\alpha_1}{2} \int_0^T \int_{\mathcal{O}_{1,d}} |y - y_{1,d}|^2 \, dx \, dt + \sum_{i=1}^{2} \frac{\mu_i}{2} \int_0^T |l^i|^2 \, dx \, dt, \tag{1.2}$$

$$\mathcal{R}(f;l,r) := \frac{\alpha_2}{2}\int_0^T\int_{\mathcal{O}_{2,d}} |y - y_{2,d}|^2 \, dx\,dt + \sum_{i=1}^{2}\frac{\nu_i}{2}\int_0^T |r^i|^2 \, dx\,dt. \tag{1.3}$$

The structure of the control process we will follow is described as follows:

1. The *left followers* $l = (l_1, l_2)$ and the *right followers* $r = (r_1, r_2)$ assume that the leader f has made a choice and intend to be a *Nash equilibrium* for the costs $\mathcal{L}$ and $\mathcal{R}$ respectively, that is, once f has been fixed, we look for controls l and r in $[L^2(0,T)]^2$ such that

$$\mathcal{L}(f;l,r) = \min_{\hat{l}} \mathcal{L}(f;\hat{l},r), \quad \mathcal{R}(f;l,r) = \min_{\hat{r}} \mathcal{R}(f;l,\hat{r}). \tag{1.4}$$

2. Let us fix an uncontrolled trajectory of (1.1), that is, a sufficiently regular solution to the system

$$\begin{cases} \bar{y}_t + \bar{y}_{xxxx} + \nu\bar{y}_{xx} + \bar{y}\bar{y}_x = 0 & (t,x) \in [0,T]\times[0,L], \\ \bar{y}(t,0) = \bar{y}(t,L) = 0 & t \in [0,T], \\ \bar{y}_x(t,0) = \bar{y}_x(t,L) = 0 & t \in [0,T], \\ \bar{y}(x,0) = \bar{y}^0(x) & x \in [0,L]. \end{cases} \tag{1.5}$$

Once the Nash equilibrium has been identified and fixed for each f, we look for a leader control $\hat{f} \in L^2(\mathcal{O}\times(0,T))$ such that

$$y(x,T) = \bar{y}(x,T) \text{ in } (0,L). \tag{1.6}$$

We stress that controlling the state y to a given trajectory as in (1.6) is the main objective to be accomplished and the leader control f is the one responsible for it. If there are no other objectives, then no other controls are needed, meaning that we can take $\{l^i\}_{i=1}^2$ and $\{r^i\}_{i=1}^2$ as zero in (1.1), becoming a standard controllability problem, which was already considered, for instance, in [35]. In the present paper the situation is different. Despite the fact we want to drive the state to the trajectory $\bar{y}$ in the time T, we also want the state not to be far, as much as possible, from given functions $\{y_{i,d}\}_{i=1}^2$ in the region $\{\mathcal{O}_{i,d}\}_{i=1}^2$, and that is the role of functionals $\mathcal{L}$ and $\mathcal{R}$ in (1.2) and (1.3), respectively. Then it becomes clear that the best we can do is search for controls $\{l^i\}_{i=1}^2$ and $\{r^i\}_{i=1}^2$ that somehow minimize $\mathcal{L}$ and $\mathcal{R}$.

It is important to remark that condition (1.4) says that we are applying a noncooperative optimization criterium for the determination of the followers. More precisely, the followers $\{l_i\}$ (resp. $\{r_i\}$) want to minimize functional $\mathcal{L}$ (resp. $\mathcal{R}$) without being concerned about what may happen to the cost $\mathcal{R}$ (resp. $\mathcal{L}$). This kind of concept was introduced by John Nash, in [28], in the context of game theory and economy. Then, it becomes clear why optimal solutions to (1.4) are usually called Nash equilibrium. There are other concepts of equilibrium, and we refer to Section 6.3 for some comments about this.

Now, let $z := y - \bar{y}$. It is clear that property (1.6) is equivalent to a null controllability property for z, that is,

$$z(x,T) = 0 \text{ in } (0,L), \tag{1.7}$$

where z is the solution of the equation

$$\begin{cases} z_t + z_{xxxx} + \nu z_{xx} + zz_x + (\bar{y}z)_x = f\mathbb{1}_{\mathcal{O}} & (x,t) \in Q, \\ z(0,t) = l_1(t), \quad z(L,t) = r_1(t) & t \in (0,T), \\ z_x(0,t) = l_2(t), \quad z_x(L,t) = r_2(t) & t \in (0,T), \\ z(x,0) = z^0(x) & x \in (0,L), \end{cases} \tag{1.8}$$

and $z^0(x) = y^0(x) - \bar{y}^0(x)$. From now on, we are going to work with system (1.8) instead of (1.1).

The functionals $\mathcal{L}$ and $\mathcal{R}$ can be rewritten as

$$\mathcal{L}(f;l,r) := \frac{\alpha_1}{2}\int_0^T\int_{\mathcal{O}_{i,d}} |z - z_{1,d}|^2 \, dx\,dt + \sum_{i=1}^{2}\frac{\mu_i}{2}\int_0^T |l^i|^2 \, dx\,dt \quad i = 1,2, \tag{1.9}$$

and

$$\mathcal{R}(f;l,r) := \frac{\alpha_2}{2}\int_0^T\int_{\mathcal{O}_{i,d}} |z - z_{2,d}|^2 \, dx\,dt + \sum_{i=1}^{2}\frac{\nu_i}{2}\int_0^T |r^i|^2 \, dx\,dt \quad i = 1,2, \tag{1.10}$$

where $z_{i,d} := y_{i,d} - \bar{y}$.

Now, if the functionals (1.9) and (1.10) are convex, then $(l, r) \in [L^2(0,T)]^4$ is a Nash equilibrium if and only if

$$D\mathcal{L}(f; l, r)(\hat{l}, 0) = 0, \quad \forall \hat{l} \in [L^2\,(0,T)]^2, \tag{1.11}$$

and

$$D\mathcal{R}(f; l, r)(0, \hat{r}) = 0, \quad \forall \hat{r} \in [L^2\,(0,T)]^2. \tag{1.12}$$

Following an standard procedure, see for instance [2], we can prove that a pair (v^1, v^2) satisfying (1.11)-(1.12) can be characterized by

$$l_1 = -\frac{1}{\mu_1}\phi^1_{xxx}(0), \quad l_2 = \frac{1}{\mu_2}\phi^1_{xx}(0) \tag{1.13}$$

and

$$r_1 = \frac{1}{\nu_1}\phi^2_{xxx}(L), \quad r_2 = -\frac{1}{\nu_2}\phi^2_{xx}(L), \tag{1.14}$$

where (z, ϕ^1, ϕ^2) is the solution of the optimality system

$$\begin{cases} z_t + z_{xxxx} + \nu z_{xx} + z z_x + (\bar{y} z)_x = f\,\mathbb{1}_{\mathcal{O}} & & (x,t) \in Q, \\ -\phi^i_t + \phi^i_{xxxx} + \nu\phi^i_{xx} - (z + \bar{y})\phi^i_x = \alpha_i (z - z_{i,d})\mathbb{1}_{\mathcal{O}_{i,d}} & i = 1,2 & (x,t) \in Q, \\ z(0,t) = -\frac{1}{\mu_1}\phi^1_{xxx}(0), z(L,t) = \frac{1}{\nu_1}\phi^2_{xxx}(L) & & t \in (0,T), \\ z_x(0,t) = \frac{1}{\mu_2}\phi^1_{xx}(0), z_x(L,t) = -\frac{1}{\nu_2}\phi^2_{xx}(L) & & t \in (0,T), \\ \phi^i(0,t) = \phi^i(L,t) = \phi^i_x(0,t) = \phi^i_x(L,t) = 0 \quad i = 1,2 & & t \in (0,T), \\ z(x,0) = z^0(x), \quad \phi^i(x,T) = 0 \quad i = 1,2 & & x \in (0,L). \end{cases} \tag{1.15}$$

A first step to prove the existence of a Nash equilibrium is to show that system (1.15) possesses a unique solution (at least for small data). This will guarantee the existence of a unique critical point (l, r) in the sense of (1.13), (1.14). However, since (1.8) is nonlinear, functionals (1.9)-(1.10) may fail to be convex, meaning that the critical points are not necessarily the Nash equilibrium. To overcome that, we prove that the second derivatives of $\mathcal{L}$ and $\mathcal{R}$ are positive in the canonical directions. Clearly, this will imply that l and r are local minima of $\mathcal{L}(\cdot, r)$ and $\mathcal{R}(l, \cdot)$, respectively. Since the critical points are caracterized by the unique solution of the optimality system (1.15), and the second derivative is positive, we can assure that (l, r) is the unique Nash equilibrium (1.4). Motivated by all this, we prove the following result.

From now on, we denote by $\mu = \min\{\mu_1, \mu_2, \nu_1, \nu_2\}$.

Theorem 1.1. *Let μ be sufficiently large. There exists $\delta > 0$ (independent of $\{\mu_i\}$, $\{\nu_i\}$) such that, if*

$$\|z_{i,d}\|_2 + \|f\|_{L^2(\mathcal{O}\times(0,T))} + \|z^0\|_{H^{-2}(0,L)} \leqslant \delta,$$

system (1.15) possesses a unique solution. If (l, r) is given by (1.13)-(1.14), there exists $C > 0$, independent of $\{\mu_i\}$, $\{\nu_i\}$, such that

$$\langle\langle D^2\mathcal{L}(f; l, r); (\hat{l}, 0), (\hat{l}, 0)\rangle \geq C \sum_{i=1}^{2} \int_0^T |\hat{l}^i|^2\,dt, \quad \forall \hat{l} \in [L^2(0,T)]^2, \tag{1.16}$$

and

$$\langle\langle D^2\mathcal{R}(f; l, r); (0, \hat{r}), (0, \hat{r})\rangle \geq C \sum_{i=1}^{2} \int_0^T |\hat{r}^i|^2\,dt, \quad \forall \hat{r} \in [L^2(0,T)]^2. \tag{1.17}$$

Once the Nash equilibrium is known and characterized by (1.13)-(1.14), we search for a leader control that drives the variable z appearing in (1.15) to zero. This problem gives us the main Theorem of this paper.

Theorem 1.2. *Let μ be sufficiently large. Assume that*

$$\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset \quad (i = 1, 2), \tag{1.18}$$

and that one of the following two conditions holds:

$$\mathcal{O}_{1,d} = \mathcal{O}_{2,d} \tag{1.19}$$

or

$$\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}. \tag{1.20}$$

Let $\bar{y} \in L^\infty(0,T;W^{1,\infty}((0,L)))$ *a given trajectory of the uncontrolled equation* (1.5). *Then, there exist* $\delta > 0$ *and positive functions* $\hat{\rho}_i = \hat{\rho}_i(t)$ *blowing up at* $t = T$ *such that if* z^0 *and* $\bar{y}$ *satisfy*

$$\|z^0\|^2_{H^{-2}(0,L)} + \sum_{i=1,2} \int_0^T \int_{\mathcal{O}_{i,d}} \hat{\rho}_i^2 |z_{i,d}|^2 \, dxdt < \delta, \tag{1.21}$$

there exist controls $f \in L^2(\mathcal{O} \times (0,T))$ *such that the solution of equation* (1.15)*con satisfies* $z(\cdot,T) = 0$.

Remark 1.3. *The change of roles for* l^i *or* r^i *in the minimization process of* (1.9)-(1.10) *does not significantly increase the difficulty in solving the associated multiobjective control problem. For example, interchanging* l^1 *by* r^1 *or* l^2 *by* r^2*, or even if we take some of them identically zero, as long as no function becomes with no follower acting on it, generates a problem that can be solved in a very similar way as we are going to do it here.*

Remark 1.4. *Another interesting problem is to consider the leader positioned on the boundary and the followers distributed inside the domain. The proof in this case is simpler and Section 6.1 contains a sketch of it.*

Let us give some guidelines of the proof. First, we prove the null controllability of the linearized system around zero

$$\begin{cases} z_t + z_{xxxx} + \nu z_{xx} + (\bar{y}z)_x = f^0 + f\,\mathbb{1}_{\mathcal{O}} & (x,t) \in Q, \\ -\phi^i_t + \phi^i_{xxxx} + \nu\phi^i_{xx} - \bar{y}\phi^i_x = f^i + \alpha_i z \mathbb{1}_{\mathcal{O}_{i,d}} \quad i = 1,2 & (x,t) \in Q, \\ z(0,t) = -\frac{1}{\mu_1}\phi^1_{xxx}(0,t) + g^1(t), z(L,t) = \frac{1}{\nu_1}\phi^2_{xxx}(L,t) + g^2(t) & t \in (0,T), \\ z_x(0,t) = \frac{1}{\mu_2}\phi^1_{xx}(0,t) + h^1(t), z_x(L,t) = -\frac{1}{\nu_2}\phi^2_{xx}(L,t) + h^2(t) & t \in (0,T), \\ \phi^i(0,t) = \phi^i(L,t) = \phi^i_x(0,t) = \phi^i_x(L,t) = 0 \quad i = 1,2 & t \in (0,T), \\ z(x,0) = z^0(x), \quad \phi^i(x,T) = 0 \quad i = 1,2 & x \in (0,L). \end{cases} \tag{1.22}$$

Note that (1.22) is nonhomogeneous, where $\{f^i\}$, $\{g^i\}$ and $\{h^i\}$ are (arbitrary) L^2-functions decaying exponentially to zero at $t = T$.

It is well known by now that, with the help of a classical duality argument, the null controllability of system (1.22) can be deduced from an observability inequality for the solutions of the adjoint system, which in this case is given by

$$\begin{cases} -\psi_t + \psi_{xxxx} + \nu\psi_{xx} - \bar{y}\psi_x = g^0 + \alpha_1\gamma^1\mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2\gamma^2\mathbb{1}_{\mathcal{O}_{2,d}} & (x,y) \in Q, \\ \gamma^i_t + \gamma^i_{xxxx} + \nu\gamma^i_{xx} + (\bar{y}\gamma^i)_x = g^i \quad i = 1,2 & (x,y) \in Q, \\ \psi(0,t) = \psi(L,t) = 0 & t \in (0,T), \\ \psi_x(0,t) = \psi_x(L,t) = 0 & t \in (0,T), \\ \gamma^1(0,t) = \frac{1}{\mu_1}\psi_{xxx}(0,t), \quad \gamma^1_x(0,t) = -\frac{1}{\mu_2}\psi_{xx}(0,t) & t \in (0,T), \\ \gamma^1(L,t) = \gamma^1_x(L,t) = 0 & t \in (0,T), \\ \gamma^2(0,t) = 0, \quad \gamma^2_x(0,t) = 0 & t \in (0,T), \\ \gamma^2(L,t) = \frac{1}{\nu_1}\psi_{xxx}(L,t), \quad \gamma^2_x(L,t) = -\frac{1}{\nu_2}\psi_{xx}(L,t) & t \in (0,T), \\ \psi(x,T) = \psi^T(x), \quad h(x,0) = 0 & x \in (0,L). \end{cases} \tag{1.23}$$

More precisely, to prove the null controllability of system (1.22), we prove that there exists $C > 0$ such that

$$\begin{aligned} &\int_0^T \bar{\rho}_0^2 \big(|\psi_{xxx}(0,t)|^2 + |\psi_{xxx}(L,t)|^2 + (|\psi_{xx}(0,t)|^2 + |\psi_{xx}(L,t)|^2)\big)\,dt \\ &\qquad + \int_0^L |\psi_{xx}(x,0)|^2 dx + \int_0^T\int_0^L \bar{\rho}_1^2|\psi_{xxxx}|^2\,dx\,dt + \sum_{i=2}^3 \int_0^T\int_0^L \bar{\rho}_i^2|\gamma^i|^2\,dx\,dt \\ &\qquad\qquad \leqslant C\sum_{i=0}^2 \int_0^T\int_0^L \widetilde{\rho}_i^2|g^i|^2\,dx\,dt + C\int_0^T\int_{\mathcal{O}} \widetilde{\rho}_3^2|\psi|^2\,dx\,dt, \end{aligned} \tag{1.24}$$

where $\{\hat{\rho}_i\}_{i=0}^3$ and $\{\tilde{\rho}_i\}_{i=0}^3$ are weight functions depending only on t and blowing up as $t \to T$. After proving this linear controllability result, we apply a suitable inversion mapping theorem in some very particular functional spaces to recover a local controllability result for the nonlinear system (1.15).

Observability inequality (1.24) is proved using Carleman estimates. The idea is to combine suitable Carleman estimates for each equation in system (1.23). The geometric assumptions (1.18)-(1.20) allow us to estimate the local terms (terms in $\mathcal{O}$) of γ^1 and γ^2 by the local term of ψ in the right-hand side of (1.24). The main idea is that, under (1.19), we are able to reduce the number of equations in (1.23) to only two. This is done considering the function $h = \alpha_1\gamma^1 + \alpha_2\gamma^2$. In the case that (1.19) does not hold, this reduction is no longer possible. Therefore, we have to deal with both γ^1 and γ^2. Assumption (1.20) allows to consider two different weight functions so that γ^1 and γ^2 are localized in disjoint sets in order to separate the couplings in the equation satisfied by ψ. Unfortunately, when $\mathcal{O}_{1,d}$ and $\mathcal{O}_{2,d}$ are different only outside of $\mathcal{O}$ (the only case that (1.19) and 1.20 do not cover), the previous ideas do not work. All the technical details are presented more extensively in Section 3.2.

The fact that the followers are located on the boundary translates in having the equations in (1.23) coupled on the boundary. More precisely, the boundary conditions for γ^1 and γ^2 are nonhomogeneous, which is the main difference with the full internal case in [6]. This means that the Carleman estimates applied to γ^1 and γ^2 need to be of a special kind, namely for equations with nonhomogeneous boundary conditions. This kind of Carleman estimates are in general more restrictive and give some weighted boundary integrals that need to be estimated, in addition to the local ones, which is a much technically difficult problem than the one in [6].

1.2 Organization of the paper

The paper is organized as follows. On Section 2, we prove Theorem 1.1. We start by establishing the well-posedness of system (1.15), and then we prove the inequalities (1.16)-(1.17). Section 3 is devoted to Carleman estimates, where we recall some known results and prove some new Carleman estimates for the adjoint system (1.23). In Section (4), we prove a null controllability result for the linearized system (1.22). Finally, in Section 5, we prove Theorem 1.2. In Appendix A, we have included some technical well-posedness results concerning equations and systems of fourth-order parabolic equations.

2 On the existence of the Nash equilibrium

This section is dedicated to the proof of Theorem 1.1. We start proving that (1.15) possesses a unique solution.

Proposition 2.1. *For any* $z_0 \in H^{-2}(0,L)$*,* f^0, f^1, f^2 *in* $L^2(0,T;H^{-2}(0,L))$ *and* $f \in L^2((0,L)\times(0,T))$*, and if* μ *is sufficiently large, there exists a unique triplet* (z,ϕ^1,ϕ^2) *such that*

$$\begin{gathered} z \in C([0,T];H^{-2}(0,L)) \cap L^2((0,L)\times(0,T)), \\ (\phi^1,\phi^2) \in [C([0,T];H_0^2(0,L)) \cap L^2(0,T;H^4(0,L))]^2, \end{gathered} \tag{2.1}$$

solution of (1.15) *that satisfies*

$$\begin{aligned} \|z\|^2_{L^\infty(0,T;H^{-2}(0,L))} + \iint_Q |z|^2\,dxdt \leqslant C\Big(& \|f^0\|^2_{L^2(0,T;H^{-2}(0,L))} + \|f\|^2_{L^2((0,L)\times(0,T)))} + \|z^0\|^2_{H^{-2}(0,L)} \\ & + \sum_{i=1}^2 \|f^i\|^2_{L^2(0,T;H^{-2}(0,L))}\Big). \end{aligned} \tag{2.2}$$

Proof. We fix $\hat{z} \in L^2(Q)$ and $\hat{\phi}^i \in L^\infty(0,T;H_0^2(0,L))$, and we consider the system

$$\left\{\begin{array}{ll} z_t + z_{xxxx} + \nu z_{xx} + (\bar{y}z)_x = f^0 + f\,\mathbb{1}_{\mathcal{O}} - \dfrac{1}{2}(\hat{z}^2)_x & (x,t) \in Q, \\ -\phi^i_t + \phi^i_{xxxx} + \nu\phi^i_{xx} - \bar{y}\phi^i_x = f^i + \alpha_i(z - z_{i,d})\mathbb{1}_{\mathcal{O}_{i,d}} + \hat{z}\hat{\phi}^i_x,\ i = 1,2, & (x,t) \in Q, \\ z(0,t) = -\frac{1}{\mu_1}\phi^1_{xxx}(0), z(L,t) = \frac{1}{\nu_1}\phi^2_{xxx}(L) & t \in (0,T), \\ z_x(0,t) = \frac{1}{\mu_2}\phi^1_{xx}(0), z_x(L,t) = -\frac{1}{\nu_2}\phi^2_{xx}(L) & t \in (0,T), \\ \phi^i(0,t) = \phi^i(L,t) = \phi^i_x(0,t) = \phi^i_x(L,t) = 0 \quad i = 1,2 & t \in (0,T), \\ z(x,0) = z^0(x), \quad \phi^i(x,T) = 0 \quad i = 1,2 & x \in (0,L). \end{array}\right. \tag{2.3}$$

From Proposition A.9, for μ sufficiently large, the mapping $\Pi : L^2(Q) \times [L^\infty(0,T;H_0^2(0,L))]^2 \to L^2(Q) \times [L^\infty(0,T;H_0^2(0,L))]^2$ such that $\Pi(\hat{z},\hat{\phi}^1,\hat{\phi}^2) = (z,\phi^1,\phi^2)$ is well defined. Moreover, since $\|(\hat{z}^2)_x\|_{L^1(0,T;H^{-2}(0,L))} \leqslant \|\hat{z}\|_2^2$ and $\|\hat{z}\hat{\phi}^i_x\|_2 \leqslant \|\hat{\phi}^i\|_{L^\infty(0,T;H_0^2(0,L))}\|\hat{z}\|_2$, we obtain from (A.21) that

$$\begin{aligned} \|\Pi(z,\phi^1,\phi^2)\|^2 \leqslant C(T,\nu,\|\bar{y}\|_2,\|\bar{y}_x\|_2)\Big(&\|f^0\|^2_{L^2(0,T;H^{-2}(0,L))} + \|f\|^2_{L^2((0,L)\times(0,T)))} + \|z^0\|^2_{H^{-2}(0,L)} \\ &+ \sum_{i=1}^{2}\|f^i\|^2_{L^2(0,T;H^{-2}(0,L))} + \|\hat{z}\|_2^4 + \|\hat{\phi}^i\|^2_{L^\infty(0,T;H_0^2(0,L))}\|\hat{z}\|_2^2\Big). \end{aligned} \tag{2.4}$$

Let

$$B(0,R) = \{(\hat{z},\hat{\phi}^1,\hat{\phi}^2) \in L^2(Q) \times [L^\infty(0,T;H_0^2(0,L))]^2; \|(\hat{z},\hat{\phi}^1,\hat{\phi}^2)\|_{L^2(Q)\times[L^\infty(0,T;H_0^2(0,L))]^2} \leqslant R\},$$

and let $r > 0$ such that

$$\|f^0\|^2_{L^2(0,T;H^{-2}(0,L))} + \|f\|^2_{L^2((0,L)\times(0,T)))} + \|z^0\|^2_{H^{-2}(0,L)} + \sum_{i=1}^{2}\|f^i\|^2_{L^2(0,T;H^{-2}(0,L))} \leqslant r.$$

Then,

$$\|\Pi(z,\phi^1,\phi^2)\|^2 \leqslant C(T,\nu,\|\bar{y}\|_2,\|\bar{y}_x\|_2)\left(r + R^4\right). \tag{2.5}$$

Then, we just have to choose r and R in such a way that $C(T,\nu,\|\bar{y}\|_2,\|\bar{y}_x\|_2)\left(r + R^4\right) \leqslant R^2$, for instance, $r \leqslant R^4$ and $R \leqslant 1/\sqrt{2C(T,\nu,\|\bar{y}\|_2,\|\bar{y}_x\|_2)}$. Note that, in this case, the parameter R is bounded uniformly in terms of $\{\mu_i\}$, $\{\nu_i\}$. Hence, we have that Π is well defined from $B(0,R)$ to $B(0,R)$. Moreover

$$\begin{aligned} &\|\Pi(z,\phi^1,\phi^2) - \Pi(z_*,\phi^1_*,\phi^2_*)\|^2 \\ &\quad\leqslant C(T,\nu,\|\bar{y}\|_2,\|\bar{y}_x\|_2)\Big(\|z - z_*\|_2^2\|z + z_*\|_2^2 + \|\phi^i_x\|_\infty^2\|z - z_*\|_2^2 + \|\phi^i_x - \phi^i_{*\,x}\|_\infty^2\|z_*\|_2^2\Big) \\ &\quad\leqslant R^2 C(T,\nu,\|\bar{y}\|_2,\|\bar{y}_x\|_2)\Big(\|z - z_*\|_2^2 + \|\phi^i_x - \phi^i_{*\,x}\|_\infty^2\Big). \end{aligned} \tag{2.6}$$

Finally, if R is sufficiently small depending only on T, ν, $\|\bar{y}\|_2$, $\|\bar{y}_x\|_2$, then Π is a contraction, and the conclusion follows from the Banach Fixed Point Theorem. Note that estimate (2.6) also provides the uniqueness of solution, while estimate (2.4) implies (2.2), all that by taking R sufficiently small. ☐

In Proposition 2.1 we proved the existence of solution to (1.15), which implies the existence of critical points in the sense (1.11)-(1.12). To finish the proof of Theorem 1.1, we just have to prove that (1.16)-(1.17) are true. For simplicity, we are going to concentrate on the proof of (1.16). The proof of (1.17) is completely analogous.

Let ϕ^1 be a solution of

$$\left\{\begin{array}{ll} -\phi^1_t + \phi^1_{xxxx} + \nu\phi^1_{xx} - (z + \bar{y})\phi^1_x = \alpha_1(z - z_{1,d})\mathbb{1}_{\mathcal{O}_{1,d}} & (x,t) \in Q, \\ \phi^1(0,t) = \phi^1(L,t) = \phi^1_x(0,t) = \phi^1_x(L,t) = 0 & t \in (0,T), \\ \phi^1(x,T) = 0 & x \in (0,L). \end{array}\right. \tag{2.7}$$

It is standard to prove that (see [2])

$$\langle D^2J_1(f;l,r);(\hat{l},0),(\hat{l},0)\rangle = \mu_1\sum_{i=1}^{2}\int_0^T\!\!\int_{\mathcal{O}_1}|\hat{l}^i|^2\,dt - \int_0^T\int_0^T \eta^1_{xxx}(0,t)\hat{l}^1(t)\,dt + \int_0^T\int_0^T \eta^1_{xx}(0,t)\hat{l}^2(t)\,dt, \quad (2.8)$$

where

$$\begin{cases} -\eta^1_t + \eta^1_{xxxx} + \nu\eta^1_{xx} - h\phi^1_x - (z+\bar{y})\eta^1_x = \alpha_1 h\mathbb{1}_{\mathcal{O}_{1,d}} & (x,t)\in Q,\\ h_t + h_{xxxx} + \nu h_{xx} + (hz + h\bar{y})_x = 0 & (x,t)\in Q,\\ \eta^1(0,t) = \eta^1(L,t) = \eta^1_x(0,t) = \eta^1_x(L,t) = 0 & t\in(0,T),\\ h(0,t) = \hat{l}^1(t),\ h_x(0,t) = \hat{l}^2(t),\ h(L,t)=0,\ h_x(L,t)=0 & t\in(0,T),\\ \eta^1(\cdot,T) = h(\cdot,0) = 0 & x\in(0,L). \end{cases} \quad (2.9)$$

We take, in particular, (l,r) a critical point for the functionals (1.9)-(1.10), characterized by (1.13)-(1.15), and such that (z,ϕ^1,ϕ^2) is uniformly bounded in terms of $\{\mu_i\}$, $\{\nu_i\}$. From Proposition 2.1 this is possible at least for small data. Then, from (A.18), we have that

$$\iint_Q |h|^2\,dxdt \leqslant 2e^{2\nu^2T+\|z\|_2^2+\|\bar{y}\|_2^2}\sum_{i=1}^{2}\int_0^T|\hat{l}^i(t)|^2\,dt, \quad (2.10)$$

while that for η^1 we obtain that

$$\iint_Q |\eta^1_{xxxx}|^2\,dxdt \leqslant 2e^{2\nu^2T+\|z\|_2^2+\|\bar{y}\|_2^2}\|h\|_2^2\big(\alpha_1^2 + \|\phi^1\|^2_{L^\infty(0,T;H_0^2(0,L))}\big). \quad (2.11)$$

By combining (2.10) and (2.11), we obtain that

$$\iint_Q |\eta^1_{xxxx}|^2\,dxdt \leqslant C(\nu,T,\|z\|_2^2,\|\bar{y}\|_2^2)\big(\alpha_1^2 + \|\phi^1\|^2_{L^\infty(0,T;H_0^2(0,L))}\big)\sum_{i=1}^{2}\int_0^T|\hat{l}^i(t)|^2\,dt. \quad (2.12)$$

From (2.8) and using a trace estimate, we have that

$$\begin{aligned}\langle D^2J_1(f;l,r);(\hat{l},0),(\hat{l},0)\rangle &\geq \mu_1\sum_{i=1}^{2}\int_0^T\!\!\int_{\mathcal{O}_1}|\hat{l}^i|^2\,dt - C\|\eta^1_{xxxx}\|_2\left(\sum_{i=1}^{2}\int_0^T|\hat{l}^i(t)|^2\,dt\right)^{1/2}\\ &\geq \big(\mu_1 - C(\nu,T,\|z\|_2^2,\|\bar{y}\|_2^2)\big(\alpha_1^2 + \|\phi^1\|^2_{L^\infty(0,T;H_0^2(0,L))}\big)\big)\sum_{i=1}^{2}\int_0^T|\hat{l}^i(t)|^2\,dt. \quad (2.13)\end{aligned}$$

In this way, if we take μ_1 sufficiently large, the second derivative is positive, and then Theorem 1.1 is proved.

Now we turn to the proof of Theorem 1.2, that is, a null controllability result for system (1.15).

3 Carleman estimates

In this section, we prove some Carleman estimates for the solutions of (1.23). Before that, we show some notations and preliminary results.

3.1 Notation and preliminary results

Let ω and ω_0 be non-empty open subsets of $(0,L)$ such that $\omega_0 \subset\subset \omega$. Let η_0 be a $C^4([0,L])$-function such that

$$\begin{cases} \eta_0 > 0 \text{ in } (0,L),\ \eta_0(0) = \eta_0(L) = 0,\\ |\eta_0'| > 0 \text{ in } [0,L]\setminus\overline{\omega}_0. \end{cases} \quad (3.1)$$

The existence of such a function in dimension higher than one is proved in [19, Lemma 1.1].

Next, we introduce the (positive) weight functions

$$\sigma(x,t) := \frac{\exp\big(4\lambda\|\eta_0\|_\infty\big) - \exp\big(\lambda(\|\eta_0\|_\infty + \eta_0(x))\big)}{t^{1/3}(T-t)^{1/3}}, \quad \xi(x,t) := \frac{\exp\big(\lambda(\|\eta_0\|_\infty + \eta_0(x))\big)}{t^{1/3}(T-t)^{1/3}}, \tag{3.2}$$

where $\lambda > 1$.

Consider the following notations:

$$\sigma^*(t) = \max_{x\in[0,L]} \sigma(x,t), \quad \hat{\sigma}(t) = \min_{x\in[0,L]} \sigma(x,t), \quad \xi^*(t) = \min_{x\in[0,L]} \xi(x,t), \quad \hat{\xi}(t) = \max_{x\in[0,L]} \xi(x,t).$$

Notice that

$$\begin{aligned} \sigma^*(t) &= \frac{\exp\big(4\lambda\|\eta_0\|_\infty\big) - \exp\big(\lambda\|\eta_0\|_\infty\big)}{t^{1/3}(T-t)^{1/3}}, \quad \xi^*(t) = \frac{\exp\big(\lambda\|\eta_0\|_\infty\big)}{t^{1/3}(T-t)^{1/3}}, \\ \hat{\sigma}(t) &= \frac{\exp\big(4\lambda\|\eta_0\|_\infty\big) - \exp\big(2\lambda\|\eta_0\|_\infty\big)}{t^{1/3}(T-t)^{1/3}}, \quad \hat{\xi}(t) = \frac{\exp\big(2\lambda\|\eta_0\|_\infty\big)}{t^{1/3}(T-t)^{1/3}}. \end{aligned} \tag{3.3}$$

A simple computation shows that, for any $m, n \in \mathbb{N}$, we have

$$1 \leqslant T^{2/3}\xi, \quad |\partial_t^m \partial_x^n \xi| \leqslant CT^m \lambda^n \xi^{1+3m}, \quad |\partial_t^m \partial_x^n \sigma| \leqslant CT^m \lambda^n \xi^{1+3m}, \tag{3.4}$$

for every $\lambda > 1$. These properties will be used several times in this paper.

For F and F^i in $L^2(Q)$, $i = 1, \dots, 4$, consider the following fourth order equation

$$\begin{cases} u_t + u_{xxxx} = F + \sum_{i=1}^4 \partial_x^i F^i & (x,t) \in Q, \\ u(0,t) = u(L,t) = 0 & t \in (0,T), \\ u_x(0,t) = u_x(L,t) = 0 & t \in (0,T), \\ u(x,0) = u^0(x) & x \in (0,L), \end{cases} \tag{3.5}$$

and define the following weighted energy

$$\begin{aligned} I(u) := &\int_0^T\!\!\int_0^L e^{-2s\sigma}\big(s^6\lambda^7\xi^6(|u_t|^2 + |u_{xxxx}|^2) + s^8\lambda^9\xi^8|u_{xxx}|^2\big)\,dx\,dt \\ &+ \int_0^T\!\!\int_0^L e^{-2s\sigma}\big(s^{10}\lambda^{11}\xi^{10}|u_{xx}|^2 + s^{12}\lambda^{13}\xi^{12}|u_x|^2 + s^{14}\lambda^{15}\xi^{14}|u|^2\big)\,dx\,dt. \end{aligned} \tag{3.6}$$

For the case where the right-hand side of (3.5) is $L^2(Q)$ only, we have the following result:

Proposition 3.1. *Let $F \in L^2(Q)$, $F^i = 0$ for all $i = 1, \dots, 4$ and $\omega \subset (0,L)$. Then, there exists $C(L,\omega) > 0$ such that for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$, we have*

$$I(u) \leqslant C\left(s^{14}\lambda^{15}\int_0^T\!\!\int_\omega e^{-2s\sigma}\xi^{14}|u|^2\,dx\,dt + s^7\lambda^7\int_0^T\!\!\int_0^L e^{-2s\sigma}\xi^7|F|^2\,dx\,dt\right), \tag{3.7}$$

for every solution u of (3.5).

The proof of Proposition 3.1 can be found in [10].

Now, let $H = H^4(0,L) \cap H_0^2(0,L)$ and assume that the right-hand side of equation (3.5) satisfies

$$F, F^i \in L^2((0,L)\times(0,T))\,(i = 1, \dots, 4) \quad \text{and} \quad F^3_{xxx}, F^4_{xxxx} \in L^2(0,T;H'). \tag{3.8}$$

We have the following result, whose proof can be found in [6].

Proposition 3.2. *Let F, F^i $(i = 1, \dots, 4)$ satisfying* (3.8) *and $\omega \subset (0, L)$. Then, there exists $C(L, \omega) > 0$ such that for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$, we have*

$$
\begin{aligned}
s^{14}\lambda^{15}\int_0^T\!\!\int_0^L e^{-2s\sigma}\xi^{14}|u|^2\,dx\,dt \leqslant C\bigg(& s^{14}\lambda^{15}\int_0^T\!\!\int_\omega e^{-2s\sigma}\xi^{14}|u|^2\,dx\,dt + s^7\lambda^7\int_0^T\!\!\int_0^L e^{-2s\sigma}\xi^7|F|^2\,dx\,dt \\
&+ s^9\lambda^9\int_0^T\!\!\int_0^L e^{-2s\sigma}\xi^9\big(|F^1|^2 + s^2\lambda^2\xi^2|F^2|^2 + s^4\lambda^4\xi^4|F^3|^2 + s^6\lambda^6\xi^6|F^4|^2\big)\,dx\,dt \\
&+ s^{12}\lambda^{12}\int_0^T e^{-2s\sigma^*}(\xi^*)^{12}\big(\big|F^3(L,t)\big|^2 + \big|F^3(0,t)\big|^2 + \big|F^4_x(L,t)\big|^2 + \big|F^4_x(0,t)\big|^2\big)\,dt \\
&+ s^{14}\lambda^{14}\int_0^T e^{-2s\sigma^*}(\xi^*)^{14}\big(\big|F^4(L,t)\big|^2 + \big|F^4(0,t)\big|^2\big)\,dt\bigg), \qquad (3.9)
\end{aligned}
$$

for every solution u of (3.5).

To give a sense to the trace terms of F^3 and F^4, let us recall a remark made in [6] about F^3 and F^4 (see also [16] where a similar situation occurs in the context of the heat equation). Observe that under (3.8), we can consider the dual product of the distribution F^4_{xxxx} with an element of $L^2(0, T; H)$ in the sense that there exist some functions g_1, g_2, h_1 and h_2 in $L^2(0, T)$ such that

$$
\begin{aligned}
\langle F^4_{xxxx}, u\rangle = &\int_0^T\!\!\int_0^L F^4 u_{xxxx}\,dx\,dt + \int_0^T \big(g_2(t)u_{xx}(L,t) - g_1(t)u_{xx}(0,t)\big)\,dt \\
&- \int_0^T \big(h_2(t)u_{xxx}(L,t) - h_1(t)u_{xxx}(0,t)\big)\,dt, \quad \forall u \in L^2(0, T; H).
\end{aligned}
$$

If F^4 were regular enough, we would have

$$
g_1(t) = F^4_x(0,t), \quad g_2(t) = F^4_x(L,t), \quad h_1(t) = F^4(0,t) \quad \text{and} \quad h_2(t) = F^4(L,t).
$$

The arguments also apply to F^3. Therefore, we use this notation for the trace terms of F^3 and F^4.

When the right-hand side is in $L^2(0, T; H^{-2}(0, L))$ and we have nonhomogeneous boundary conditions, then we have the following system

$$
\begin{cases}
u_t + u_{xxxx} = B_0 + \partial_x B_1 + \partial^2_{xx} B_2 & (x,t) \in Q, \\
u(0,t) = b_1(t), \quad u(L,t) = b_2(t) & t \in (0,T), \\
u_x(0,t) = b_3(t), \quad u_x(L,t) = b_4(t) & t \in (0,T), \\
u(x,0) = u^0(x) & x \in (0,L).
\end{cases} \qquad (3.10)
$$

In this case, the following Carleman estimate holds.

Proposition 3.3. *Let $B_0, B_1, B_2 \in L^2(Q)$ and $b_1, b_2, b_3, b_4 \in L^2(0, T)$, and $\omega \subset (0, L)$. Then, there exists $C(L, \omega) > 0$ such that for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$ we have*

$$
\begin{aligned}
s^7\lambda^8\iint_Q e^{-2s\sigma}\xi^7|u|^2\,dxdt \leqslant C\Bigg(&\iint_Q e^{-2s\sigma}\left(|B_0|^2 + s^2\lambda^2\xi^2|B_1|^2 + s^4\lambda^4\xi^4|B_2|^2\right)\,dxdt \\
&+ s^7\lambda^7\int_0^T e^{-2s\sigma^*}(\xi^*)^7\left(|b_1(t)|^2 + |b_2(t)|^2\right)\,dt + s^5\lambda^5\int_0^T e^{-2s\sigma^*}(\xi^*)^5\left(|b_3(t)|^2 + |b_4(t)|^2\right)\,dt \\
&+ s^7\lambda^8\int_0^T\!\!\int_\omega e^{-2s\sigma}\xi^7|u|^2\,dxdt\Bigg). \qquad (3.11)
\end{aligned}
$$

The proof of Proposition 3.3 can be found in [5].

The next steps are based in combining Carleman estimates (3.7), (3.9) and (3.11) to proving a new Carleman estimate to the solutions of (1.23). Before that, we prove three technical lemmas.

Lemma 3.4 (Local Term Estimate). *Let $\omega \subset (0,L)$ an open set, $\nu > 0$, $Y \in L^{\infty}(Q)$, F and G in $L^2(Q)$, and p, q two sufficiently regular functions satisfying*

$$\begin{cases} -p_t + p_{xxxx} + \nu p_{xx} - Y p_x = q + F & (x,t) \in \tilde{\omega} \times (0,T), \\ q_t + q_{xxxx} + \nu q_{xx} + (Yq)_x = G & (x,t) \in \tilde{\omega} \times (0,T), \end{cases} \tag{3.12}$$

where $\tilde{\omega}$ is any open subset of $(0,L)$ such that $\overline{\omega} \subset \tilde{\omega}$. Consider any function η satisfying (3.1), and weight functions σ, ξ defined by (3.2). Then, for $\epsilon > 0$, there exists a positive constant $C = C(T, \nu, \|Y\|_{\infty})$ such that

$$\int_0^T\!\!\int_{\omega} e^{-2s\sigma}\xi^7|q|^2\,dxdt \leqslant \epsilon \iint_Q e^{-2s\sigma}\xi^7|q|^2\,dxdt + C\left(\iint_Q e^{-2s\sigma}\xi^7(|F|^2+|G|^2)\,dxdt + \right.$$
$$\left. + s^2\lambda^2\int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^9\Big(s^6\lambda^6\xi^6|p|^2 + s^4\lambda^4\xi^4|p_x|^2 + s^2\lambda^2\xi^2|p_{xx}|^2 + |p_{xxx}|^2\Big)\,dxdt\right).$$

Proof. Let $\tilde{\theta} \in C_0^4(\tilde{\omega})$ such that $\tilde{\theta} = 1$ in ω. Then, from (3.12), we have

$$\int_0^T\!\!\int_{\omega} e^{-2s\sigma}\xi^7|q|^2\,dxdt \leqslant \int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^7\tilde{\theta}|q|^2\,dxdt$$
$$= \int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^7\tilde{\theta}q\big(-p_t + p_{xxxx} + \nu p_{xx} - Yp_x - F\big)\,dxdt. \tag{3.13}$$

By writing the right-hand side of (3.13) as a sum of five integrals, we denote each of them, respectively, by $A_1, \ldots, A_5$.

Let us estimate the terms A_i for $i = 1, \ldots, 5$. For the first one, we integrate by parts in time, and we get that

$$A_1 = \int_0^T\!\!\int_{\tilde{\omega}} \big(e^{-2s\sigma}\xi^7\big)_t\tilde{\theta}qp\,dxdt + \int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^7\tilde{\theta}q_tp\,dxdt = A_{1\,1} + A_{1\,2}.$$

From now on, ϵ will represent a positive real number, sufficiently small.

Using properties (3.4), we estimate $A_{1\,1}$ as follows

$$|A_{1\,1}| \leqslant \epsilon \iint_Q e^{-2s\sigma}\xi^7|q|^2\,dxdt + CT^2s^2\int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^{15}|p|^2\,dxdt. \tag{3.14}$$

The term $A_{1\,2}$ will be considered later.

For the term A_2, we integrate by parts several times and we obtain

$$A_2 = \int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^7\tilde{\theta}qp_{xxxx}\,dxdt$$
$$= \int_0^T\!\!\int_{\tilde{\omega}} \Big((e^{-2s\sigma}\xi^7\tilde{\theta})_{xxxx}q + 4(e^{-2s\sigma}\xi^7\tilde{\theta})_{xxx}q_x + 6(e^{-2s\sigma}\xi^7\tilde{\theta})_{xx}q_{xx}\Big)p\,dxdt$$
$$+ \int_0^T\!\!\int_{\tilde{\omega}} \Big(4(e^{-2s\sigma}\xi^7\tilde{\theta})_xq_{xxx} + e^{-2s\sigma}\xi^7\tilde{\theta}q_{xxxx}\Big)p\,dxdt = A_{21} + 4A_{22} + 6A_{23} + 4A_{24} + A_{25}. \tag{3.15}$$

Let us estimate each of the A_{2j} terms, for $j = 1, \ldots, 5$. For the first one, we use (3.4) and Young's inequality, to get

$$A_{21} \leqslant \epsilon \iint_Q e^{-2s\sigma}\xi^7|q|^2\,dxdt + Cs^8\lambda^8\int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^{15}|p|^2\,dxdt. \tag{3.16}$$

For the terms A_{2k} for $k = 2 \ldots 4$, we integrate by parts $k-1$ times and we use again Young's inequality to have

$$\sum_{k=2}^{4} A_{2k} \leqslant \epsilon \iint_Q e^{-2s\sigma}\xi^7|q|^2\,dxdt$$
$$+ Cs^2\lambda^2\int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^9\Big(s^6\lambda^6\xi^6|p|^2 + s^4\lambda^4\xi^4|p_x|^2 + s^2\lambda^2\xi^2|p_{xx}|^2 + |p_{xxx}|^2\Big)\,dxdt. \tag{3.17}$$

Now, we sum A_{12} and A_{25}, and use the second equation in (3.12) to have

$$\begin{aligned}A_{12}+A_{25}=\int_0^T\!\!\int_{\tilde\omega} e^{-2s\sigma}\xi^7\tilde\theta(q_t+q_{xxxx})p\,dxdt=\int_0^T\!\!\int_{\tilde\omega} e^{-2s\sigma}\xi^7\tilde\theta(-\nu q_{xx}-(Yq)_x+G)p\,dxdt\\ \leqslant \epsilon\iint_Q e^{-2s\sigma}\xi^7|q|^2\,dxdt+C(1+\nu^2+\|Y\|_\infty^2)s^4\lambda^4\int_0^T\!\!\int_{\tilde\omega} e^{-2s\sigma}\xi^{11}|p|^2\,dxdt\\ +C(\nu^2+\|Y\|_\infty^2)s^2\lambda^2\int_0^T\!\!\int_{\tilde\omega} e^{-2s\sigma}\xi^9|p_x|^2\,dxdt+C\nu^2\int_0^T\!\!\int_{\tilde\omega} e^{-2s\sigma}\xi^7|p_{xx}|^2\,dxdt\\ +C\iint_Q e^{-2s\sigma}\xi^7|G|^2\,dxdt.\quad(3.18)\end{aligned}$$

To estimate A_3, A_4 and A_5, we simply use Young's inequality, and we get that

$$\begin{aligned}\sum_{j=3}^5 A_j\leqslant \epsilon\iint_Q e^{-2s\sigma}\xi^7|q|^2\,dxdt+C\nu^2\int_0^T\!\!\int_{\tilde\omega} e^{-2s\sigma}\xi^7|p_{xx}|^2\,dxdt+C\|Y\|_\infty^2\int_0^T\!\!\int_{\tilde\omega} e^{-2s\sigma}\xi^7|p_x|^2\,dxdt\\ +C\iint_Q e^{-2s\sigma}\xi^7|F|^2\,dxdt.\quad(3.19)\end{aligned}$$

To finish the proof, we just combine (3.13)-(3.19). ☐

The next Lemma is an estimate of boundary terms:

Lemma 3.5. *Let p a sufficiently regular function, then*

$$\begin{aligned}\int_0^T e^{-2s\sigma^*}(\xi^*)^7\Big(|p_{xxx}(0,t)|^2+|p_{xxx}(L,t)|^2\Big)\,dt\\ \leqslant Cs\lambda\iint_Q e^{-2s\sigma^*}(\xi^*)^8|p_{xxx}|^2\,dxdt+\frac12 s^{-1}\lambda^{-1}\iint_Q e^{-2s\sigma^*}(\xi^*)^6|p_{xxxx}|^2\,dxdt,\quad(3.20)\end{aligned}$$

and

$$\begin{aligned}\int_0^T e^{-2s\sigma^*}(\xi^*)^5\left(|p_{xx}(0,t)|^2+|p_{xx}(L,t)|^2\right)\,dt\\ \leqslant C\left(\iint_Q e^{-2s\sigma^*}(\xi^*)^5|p_{xx}|^2\,dxdt+\iint_Q e^{-2s\sigma^*}(\xi^*)^5|p_{xxx}|^2\,dxdt\right).\quad(3.21)\end{aligned}$$

Proof. Let $r\in C^\infty([0,L])$ such that $r(L)=0$ and $r(0)=1$. Then,

$$\begin{aligned}\int_0^T e^{-2s\sigma^*}(\xi^*)^7|p_{xxx}(0,t)|^2\,dt=-\iint_Q e^{-2s\sigma^*}(\xi^*)^7\partial_x(r(x)|p_{xxx}|^2)\,dxdt\\ =-\iint_Q e^{-2s\sigma^*}(\xi^*)^7 r_x(x)|p_{xxx}|^2\,dxdt-2\iint_Q e^{-2s\sigma^*}(\xi^*)^7 r(x)p_{xxx}p_{xxxx}\,dt\\ \leqslant Cs\lambda\iint_Q e^{-2s\sigma^*}(\xi^*)^8|p_{xxx}|^2\,dxdt+\frac14 s^{-1}\lambda^{-1}\iint_Q e^{-2s\sigma^*}(\xi^*)^6|p_{xxxx}|^2\,dxdt.\quad(3.22)\end{aligned}$$

In a completely analogous way, we can prove the estimate for $p_{xxx}(L,t), p_{xx}(L,t)$ and $p_{xx}(0,t)$ to obtain (3.20) and (3.21). In that case, we use a function $\tilde r$ such that $\tilde r(L)=1$ and $\tilde r(0)=0$ instead of r. ☐

Now, we present the third, and last, technical Lemma.

Lemma 3.6. *For $i = 1, 2, 3$, let s_i and r_i be positive real numbers. Let ϵ a positive real number, $\tilde{\omega}$ and $\hat{\omega}$ two non-empty subsets of $(0, L)$ such that $\overline{\tilde{\omega}} \subset \hat{\omega}$, and p a regular function. There exists $C > 0$ such that*

$$\int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^{s_1}|p_x|^2\,dxdt \leqslant C(1+\epsilon^{-1})\int_0^T\!\!\int_{\hat{\omega}} e^{-2s(2\hat{\sigma}-\sigma^*)}\hat{\xi}^{2s_1}|p|^2\,dxdt + \epsilon\int_0^T\!\!\int_Q e^{-2s\sigma^*}(\xi^*)^{r_1}|p_{xx}|^2\,dxdt, \tag{3.23}$$

$$\begin{aligned}\int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^{s_2}|p_{xx}|^2\,dxdt \leqslant{}& C(1+\epsilon^{-3})\int_0^T\!\!\int_{\hat{\omega}} e^{-2s(4\hat{\sigma}-3\sigma^*)}\hat{\xi}^{4s_2}|p|^2\,dxdt\\ &+\epsilon\int_0^T\!\!\int_Q e^{-2s\sigma^*}(\xi^*)^{r_1}|p_{xx}|^2\,dxdt + \epsilon\int_0^T\!\!\int_Q e^{-2s\sigma^*}(\xi^*)^{r_2}|p_{xxx}|^2\,dxdt,\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}\int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^{s_3}|p_{xxx}|^2\,dxdt \leqslant{}& C(1+\epsilon^{-7})\int_0^T\!\!\int_{\hat{\omega}} e^{-2s(8\hat{\sigma}-7\sigma^*)}\hat{\xi}^{8s_3}|p|^2\,dxdt\\ \epsilon\int_0^T\!\!\int_Q e^{-2s\sigma^*}(\xi^*)^{r_3}|p_{xxxx}|^2\,dxdt &+ \epsilon\int_0^T\!\!\int_Q e^{-2s\sigma^*}(\xi^*)^{r_2}|p_{xxx}|^2\,dxdt + \epsilon\int_0^T\!\!\int_Q e^{-2s\sigma^*}(\xi^*)^{r^1}|p_{xx}|^2\,dxdt.\end{aligned} \tag{3.25}$$

Proof. Let ϵ a sufficiently small number and ω_i $(i = 1, \ldots, 4)$ be open subsets of $(0, L)$ such that

$$\tilde{\omega} \subset \omega_i \subset \overline{\omega_i} \subset \omega_{i+1} \subset \overline{\omega_{i+1}} \subset \hat{\omega},\ \ i = 1, 2, 3.$$

Consider also $\theta_i \in C_0^4(\omega_i)$, positive functions, such that $\theta_i = 1$ in ω_{i-1} for $i = 2, \ldots, 4$ and $\theta_1 = 1$ in $\tilde{\omega}$. In this way,

$$\begin{aligned}\int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^{s_1}|p_x|^2\,dxdt &\leqslant \int_0^T\!\!\int_{\omega_1} e^{-2s\hat{\sigma}}\hat{\xi}^{s_1}\theta_1|p_x|^2\,dxdt\\ &= \frac{1}{2}\int_0^T\!\!\int_{\omega_1} e^{-2s\hat{\sigma}}\hat{\xi}^{s_1}\theta_{1\,xx}|p|^2\,dxdt - \int_0^T\!\!\int_{\omega_1} e^{-2s\hat{\sigma}}\hat{\xi}^{s_1}\theta_1 p_{xx}p\,dxdt\\ &\leqslant C(1+\epsilon^{-1})\int_0^T\!\!\int_{\omega_1} e^{-2s(2\hat{\sigma}-\sigma^*)}\hat{\xi}^{2s_1}|p|^2\,dxdt + \epsilon\int_0^T\!\!\int_Q e^{-2s\sigma^*}(\xi^*)^{r_1}|p_{xx}|^2\,dxdt.\end{aligned} \tag{3.26}$$

Proceeding in a very similar way, we can estimate the term of second order derivative in the following way

$$\begin{aligned}\int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^{s_2}|p_{xx}|^2\,dxdt &\leqslant \int_0^T\!\!\int_{\omega_1} e^{-2s\hat{\sigma}}\hat{\xi}^{s_2}\theta_1|p_{xx}|^2\,dxdt\\ &\leqslant C(1+\epsilon^{-1})\int_0^T\!\!\int_{\omega_1} e^{-2s(2\hat{\sigma}-\sigma^*)}\hat{\xi}^{2s_2}|p_x|^2\,dxdt + \epsilon\int_0^T\!\!\int_Q e^{-2s\sigma\ *}(\xi^*)^{r_2}|p_{xxx}|^2\,dxdt.\end{aligned} \tag{3.27}$$

By similar computations as in (3.26)

$$\begin{aligned}&\int_0^T\!\!\int_{\omega_1} e^{-2s(2\hat{\sigma}-\sigma^*)}\hat{\xi}^{2s_2}|p_x|^2\,dxdt \leqslant \int_0^T\!\!\int_{\omega_2} e^{-2s(2\hat{\sigma}-\sigma^*)}\hat{\xi}^{2s_2}\theta_2|p_x|^2\,dxdt\\ &\leqslant C(1+\epsilon^{-2})\int_0^T\!\!\int_{\omega_2}\Big(e^{-2s(2\hat{\sigma}-\sigma^*)} + e^{-2s(4\hat{\sigma}-3\sigma^*)}\hat{\xi}^{2s_2}\Big)\hat{\xi}^{2s_2}|p|^2\,dxdt + \epsilon^2\int_0^T\!\!\int_Q e^{-2s\sigma^*}(\xi^*)^{r_1}|p_{xx}|^2\,dxdt.\end{aligned} \tag{3.28}$$

Then, combining (3.27) and (3.28), we get that

$$\int_0^T\int_{\tilde{\omega}} e^{-2s\sigma}\xi^{s_2}|p_{xx}|^2\,dxdt \leqslant C(1+\epsilon^{-3})\int_0^T\int_{\omega_2} e^{-2s(4\hat{\sigma}-3\sigma^*)}\hat{\xi}^{4s_2}|p|^2\,dxdt \\ +\epsilon\int_0^T\int_Q e^{-2s\sigma^*}(\xi^*)^{r_1}|p_{xx}|^2\,dxdt+\epsilon\int_0^T\int_Q e^{-2s\sigma^*}(\xi^*)^{r_2}|p_{xxx}|^2\,dxdt. \quad (3.29)$$

The third order term can be estimated in a very similar way.

$$\int_0^T\int_{\tilde{\omega}} e^{-2s\sigma}\xi^{s_3}|p_{xxx}|^2\,dxdt \leqslant \int_0^T\int_{\omega_1} e^{-2s\hat{\sigma}}\hat{\xi}^{s_3}\theta_1|p_{xxx}|^2\,dxdt \\ \leqslant C(1+\epsilon^{-1})\int_0^T\int_{\omega_1} e^{-2s(2\hat{\sigma}-\sigma^*)}\hat{\xi}^{2s_3}|p_{xx}|^2\,dxdt+\epsilon\int_0^T\int_Q e^{-2s\sigma^*}(\xi^*)^{r_3}|p_{xxxx}|^2\,dxdt. \quad (3.30)$$

Making similar computations as in (3.28) and (3.29)

$$\int_0^T\int_{\omega_1} e^{-2s(2\hat{\sigma}-\sigma^*)}\hat{\xi}^{2s_3}|p_{xx}|^2\,dxdt \leqslant \int_0^T\int_{\omega_2} e^{-2s(2\hat{\sigma}-\sigma^*)}\hat{\xi}^{2s_3}\theta_1|p_{xx}|^2\,dxdt \\ \leqslant C(1+\epsilon^{-2})\int_0^T\int_{\omega_2} e^{-2s(4\hat{\sigma}-3\sigma^*)}\hat{\xi}^{4s_3}|p_x|^2\,dxdt+\epsilon^2\int_0^T\int_Q e^{-2s\sigma^*}(\xi^*)^{r_2}|p_{xxx}|^2\,dxdt \\ \leqslant C(1+\epsilon^{-6})\int_0^T\int_{\omega_3} e^{-2s(8\hat{\sigma}-7\sigma^*)}\hat{\xi}^{8s_3}|p|^2\,dxdt+\epsilon^2\int_0^T\int_Q e^{-2s\sigma^*}(\xi^*)^{r_1}|p_{xx}|^2\,dxdt \\ +\epsilon^2\int_0^T\int_Q e^{-2s\sigma^*}(\xi^*)^{r_2}|p_{xxx}|^2\,dxdt \quad (3.31)$$

Combining (3.30) and (3.31), we see that

$$\int_0^T\int_{\tilde{\omega}} e^{-2s\sigma}\xi^{s_3}|p_{xxx}|^2\,dxdt \leqslant C(1+\epsilon^{-7})\int_0^T\int_{\omega_3} e^{-2s(8\hat{\sigma}-7\sigma^*)}\hat{\xi}^{8s_3}|p|^2\,dxdt \\ \epsilon\int_0^T\int_Q e^{-2s\sigma^*}(\xi^*)^{r_3}|p_{xxxx}|^2\,dxdt+\epsilon\int_0^T\int_Q e^{-2s\sigma^*}(\xi^*)^{r_2}|p_{xxx}|^2\,dxdt+\epsilon\int_0^T\int_Q e^{-2s\sigma^*}(\xi^*)^{r_1}|p_{xx}|^2\,dxdt. \quad (3.32)$$

□

3.2 New Carleman estimates

Now, we are ready to prove a new Carleman estimate to the solutions of (1.23).

3.2.1 Case 1: $\mathcal{O}_{1,d}=\mathcal{O}_{2,d}$

In this case, we define $\mathcal{O}_d=\mathcal{O}_{1,d}=\mathcal{O}_{2,d}$ and we set

$$h=\alpha_1\gamma^1+\alpha_2\gamma^2. \quad (3.33)$$

Then, it is easy to see that (ψ,h) satisfies

$$\begin{cases} -\psi_t+\psi_{xxxx}+\nu\psi_{xx}-\bar{y}\psi_x=g^0+h1\!\!1_{\mathcal{O}_d} & (x,t)\in Q, \\ h_t+h_{xxxx}+\nu h_{xx}+(\bar{y}h)_x=\sum_{i=1}^2\alpha_i g^i & (x,t)\in Q, \\ \psi(0,t)=\psi(L,t)=0 & t\in(0,T), \\ \psi_x(0,t)=\psi_x(L,t)=0 & t\in(0,T), \\ h(0,t)=\frac{\alpha_1}{\mu_1}\psi_{xxx}(0,t), \quad h_x(0,t)=-\frac{\alpha_1}{\mu_2}\psi_{xx}(0,t) & t\in(0,T), \\ h(L,t)=\frac{\alpha_2}{\nu_1}\psi_{xxx}(L,t), \quad h_x(L,t)=-\frac{\alpha_2}{\nu_2}\psi_{xx}(L,t) & t\in(0,T), \\ \psi(x,T)=\psi^T(x), \quad h(x,0)=0 & x\in(0,L). \end{cases} \quad (3.34)$$

Then, in this case, the following Carleman estimate holds

Proposition 3.7. *Assume that conditions* (1.18) *and* (1.19) *hold, and* $\bar{y} \in L^\infty(0,T;W^{1,\infty}((0,L)))$*. Then, there exists* $C = C(L, \mathcal{O}, \mathcal{O}_d, \alpha_i, \mu_i, \|\bar{y}\|_\infty, \|\bar{y}_x\|_\infty) > 0$ *such that for every* $s \geq C((1+\nu^{2/3}+\|\bar{y}\|_\infty^{2/5})T^{2/3} + T^{1/3})$ *and every* $\lambda \geq C$ *we have*

$$\begin{aligned}s^8\lambda^9 \int_0^T e^{-2s\sigma^*}(\xi^*)^8\Big(|\psi_{xxx}(0,t)|^2 + |\psi_{xxx}(L,t)|^2 + s^2\lambda^2(\xi^*)^2(|\psi_{xx}(0,t)|^2 + |\psi_{xx}(L,t)|^2)\Big)\,dt \\ + I(\psi) + s^7\lambda^8 \iint_Q e^{-2s\sigma}\xi^7|h|^2\,dxdt \leqslant C\Big(s^7\lambda^8 \iint_Q e^{-2s\sigma}\xi^7\Big(|g^0|^2+|g^1|^2+|g^2|^2\Big)\,dxdt \\ + Cs^{30}\lambda^{31}\int_0^T\int_{\mathcal{O}} e^{-2s(8\hat{\sigma}-7\sigma^*)}\hat{\xi}^{72}|\psi|^2\,dxdt\Big), \quad (3.35)\end{aligned}$$

for every (ψ, h) *solution of* (3.34).

Proof. Let $\omega \subset \mathcal{O}_d \cap \mathcal{O}$. By using (3.7) for ψ, we obtain that

$$\begin{aligned}I(\psi) \leqslant Cs^7\lambda^7\Bigg(s^7\lambda^8\int_0^L\int_\omega e^{-2s\sigma}\xi^{14}|\psi|^2\,dx\,dt \\ + \iint_Q e^{-2s\sigma}\xi^7\Big(|g^0|^2+|h|^2+\nu^2|\psi_{xx}|^2+\|\bar{y}\|_\infty^2|\psi_x|^2\Big)\,dxdt\Bigg), \quad (3.36)\end{aligned}$$

for every $s \geq C(T^{\frac{2}{3}} + T^{\frac{1}{3}})$ and every $\lambda \geq C$.

For $\epsilon > 0$ sufficiently small, we can take $\lambda \geq C$ and $s \geq C((\nu^{2/3} + \|\bar{y}\|^{2/5})T^{2/3})$, to get that

$$Cs^7\lambda^7\iint_Q\Big(\nu^2|\psi_{xx}|^2+\|\bar{y}\|_\infty^2|\psi_x|^2\Big)\,dxdt \leqslant \epsilon I(\psi). \quad (3.37)$$

Hence, we combine (3.36) and (3.37), obtaining

$$I(\psi) \leqslant C\left(s^{14}\lambda^{15}\int_0^L\int_\omega e^{-2s\sigma}\xi^{14}|\psi|^2\,dxdt + s^7\lambda^7\iint_Q e^{-2s\sigma}\xi^7\Big(|g^0|^2+|h|^2\Big)\,dxdt\right), \quad (3.38)$$

for every $s \geq C((1+\nu^{2/3}+\|\bar{y}\|_\infty^{2/5})T^{2/3} + T^{1/3})$ and every $\lambda \geq C$.

Now, using that $\bar{y} \in L^\infty(Q)$, and (3.11) for the function h, we get that

$$\begin{aligned}s^7\lambda^8\iint_Q e^{-2s\sigma}\xi^7|h|^2\,dxdt \leqslant C\Bigg(\iint_Q e^{-2s\sigma}|\alpha_1 g^1+\alpha_2 g^2|^2\,dxdt \\ + s^7\lambda^7\int_0^T e^{-2s\sigma^*}(\xi^*)^7\left(\frac{\alpha_1^2}{\mu_1^2}|\psi_{xxx}(0,t)|^2 + \frac{\alpha_2^2}{\nu_1^2}|\psi_{xxx}(L,t)|^2\right)dt \\ + s^5\lambda^5\int_0^T e^{-2s\sigma^*}(\xi^*)^5\left(\frac{\alpha_1^2}{\mu_2^2}|\psi_{xx}(0,t)|^2 + \frac{\alpha_2^2}{\nu_2^2}|\psi_{xx}(L,t)|^2\right)dt \\ + s^7\lambda^8\int_0^T\int_\omega e^{-2s\sigma}\xi^7|h|^2\,dxdt\Bigg), \quad (3.39)\end{aligned}$$

for every $s \geq C((1+\nu^{2/3}+\|\bar{y}\|_\infty^{2/5})T^{2/3} + T^{1/3})$ and every $\lambda \geq C$.

To estimate the local term of h in the right-hand side of (3.39), we use Lemma 3.4. Indeed, from (3.34), we have that

$$\begin{cases} -\psi_t + \psi_{xxxx} + \nu\psi_{xx} - \bar{y}\psi_x = h + g^0 & (x,t) \in \tilde{\omega}\times(0,T), \\ h_t + h_{xxxx} + \nu h_{xx} + (\bar{y}h)_x = \sum_{i=1}^2 \alpha_i g^i & (x,t) \in \tilde{\omega}\times(0,T), \end{cases} \quad (3.40)$$

for any $\tilde{\omega} \subset \mathcal{O}_d \cap \mathcal{O}$ strictly containing ω, more precisely, $\overline{\omega} \subset \tilde{\omega}$. Then, using Lemma 3.4 for $(q, p, Y, F, G) = (h, \psi, \bar{y}, g^0, \sum_{i=1}^{2} \alpha_i g^i)$, we get

$$\begin{aligned}\int_0^T\!\!\int_\omega e^{-2s\sigma}\xi^7|h|^2\,dxdt \leqslant C\iint_Q e^{-2s\sigma}\xi^7(|g^0|^2 + \sum_{i=1}^{2}\alpha_i^2|g^i|^2)\,dxdt + \epsilon\iint_Q e^{-2s\sigma}\xi^7|h|^2\,dxdt \\ + Cs^2\lambda^2\int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^9\Big(s^6\lambda^6\xi^6|\psi|^2 + s^4\lambda^4\xi^4|\psi_x|^2 + s^2\lambda^2\xi^2|\psi_{xx}|^2 + |\psi_{xxx}|^2\Big)\,dxdt. \qquad (3.41)\end{aligned}$$

Hence, from (3.39) and (3.41), we obtain

$$\begin{aligned}s^7\lambda^8\iint_Q e^{-2s\sigma}\xi^7|h|^2\,dxdt \leqslant C\Bigg(s^7\lambda^8\iint_Q e^{-2s\sigma}\xi^7\Big(|g^0|^2 + |g^1|^2 + |g^2|^2\Big)\,dxdt \\ + s^7\lambda^7\int_0^T e^{-2s\sigma^*}(\xi^*)^7\left(\frac{\alpha_1^2}{\mu_1^2}|\psi_{xxx}(0,t)|^2 + \frac{\alpha_2^2}{\nu_1^2}|\psi_{xxx}(L,t)|^2\right)\,dt \\ + s^5\lambda^5\int_0^T e^{-2s\sigma^*}(\xi^*)^5\left(\frac{\alpha_1^2}{\mu_2^2}|\psi_{xx}(0,t)|^2 + \frac{\alpha_2^2}{\nu_2^2}|\psi_{xx}(L,t)|^2\right)\,dt \\ +s^9\lambda^{10}\int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^9\Big(s^6\lambda^6\xi^6|\psi|^2 + s^4\lambda^4\xi^4|\psi_x|^2 + s^2\lambda^2\xi^2|\psi_{xx}|^2 + |\psi_{xxx}|^2\Big)\,dxdt\Bigg), \qquad (3.42)\end{aligned}$$

for every $s \geq C((1 + \nu^{2/3} + \|\bar{y}\|_\infty^{2/5})T^{2/3} + T^{1/3})$ and every $\lambda \geq C$.

Now, we use Lemma 3.5 to estimate the boundary terms in (3.42), obtaining

$$\begin{aligned}s^7\lambda^8\iint_Q e^{-2s\sigma}\xi^7|h|^2\,dxdt \leqslant C\Bigg(s^7\lambda^8\iint_Q e^{-2s\sigma}\xi^7\Big(|g^0|^2 + |g^1|^2 + |g^2|^2\Big)\,dxdt \\ + \frac{C}{\mu}s^5\lambda^5\Big(s\lambda\iint_Q e^{-2s\alpha}(\xi)^6|\psi_{xxxx}|^2\,dxdt + s^3\lambda^3\iint_Q e^{-2s\alpha}(\xi)^8|\psi_{xxx}|^2\,dxdt + \iint_Q e^{-2s\alpha}(\xi)^5|\psi_{xx}|^2\,dxdt\Big) \\ +s^9\lambda^{10}\int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^9\Big(s^6\lambda^6\xi^6|\psi|^2 + s^4\lambda^4\xi^4|\psi_x|^2 + s^2\lambda^2\xi^2|\psi_{xx}|^2 + |\psi_{xxx}|^2\Big)\,dxdt\Bigg), \qquad (3.43)\end{aligned}$$

for every $s \geq C((1 + \nu^{2/3} + \|\bar{y}\|_\infty^{2/5})T^{2/3} + T^{1/3})$ and every $\lambda \geq C$, where $\mu = \min\{\mu_1, \mu_2, \nu_1, \nu_2\}$. Hence, we can take μ sufficiently large such that

$$\begin{aligned}s^7\lambda^8\iint_Q e^{-2s\sigma}\xi^7|h|^2\,dxdt \leqslant C\Bigg(s^7\lambda^8\iint_Q e^{-2s\sigma}\xi^7\Big(|g^0|^2 + |g^1|^2 + |g^2|^2\Big)\,dxdt + \frac{1}{4}I(\psi) \\ +s^9\lambda^{10}\int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^9\Big(s^6\lambda^6\xi^6|\psi|^2 + s^4\lambda^4\xi^4|\psi_x|^2 + s^2\lambda^2\xi^2|\psi_{xx}|^2 + |\psi_{xxx}|^2\Big)\,dxdt\Bigg), \qquad (3.44)\end{aligned}$$

for every $s \geq C((1 + \nu^{2/3} + \|\bar{y}\|_\infty^{2/5})T^{2/3} + T^{1/3})$ and $\lambda \geq C$.

Now, we take $\hat{\omega} \subset \mathcal{O}$ an open subset such that $\overline{\tilde{\omega}} \subset \hat{\omega}$ and we use Lemma 3.6 to estimate the local terms of $\partial_x^i\psi$, $i = 1, 2, 3$, on the right-hand side of (3.44). In this way, by taking $\epsilon < \frac{1}{4}s^{-3}\lambda^{-3}$, we obtain

$$\begin{aligned}s^9\lambda^{10}\int_0^T\!\!\int_{\tilde{\omega}} e^{-2s\sigma}\xi^9\Big(s^6\lambda^6\xi^6|\psi|^2 + s^4\lambda^4\xi^4|\psi_x|^2 + s^2\lambda^2\xi^2|\psi_{xx}|^2 + |\psi_{xxx}|^2\Big)\,dxdt \\ \leqslant Cs^{30}\lambda^{31}\int_0^T\!\!\int_{\hat{\omega}} e^{-2s(8\hat{\sigma} - 7\sigma^*)}\hat{\xi}^{72}|\psi|^2\,dxdt + \frac{1}{4}I(\psi). \qquad (3.45)\end{aligned}$$

Combining (3.45) and (3.44), we get that

$$\begin{aligned}s^7\lambda^8\iint_Q e^{-2s\sigma}\xi^7|h|^2\,dxdt \leqslant C\Bigg(s^7\lambda^8\iint_Q e^{-2s\sigma}\xi^7\Big(|g^0|^2 + |g^1|^2 + |g^2|^2\Big)\,dxdt \\ Cs^{30}\lambda^{31}\int_0^T\!\!\int_{\hat{\omega}} e^{-2s(8\hat{\sigma} - 7\sigma^*)}\hat{\xi}^{72}|\psi|^2\,dxdt + \frac{1}{2}I(\psi), \qquad (3.46)\end{aligned}$$

for every $s \geq C((1+\nu^{2/3}+\|\overline{y}\|_\infty^{2/5})T^{2/3}+T^{1/3})$ and $\lambda \geq C$.

Now, we sum (3.46) and (3.38) to obtain that

$$I(\psi)+s^7\lambda^8\iint_Q e^{-2s\sigma}\xi^7|h|^2\,dxdt \leqslant C\Big(s^7\lambda^8\iint_Q e^{-2s\sigma}\xi^7\Big(|g^0|^2+|g^1|^2+|g^2|^2\Big)\,dxdt \\ +Cs^{30}\lambda^{31}\int_0^T\int_{\hat{\omega}} e^{-2s(8\hat{\sigma}-7\sigma^*)}\hat{\xi}^{72}|\psi|^2\,dxdt\Big), \quad (3.47)$$

for every $s \geq C((1+\nu^{2/3}+\|\overline{y}\|_\infty^{2/5})T^{2/3}+T^{1/3})$ and $\lambda \geq C$.

Now, a simple localization argument shows that

$$s^8\lambda^9\int_0^T e^{-2s\sigma^*}(\xi^*)^8\Big(|\psi_{xxx}(0,t)|^2+|\psi_{xxx}(L,t)|^2+s^2\lambda^2(\xi^*)^2(|\psi_{xx}(0,t)|^2+|\psi_{xx}(L,t)|^2)\Big)\,dt \leqslant CI(\psi). \quad (3.48)$$

Finally, we combine (3.48) and (3.47) obtaining (3.35).

□

3.2.2 Case 2: $\mathcal{O}_{1,d} \neq \mathcal{O}_{2,d}$

In Case 1, since $\mathcal{O}_{1,d}=\mathcal{O}_{2,d}$, we could reduce the quantity of variables from three to two by defining h as in (3.33). Now, the situation is different and the arguments presented to prove Carleman estimate (3.35) are no longer viable.

Since we are assuming (1.18), there are ω_1 and ω_2 two disjoint non-empty open subsets of $(0,L)$ such that

$$\overline{\omega}_1 \subset \mathcal{O}_{1,d}\cap\mathcal{O} \quad \text{and} \quad \overline{\omega}_2 \subset \mathcal{O}_{2,d}\cap\mathcal{O}. \quad (3.49)$$

Let $\mathcal{O}_0$ be a non-empty subset of $\mathcal{O}$ such that $\overline{\mathcal{O}_0}\subset\mathcal{O}$ and

$$\overline{\omega}_1 \subset \mathcal{O}_0 \quad \text{and} \quad \overline{\omega}_2 \subset \mathcal{O}_0. \quad (3.50)$$

For $i=1,2$, let η_i be $C^4([0,L])$-functions such that

$$\begin{cases} \eta_i>0 \quad \text{in } (0,L), \quad \eta_i(0)=\eta_i(L)=0, \\ |\eta_i'|>0 \quad \text{in } [0,L]\setminus\overline{\omega}_i, \\ \eta_1\equiv\eta_2 \quad \text{in } [0,L]\setminus\mathcal{O}_0, \\ \|\eta_1\|_\infty=\|\eta_2\|_\infty. \end{cases} \quad (3.51)$$

Conditions in (3.51) say that these functions have their critical points in disjoint sets, but coincide outside a set containing them. This property will be crucial in our argument. The existence of such functions in this case (one dimension in space) does not require much discussion. However, a proof in dimension higher than one can be found in [3].

Analogously as in (3.2), for $i=1,2$, we define the weight functions

$$\sigma_i(x,t):=\frac{\exp\big(4\lambda\|\eta_i\|_\infty\big)-\exp\big(\lambda(\|\eta_i\|_\infty+\eta_i(x))\big)}{t^{1/3}(T-t)^{1/3}}, \quad \xi_i(x,t):=\frac{\exp\big(\lambda(\|\eta_i\|_\infty+\eta_i(x))\big)}{t^{1/3}(T-t)^{1/3}}, \quad (3.52)$$

where $\lambda>1$, and $\hat{\sigma}_i$, σ_i^*, $\hat{\xi}_i$ and ξ_i^* as in (3.3). We denote by $I_i(\cdot)$ the corresponding weighted energy as in (3.6). Of course, the set $\mathcal{O}_0$ and the functions in (3.51)-(3.52) depend on the choice of ω_1 and ω_2 satisfying (3.49). They will be chosen appropriately in the proof of Proposition 3.8 below.

Assumption (1.20) (in addition to (1.18)) means that the following property holds:

$$[\mathcal{O}_{i,d}\cap\mathcal{O}]\setminus[\mathcal{O}_{j,d}\cap\mathcal{O}]\neq\emptyset \quad \text{for some } i,j\in\{1,2\},\ i\neq j. \quad (3.53)$$

Since now the function h in (3.33) is no longer useful in this case, we need to deal with γ^1 and γ^2 directly. The property (3.53) allows us to have local terms of γ^1 and γ^2 (in ω_1 and ω_2, respectively) in such a way that one do not interfere with the other when using Lemma 3.4.

For the previous reason, the case where (1.19) and (1.20) do not hold, that is, the case where $\mathcal{O}_{1,d}$ and $\mathcal{O}_{2,d}$ differ only outside $\mathcal{O}$ is completely open. However, this is only a technical limitation of the method, so other techniques may be applied to deal with this special case.

Now we have the following result.

Proposition 3.8. *Assume that conditions* (1.18) *and* (1.20) *hold, and* $\bar{y} \in L^\infty(0,T;W^{1,\infty}((0,L)))$. *Then, there exists* $C = C(L, \mathcal{O}, \mathcal{O}_{i,d}, \alpha_i, \mu_i, \|\bar{y}\|_\infty, \|\bar{y}_x\|_\infty) > 0$ *such that for every* $s \geq C((1+\nu^2+\|\bar{y}\|^2_\infty+\|\bar{y}_x\|^2_\infty)T^{\frac{2}{3}} + T^{\frac{1}{3}})$, $\lambda \geq C$ *we have*

$$\begin{aligned}
s^{\frac{21}{2}}\lambda^{\frac{23}{2}} &\int_0^T e^{-2s\sigma_1^*}(\xi_1^*)^8\big(|\psi_{xxx}(0,t)|^2 + |\psi_{xxx}(L,t)|^2 + (\xi_1^*)^2(|\psi_{xx}(0,t)|^2 + |\psi_{xx}(L,t)|^2)\big)\,dt \\
&+ \sum_{i=1}^4 s^{14-\frac{7i}{4}}\lambda^{15-\frac{7i}{4}} \iint_Q e^{-2s\sigma_1^*}(\xi_1^*)^{14-2i}|\partial_x^i\psi|^2\,dxdt + s^{14}\lambda^{15}\int_0^T\int_0^L e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dxdt \\
&+ s^7\lambda^8\sum_{i=1}^2\iint_Q e^{-2s\alpha_i}\xi_i^7|\gamma^i|^2\,dxdt \leqslant C\left(s^{30}\lambda^{31}\int_0^T\int_{\mathcal{O}} e^{-2s(8\hat{\sigma}-7\sigma_1^*)}\hat{\xi}_1^{72}|\psi|^2\,dxdt + \right. \\
&\left. s^7\lambda^8\iint_Q\Big(\sum_{i=1}^2 e^{-2s\sigma_i}\xi_i^7|g^0|^2 + e^{-2s\sigma_1}\xi_1^7|g^1|^2 + e^{-2s\sigma_2}\xi_2^7|g^2|^2\Big)\,dxdt\right), \qquad (3.54)
\end{aligned}$$

for every solution $(\psi, \gamma^1, \gamma^2)$ *of system* (1.23).

Proof. Let $\Lambda \in C^\infty(\mathbb{R})$ such that

$$\Lambda(x) = \begin{cases} 0, & x \in \mathcal{O}_0, \\ 1, & x \in \mathbb{R}\setminus\mathcal{O}. \end{cases} \qquad (3.55)$$

Noticing that $\Lambda\psi$ satisfies

$$-(\Lambda\psi)_t + (\Lambda\psi)_{xxxx} = \Lambda g^0 + \alpha_1\Lambda\gamma^1\mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2\Lambda\gamma^2\mathbb{1}_{\mathcal{O}_{2,d}} + R(\psi) \qquad (3.56)$$

where

$$R(\psi) = -2\nu\Lambda''\psi - \bar{y}\Lambda'\psi - \bar{y}_x\Lambda\psi - \Lambda''''\psi + (2\nu\Lambda'\psi + \bar{y}\Lambda\psi + 4\Lambda'''\psi)_x - (\nu\Lambda\psi + 6\Lambda''\psi)_{xx} + 4(\Lambda'\psi)_{xxx},$$

and using Proposition 3.2, we get

$$\begin{aligned}
s^{14}\lambda^{15}\int_0^T\int_0^L e^{-2s\sigma_1}\xi_1^{14}|\Lambda\psi|^2\,dxdt &\leqslant C\Big(s^7\lambda^7\int_0^T\int_0^L e^{-2s\sigma_1}\xi_1^7|g^0|^2\,dxdt \\
&+ s^7\lambda^7\int_0^T\int_0^L e^{-2s\sigma_1}\xi_1^7\big(|\Lambda\gamma^1|^2 + |\Lambda\gamma^2|^2\big)\,dxdt + s^{14}\lambda^{15}\int_0^T\int_{\omega_1} e^{-2s\sigma_1}\xi_1^{14}|\Lambda\psi|^2\,dxdt \\
&+ \big(1+\nu^2+\|\bar{y}\|^2_\infty + \|\bar{y}_x\|^2_\infty\big)s^{13}\lambda^{13}\int_0^T\int_0^L e^{-2s\sigma_1}\xi_1^{13}|\psi|^2\,dx\,dt\Big), \qquad (3.57)
\end{aligned}$$

for every $s \geq C(T^{\frac{2}{3}} + T^{\frac{1}{3}})$ and every $\lambda \geq C$, where we have used properties (3.4). From the definition of Λ, we have

$$s^{14}\lambda^{15}\int_0^T\int_0^L e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dx\,dt - s^{14}\lambda^{15}\int_0^T\int_{\mathcal{O}} e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dx\,dt \leqslant s^{14}\lambda^{15}\int_0^T\int_0^L e^{-2s\sigma_1}\xi_1^{14}|\Lambda\psi|^2\,dx\,dt,$$

and since the weight functions are equal outside $\mathcal{O}_0$ (see (3.51)),

$$\int_0^T\int_0^L e^{-2s\sigma_1}\xi_1^7\big(|\Lambda\gamma^1|^2 + |\Lambda\gamma^2|^2\big)\,dx\,dt = \int_0^T\int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_1}\xi_1^7|\Lambda\gamma^1|^2\,dx\,dt + \int_0^T\int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_2}\xi_2^7|\Lambda\gamma^2|^2\,dx\,dt.$$

Going back to (3.57), we have

$$s^{14}\lambda^{15}\int_0^T\int_0^L e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dxdt \leqslant C\Big(s^7\lambda^7\int_0^T\int_0^L e^{-2s\sigma_1}\xi_1^7|g^0|^2\,dx\,dt$$
$$+\,s^7\lambda^7\sum_{i=1}^2\int_0^T\int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_i}\xi_i^7|\Lambda\gamma^i|^2\,dx\,dt + s^{14}\lambda^{15}\int_0^T\int_{\mathcal{O}} e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dx\,dt\Big), \qquad (3.58)$$

for every $s \geq C((1+\nu^2+\|\bar{y}\|_\infty^2+\|\bar{y}_x\|_\infty^2)T^{\frac{2}{3}}+T^{\frac{1}{3}})$ and every $\lambda \geq C$.

At this point, we distinguish three cases:

(i) $(\mathcal{O}_{1,d}\cap\mathcal{O})\cap(\mathcal{O}_{2,d}\cap\mathcal{O})=\emptyset$.

(ii) $(\mathcal{O}_{1,d}\cap\mathcal{O})\cap(\mathcal{O}_{2,d}\cap\mathcal{O})\neq\emptyset$ and $(\mathcal{O}_{1,d}\cap\mathcal{O})\setminus(\mathcal{O}_{2,d}\cap\mathcal{O})\neq\emptyset$.

(iii) $(\mathcal{O}_{1,d}\cap\mathcal{O})\cap(\mathcal{O}_{2,d}\cap\mathcal{O})\neq\emptyset$ and $(\mathcal{O}_{2,d}\cap\mathcal{O})\setminus(\mathcal{O}_{1,d}\cap\mathcal{O})\neq\emptyset$.

The cases (ii) and (iii) are completely analogous, so the proof for the case (iii) will be omitted.
Case (i). We take ω_1 and ω_2 satisfying (3.49) and (3.50) such that

$$\omega_1\cap\mathcal{O}_{2,d}=\emptyset \text{ and } \omega_2\cap\mathcal{O}_{1,d}=\emptyset.$$

We use (3.11) for each γ^i for the corresponding weight η_i and we get

$$s^7\lambda^8\sum_{i=1}^2\iint_Q e^{-2s\alpha_i}\xi_i^7|\gamma^i|^2\,dxdt \leqslant C\sum_{i=1}^2\bigg(\iint_Q e^{-2s\alpha_i}|g^i|^2\,dxdt$$
$$s^5\lambda^5\int_0^T e^{-2s\alpha_1^*}(\xi_1^*)^5\left(\frac{s^2\lambda^2}{\mu_1^2}(\xi_1^*)^2|\psi_{xxx}(0,t)|^2+\frac{1}{\mu_2^2}|\psi_{xx}(0,t)|^2\right)dt$$
$$+\,s^5\lambda^5\int_0^T e^{-2s\alpha_2^*}(\xi_2^*)^5\left(\frac{s^2\lambda^2}{\nu_1^2}(\xi_2^*)^2|\psi_{xxx}(L,t)|^2+\frac{1}{\nu_2^2}|\psi_{xx}(L,t)|^2\right)dt$$
$$+s^7\lambda^8\sum_{i=1}^2\int_0^T\int_{\omega_i} e^{-2s\alpha_i}\xi_i^7|\gamma^i|^2\,dxdt\bigg), \qquad (3.59)$$

for every $s\geq C(T^{2/3}+T^{1/3})$ and every $\lambda\geq C$.

Let $\tilde{\omega}_i$ an open subset of $(0,L)$ such that $\overline{\omega_i}\subset\tilde{\omega}_i$. Since (i) holds, we can use the first equation of (1.23) and get that

$$\begin{cases} -\psi_t+\psi_{xxxx}+\nu\psi_{xx}-\bar{y}\psi_x = g^0+\alpha_i\gamma^i\mathbb{1}_{\mathcal{O}_{i,d}} & (x,t)\in\tilde{\omega}_i\times(0,T),\\ \gamma_t^i+\gamma_{xxxx}^i+\nu\gamma_{xx}^i+(\bar{y}\gamma^i)_x = g^i & (x,t)\in\tilde{\omega}_i\times(0,T). \end{cases}$$

In this way, we use Lemma 3.4 with $\omega=\omega_i$ and $\tilde{\omega}=\tilde{\omega}_i$, and proceed in a similar way as in (3.41), to prove that

$$s^7\lambda^8\sum_{i=1}^2\iint_Q e^{-2s\sigma_i}\xi_i^7|\gamma_i|^2\,dxdt \leqslant C\left(s^7\lambda^8\iint_Q\Big(\sum_{i=1}^2 e^{-2s\sigma_i}|g^0|^2+e^{-2s\sigma_1}|g^1|^2+e^{-2s\sigma_2}|g^2|^2\Big)dxdt\right.$$
$$s^5\lambda^5\int_0^T e^{-2s\alpha_1^*}(\xi_1^*)^5\left(\frac{s^2\lambda^2}{\mu_1^2}(\xi_1^*)^2|\psi_{xxx}(0,t)|^2+\frac{1}{\mu_2^2}|\psi_{xx}(0,t)|^2\right)dt$$
$$+\,s^5\lambda^5\int_0^T e^{-2s\alpha_2^*}(\xi_2^*)^5\left(\frac{s^2\lambda^2}{\nu_1^2}(\xi_2^*)^2|\psi_{xxx}(L,t)|^2+\frac{1}{\nu_2^2}|\psi_{xx}(L,t)|^2\right)dt$$
$$\left.+s^9\lambda^{10}\int_0^T\int_{\tilde{\omega}}\sum_{i=1}^2 e^{-2s\sigma_i}\xi_i^9\Big(s^6\lambda^6\xi_i^6|\psi|^2+s^4\lambda^4\xi_i^4|\psi_x|^2+s^2\lambda^2\xi_i^2|\psi_{xx}|^2+|\psi_{xxx}|^2\Big)dxdt\right), \qquad (3.60)$$

for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$, where $\tilde{\omega}$ is an open set such that $\overline{\tilde{\omega}_1 \cup \tilde{\omega}_2} \subset \tilde{\omega}$.

Now, we use Lemma 3.5, combined with the fact that $\|\eta_1\| = \|\eta_2\|$, $\xi_1^* = \xi_2^*$ and $\alpha_1^* = \alpha_2^*$, and we obtain that

$$\begin{aligned}
&s^7\lambda^8 \sum_{i=1}^{2} \iint_Q e^{-2s\sigma_i} \xi_i^7 |\gamma_i|^2 \, dxdt \leqslant C \Bigg(s^7\lambda^8 \iint_Q \Big(\sum_{i=1}^{2} e^{-2s\sigma_i} |g^0|^2 + e^{-2s\sigma_1} |g^1|^2 + e^{-2s\sigma_2} |g^2|^2 \Big) \, dxdt \\
&\frac{C}{\mu^2} s^5\lambda^5 \Big(s\lambda \iint_Q e^{-2s\alpha_1^*} (\xi_1^*)^6 |\psi_{xxxx}|^2 \, dxdt + s^3\lambda^3 \iint_Q e^{-2s\alpha_1^*} (\xi_1^*)^8 |\psi_{xxx}|^2 \, dxdt + \iint_Q e^{-2s\alpha_1^*} (\xi_1^*)^5 |\psi_{xx}|^2 \, dxdt \Big) \\
&\qquad + s^9\lambda^{10} \int_0^T \!\! \int_{\tilde{\omega}} \sum_{i=1}^{2} e^{-2s\sigma_i} \xi_i^9 \Big(s^6\lambda^6 \xi_i^6 |\psi|^2 + s^4\lambda^4 \xi_i^4 |\psi_x|^2 + s^2\lambda^2 \xi_i^2 |\psi_{xx}|^2 + |\psi_{xxx}|^2 \Big) \, dxdt \Bigg), \qquad (3.61)
\end{aligned}$$

for every $s \geq C(T^{2/3} + T^{1/3})$ and every $\lambda \geq C$, where $\mu = \min\{\mu_1, \mu_2, \nu_1, \nu_2\}$.

Now, we sum (3.61) and (3.58), getting that

$$\begin{aligned}
&s^{14}\lambda^{15} \int_0^T \!\! \int_0^L e^{-2s\sigma_1} \xi_1^{14} |\psi|^2 \, dx \, dt + s^7\lambda^8 \sum_{i=1}^{2} \iint_Q e^{-2s\alpha_i} \xi_i^7 |\gamma^i|^2 \, dxdt \leqslant C \Bigg(s^{14}\lambda^{15} \int_0^T \!\! \int_{\mathcal{O}} e^{-2s\sigma_1} \xi_1^{14} |\psi|^2 \, dx \, dt \\
&\frac{C}{\mu^2} s^5\lambda^5 \Big(s\lambda \iint_Q e^{-2s\alpha_1^*} (\xi_1^*)^6 |\psi_{xxxx}|^2 \, dxdt + s^3\lambda^3 \iint_Q e^{-2s\alpha_1^*} (\xi_1^*)^8 |\psi_{xxx}|^2 \, dxdt + \iint_Q e^{-2s\alpha_1^*} (\xi_1^*)^5 |\psi_{xx}|^2 \, dxdt \Big) \\
&\qquad + s^9\lambda^{10} \int_0^T \!\! \int_{\tilde{\omega}} \sum_{i=1}^{2} e^{-2s\sigma_i} \xi_i^9 \Big(s^6\lambda^6 \xi_i^6 |\psi|^2 + s^4\lambda^4 \xi_i^4 |\psi_x|^2 + s^2\lambda^2 \xi_i^2 |\psi_{xx}|^2 + |\psi_{xxx}|^2 \Big) \, dxdt \\
&\qquad\qquad + \int_0^T \!\! \int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_1} |\Lambda\gamma^1|^2 \, dx \, dt + \int_0^T \!\! \int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_2} |\Lambda\gamma^2|^2 \, dx \, dt \\
&\qquad\qquad\quad C \Bigg(s^7\lambda^8 \iint_Q \Big((e^{-2s\sigma_1} \xi_1^7 + e^{-2s\sigma_2}) |g^0|^2 + e^{-2s\sigma_1} |g^1|^2 + e^{-2s\sigma_2} |g^2|^2 \Big) \, dxdt \Bigg), \qquad (3.62)
\end{aligned}$$

for every $s \geq C((1 + \nu^2 + \|\bar{y}\|_\infty^2 + \|\bar{y}_x\|_\infty^2) T^{\frac{2}{3}} + T^{\frac{1}{3}})$ and every $\lambda \geq C$.

For $\epsilon > 0$ sufficiently small, we take $s \geq CT^{2/3}$ and $\lambda \geq C$ sufficiently large, obtaining that

$$\int_0^T \!\! \int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_1} |\Lambda\gamma^1|^2 \, dx \, dt + \int_0^T \!\! \int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_2} |\Lambda\gamma^2|^2 \, dx \, dt \leqslant \epsilon s^7\lambda^8 \sum_{i=1}^{2} \iint_Q e^{-2s\alpha_i} \xi_i^7 |\gamma^i|^2 \, dxdt.$$

In this way, can absorb the two terms of γ^1 and γ^2 in the right-hand side of (3.62),

$$\begin{aligned}
&s^{14}\lambda^{15} \int_0^T \!\! \int_0^L e^{-2s\sigma_1} \xi_1^{14} |\psi|^2 \, dx \, dt + s^7\lambda^8 \sum_{i=1}^{2} \iint_Q e^{-2s\alpha_i} \xi_i^7 |\gamma^i|^2 \, dxdt \leqslant C \Bigg(s^{14}\lambda^{15} \int_0^T \!\! \int_{\mathcal{O}} e^{-2s\sigma_1} \xi_1^{14} |\psi|^2 \, dx \, dt + \\
&\frac{C}{\mu^2} s^5\lambda^5 \Big(s\lambda \iint_Q e^{-2s\alpha_1^*} (\xi_1^*)^6 |\psi_{xxxx}|^2 \, dxdt + s^3\lambda^3 \iint_Q e^{-2s\alpha_1^*} (\xi_1^*)^8 |\psi_{xxx}|^2 \, dxdt + \iint_Q e^{-2s\alpha_1^*} (\xi_1^*)^5 |\psi_{xx}|^2 \, dxdt \Big) \\
&\qquad + s^9\lambda^{10} \int_0^T \!\! \int_{\tilde{\omega}} \sum_{i=1}^{2} e^{-2s\sigma_i} \xi_i^9 \Big(s^6\lambda^6 \xi_i^6 |\psi|^2 + s^4\lambda^4 \xi_i^4 |\psi_x|^2 + s^2\lambda^2 \xi_i^2 |\psi_{xx}|^2 + |\psi_{xxx}|^2 \Big) \, dxdt \\
&\qquad\qquad + s^7\lambda^8 \iint_Q \Big((e^{-2s\sigma_1} \xi_1^7 + e^{-2s\sigma_2}) |g^0|^2 + e^{-2s\sigma_1} |g^1|^2 + e^{-2s\sigma_2} |g^2|^2 \Big) \, dxdt, \qquad (3.63)
\end{aligned}$$

for every $s \geq C((1 + \nu^2 + \|\bar{y}\|_\infty^2 + \|\bar{y}_x\|_\infty^2) T^{\frac{2}{3}} + T^{\frac{1}{3}})$ and every $\lambda \geq C$.

Let us add a H^4 norm of ψ in the left-hand side of (3.63). Indeed, by defining $\rho_* = e^{-s\sigma_1^*} (\xi_1^*)^3$, $R = \rho_* \psi$,

$S^i = \rho_* \gamma^i$, for $i = 1, 2$, and $g^i_* = \rho_* g^i$, for $i = 0, 1, 2$, it is not difficult to see that

$$\begin{cases} -R_t + R_{xxxx} + \nu R_{xx} - \bar{y} R_x = -(\rho_*)_t \psi - (\rho_*)_t g^0_* + \alpha_1 S^1 \mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2 S^2 \mathbb{1}_{\mathcal{O}_{2,d}} & (x,t) \in Q, \\ S^i_t + S^i_{xxxx} + \nu S^i_{xx} + (\bar{y} S^i)_x = (\rho_*)_t \gamma^i + g^i_* & (x,t) \in Q, \\ R(0,t) = R(L,t) = 0 & t \in (0,T), \\ R_x(0,t) = R_x(L,t) = 0 & t \in (0,T), \\ S^1(0,t) = \frac{1}{\mu_1} R_{xxx}(0,t), \quad S^1_x(0,t) = -\frac{1}{\mu_2} R_{xx}(0,t) & t \in (0,T), \\ S^1(L,t) = S^1_x(L,t) = 0 & t \in (0,T), \\ S^2(0,t) = 0, \quad S^2_x(0,t) = 0 & t \in (0,T), \\ S^2(L,t) = \frac{1}{\nu_1} R_{xxx}(L,t), \quad S^2_x(L,t) = -\frac{1}{\nu_2} R_{xx}(L,t) & t \in (0,T), \\ R(x,T) = 0, \quad S^i(x,0) = 0 & x \in (0,L). \end{cases} \tag{3.64}$$

Using (A.28), (3.4), and the fact that $\|\eta_1\|_\infty = \|\eta_2\|_\infty$, we get that

$$\int_0^L |R_{xx}(t)|^2 \, dx + \iint_Q |R_{xxxx}|^2 \, dxdt + \sum_{i=1}^{2} \iint_Q |S^i|^2 \, dxdt \leqslant C \Bigg(\iint_Q e^{-2s\sigma_1} \xi_1^7 |\psi|^2 \, dxdt \\ + \sum_{i=1}^{2} \iint_Q e^{-2s\sigma_i} \xi_i^7 |\gamma^i|^2 \, dxdt + \sum_{i=0}^{2} \iint_Q |g^i_*|^2 \, dxdt \Bigg). \tag{3.65}$$

Then, combining (3.65) and (3.63),

$$\|e^{-s\sigma_1^*} (\xi_1^*)^3 \psi\|^2_{L^\infty(0,T);H^2 \cap H^1_0(0,L))} + s^7 \lambda^8 \iint_Q e^{-2s\sigma_1^*} (\xi_1^*)^6 |\psi_{xxxx}|^2 \, dxdt + s^{14} \lambda^{15} \int_0^T \int_0^L e^{-2s\sigma_1} \xi_1^{14} |\psi|^2 \, dx \, dt \\ + s^7 \lambda^8 \sum_{i=1}^{2} \iint_Q e^{-2s\alpha_i} \xi_i^7 |\gamma^i|^2 \, dxdt \leqslant C \Bigg(s^{14} \lambda^{15} \int_0^T \int_{\mathcal{O}} e^{-2s\sigma_1} \xi_1^{14} |\psi|^2 \, dx \, dt + \\ \frac{C}{\mu^2} s^5 \lambda^5 \Big(s\lambda \iint_Q e^{-2s\alpha_1^*} (\xi_1^*)^6 |\psi_{xxxx}|^2 \, dxdt + s^3 \lambda^3 \iint_Q e^{-2s\alpha_1^*} (\xi_1^*)^8 |\psi_{xxx}|^2 \, dxdt + \iint_Q e^{-2s\alpha_1^*} (\xi_1^*)^5 |\psi_{xx}|^2 \, dxdt \Big) \\ + s^9 \lambda^{10} \int_0^T \int_{\tilde{\omega}} \sum_{i=1}^{2} e^{-2s\sigma_i} \xi_i^9 \Big(s^6 \lambda^6 \xi_i^6 |\psi|^2 + s^4 \lambda^4 \xi_i^4 |\psi_x|^2 + s^2 \lambda^2 \xi_i^2 |\psi_{xx}|^2 + |\psi_{xxx}|^2 \Big) \, dxdt + \\ C \Bigg(s^7 \lambda^8 \iint_Q \Big(\sum_{i=1}^{2} e^{-2s\sigma_i} \xi_i^7 |g^0|^2 + e^{-2s\sigma_1} \xi_1^7 |g^1|^2 + e^{-2s\sigma_2} \xi_2^7 |g^2|^2 \Big) \, dxdt \Bigg), \tag{3.66}$$

for every $s \geq C((1 + \nu^2 + \|\bar{y}\|^2_\infty + \|\bar{y}_x\|^2_\infty) T^{\frac{2}{3}} + T^{\frac{1}{3}})$ and every $\lambda \geq C$.

To add the other derivatives with suitable weight function, we use interpolation estimates. First, note that for every $0 < r < 4$

$$\|\psi\|_{H^r} \leqslant \|\psi\|^{\frac{4-r}{4}}_{L^2} \|\psi\|^{\frac{r}{4}}_{H^4}.$$

In this way, for any positive number a_0, a and b, we have that

$$s^a \lambda^b \int_0^T e^{-2s\sigma_1^*} (\xi_1^*)^{a_0} \|\psi\|^2_{H^r} \, dt \leqslant s^a \lambda^b \int_0^T e^{-2s\sigma_1^*} (\xi_1^*)^{a_0} \|\psi\|^{\frac{4-r}{2}}_{L^2} \|\psi\|^{\frac{r}{2}}_{H^4} \, dt = \\ s^a \lambda^b \int_0^T e^{-2s\sigma_1^*} (\xi_1^*)^{a_0} (\xi_1^*)^{\frac{6r}{4}} s^{\frac{7r}{4}} \lambda^{\frac{8r}{4}} \|\psi\|^{\frac{r}{2}}_{H^4} (\xi_1^*)^{-\frac{6r}{4}} s^{-\frac{7r}{4}} \lambda^{\frac{-8r}{4}} \|\psi\|^{\frac{4-r}{2}}_{L^2} \, dt \\ \leqslant \frac{r}{4} s^7 \lambda^8 \int_0^T e^{-2s\sigma_1^*} (\xi_1^*)^6 \|\psi\|^2_{H^4} \, dt + \frac{4-r}{4} s^{\frac{4a-7r}{4-r}} \lambda^{\frac{4b-8r}{4-r}} \int_0^T e^{-2s\sigma_1^*} (\xi_1^*)^{\frac{4a_0-6r}{4-r}} \|\psi\|^2_2 \, dt. \tag{3.67}$$

Hence, by choosing $a_0 = 14 - 2r$, $a = 14 - \frac{7r}{4}$ and $b = 15 - \frac{7r}{4}$, we obtain that

$$s^{14-\frac{7r}{4}} \lambda^{15-\frac{7r}{4}} \int_0^T e^{-2s\sigma_1^*} (\xi_1^*)^{14-2r} \|\psi\|^2_{H^r} \, dt \leqslant \frac{r}{4} s^7 \lambda^8 \int_0^T e^{-2s\sigma_1^*} (\xi_1^*)^6 \|\psi\|^2_{H^4} \, dt \\ + \frac{4-r}{4} s^{14} \lambda^{15} \int_0^T e^{-2s\sigma_1^*} (\xi_1^*)^{14} \|\psi\|^2_2 \, dt. \tag{3.68}$$

Then, combining (3.66) and (3.68), for $r = 1, 2, 3$, we get that

$$\begin{aligned}
&\|e^{-s\sigma_1^*}(\xi_1^*)^3\psi\|^2_{L^\infty(0,T);H^2\cap H_0^1(0,L))} + \sum_{i=1}^{4} s^{14-\frac{7i}{4}}\lambda^{15-\frac{7i}{4}} \iint_Q e^{-2s\sigma_1^*}(\xi_1^*)^{14-2i}|\partial_x^i\psi|^2\,dxdt+\\
&s^{14}\lambda^{15}\int_0^T\!\!\int_0^L e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dxdt + s^7\lambda^8\sum_{i=1}^{2}\iint_Q e^{-2s\alpha_i}\xi_i^7|\gamma^i|^2\,dxdt \leqslant C\left(s^{14}\lambda^{15}\int_0^T\!\!\int_{\mathcal{O}} e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dxdt+\right.\\
&\frac{C}{\mu^2}s^5\lambda^5\Big(s\lambda\iint_Q e^{-2s\alpha_1^*}(\xi_1^*)^6|\psi_{xxxx}|^2\,dxdt + s^3\lambda^3\iint_Q e^{-2s\alpha_1^*}(\xi_1^*)^8|\psi_{xxx}|^2\,dxdt + \iint_Q e^{-2s\alpha_1^*}(\xi_1^*)^5|\psi_{xx}|^2\,dxdt\Big)\\
&\qquad + s^9\lambda^{10}\int_0^T\!\!\int_{\tilde\omega}\sum_{i=1}^{2} e^{-2s\sigma_i}\xi_i^9\Big(s^6\lambda^6\xi_i^6|\psi|^2 + s^4\lambda^4\xi_i^4|\psi_x|^2 + s^2\lambda^2\xi_i^2|\psi_{xx}|^2 + |\psi_{xxx}|^2\Big)\,dxdt+\\
&\qquad\qquad \left. C\left(s^7\lambda^8\iint_Q\Big(\sum_{i=1}^{2} e^{-2s\sigma_i}\xi_i^7|g^0|^2 + e^{-2s\sigma_1}\xi_1^7|g^1|^2 + e^{-2s\sigma_2}\xi_2^7|g^2|^2\Big)\,dxdt\right)\right), \qquad (3.69)
\end{aligned}$$

for every $s \geq C((1+\nu^2+\|\bar{y}\|^2_\infty + \|\bar{y}_x\|^2_\infty)T^{\frac{2}{3}} + T^{\frac{1}{3}})$ and every $\lambda \geq C$. Consequently, the second term in the right-hand side of (3.69) can be absorbed, and we get

$$\begin{aligned}
&\|e^{-s\sigma_1^*}(\xi_1^*)^3\psi\|^2_{L^\infty(0,T);H^2\cap H_0^1(0,L))} + \sum_{i=1}^{4} s^{14-\frac{7i}{4}}\lambda^{15-\frac{7i}{4}} \iint_Q e^{-2s\sigma_1^*}(\xi_1^*)^{14-2i}|\partial_x^i\psi|^2\,dxdt\\
&+ s^{14}\lambda^{15}\int_0^T\!\!\int_0^L e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dxdt + s^7\lambda^8\sum_{i=1}^{2}\iint_Q e^{-2s\alpha_i}\xi_i^7|\gamma^i|^2\,dxdt \leqslant C\left(s^{14}\lambda^{15}\int_0^T\!\!\int_{\mathcal{O}} e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dxdt\right.\\
&\qquad + s^9\lambda^{10}\int_0^T\!\!\int_{\tilde\omega}\sum_{i=1}^{2} e^{-2s\sigma_i}\xi_i^9\Big(s^6\lambda^6\xi_i^6|\psi|^2 + s^4\lambda^4\xi_i^4|\psi_x|^2 + s^2\lambda^2\xi_i^2|\psi_{xx}|^2 + |\psi_{xxx}|^2\Big)\,dxdt\\
&\qquad\qquad \left. + C\left(s^7\lambda^8\iint_Q\Big(\sum_{i=1}^{2} e^{-2s\sigma_i}\xi_i^7|g^0|^2 + e^{-2s\sigma_1}\xi_1^7|g^1|^2 + e^{-2s\sigma_2}\xi_2^7|g^2|^2\Big)\,dxdt\right)\right), \qquad (3.70)
\end{aligned}$$

for every $s \geq C((1+\nu^2+\|\bar{y}\|^2_\infty + \|\bar{y}_x\|^2_\infty)T^{\frac{2}{3}} + T^{\frac{1}{3}})$ and every $\lambda \geq C$.

Now, we just have to estimate the local terms of ψ in the right-hand side of (3.70). In order to do that, we use Lemma 3.6, and for any open $\hat\omega$ subset of $(0, L)$, such that $\overline{\tilde\omega} \subset \hat\omega$, we proceed in a very similar way as in (3.45) to bound the local terms of $\partial_x^i\psi$, for $i = 1, 2, 3$, obtaining

$$\begin{aligned}
&s^7\lambda^{10}\int_0^T\!\!\int_{\tilde\omega}\sum_{i=1}^{2} e^{-2s\sigma_i}\xi_i^7\Big(\lambda^6\xi_i^8|\psi|^2 + \lambda^4|\psi_x|^2 + \lambda^2|\psi_{xx}|^2 + |\psi_{xxx}|^2\Big)\,dxdt\\
&\qquad \leqslant \epsilon\sum_{i=1}^{4} s^{14-\frac{7i}{4}}\lambda^{15-\frac{7i}{4}}\iint_Q e^{-2s\sigma_1^*}(\xi_1^*)^{14-2i}|\partial_x^i\psi|^2\,dxdt + Cs^{30}\lambda^{31}\int_0^T\!\!\int_{\hat\omega} e^{-2s(8\hat\sigma-7\sigma_1^*)}\hat\xi_1^{72}|\psi|^2\,dxdt, \qquad (3.71)
\end{aligned}$$

for $\epsilon > 0$ sufficiently small.

Then, we combine (3.70) and (3.71), and we get

$$\begin{aligned}
&\|e^{-s\sigma_1^*}(\xi_1^*)^3\psi\|^2_{L^\infty(0,T);H^2\cap H_0^1(0,L))} + \sum_{i=1}^{4} s^{14-\frac{7i}{4}}\lambda^{15-\frac{7i}{4}} \iint_Q e^{-2s\sigma_1^*}(\xi_1^*)^{14-2i}|\partial_x^i\psi|^2\,dxdt \\
&\quad + s^{14}\lambda^{15}\int_0^T\int_0^L e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dxdt + s^7\lambda^8\sum_{i=1}^{2}\iint_Q e^{-2s\alpha_i}\xi_i^7|\gamma^i|^2\,dxdt \\
&\qquad \leqslant C\Bigg(s^{30}\lambda^{31}\int_0^T\int_{\hat\omega} e^{-2s(8\hat\sigma-7\sigma_1^*)}\hat\xi_1^{72}|\psi|^2\,dxdt + \\
&\qquad s^7\lambda^8\iint_Q\Big(\sum_{i=1}^{2}e^{-2s\sigma_i}\xi_i^7|g^0|^2 + e^{-2s\sigma_1}\xi_1^7|g^1|^2 + e^{-2s\sigma_2}\xi_2^7|g^2|^2\Big)\,dxdt\Bigg). \qquad (3.72)
\end{aligned}$$

To finish, we use a localization argument and prove that

$$\begin{aligned}
&s^{\frac{21}{2}}\lambda^{\frac{23}{2}}\int_0^T e^{-2s\sigma_1^*}(\xi_1^*)^8\big(|\psi_{xxx}(0,t)|^2 + |\psi_{xxx}(L,t)|^2 + (\xi_1^*)^2(|\psi_{xx}(0,t)|^2 + |\psi_{xx}(L,t)|^2)\big)\,dt \\
&\qquad \leqslant C\sum_{i=1}^{4} s^{14-\frac{7i}{4}}\lambda^{15-\frac{7i}{4}}\iint_Q e^{-2s\sigma_1^*}(\xi_1^*)^{14-2i}|\partial_x^i\psi|^2\,dxdt. \qquad (3.73)
\end{aligned}$$

In this way, we combine (3.72) and (3.73) to get (3.54).
Case (ii). Here, we can take ω_1 and ω_2 satisfying (3.49) and (3.50) such that

$$\omega_1\cap\mathcal{O}_{2,d} = \emptyset \text{ and } \omega_2\subset\mathcal{O}_{1,d}.$$

Now, we proceed in the following way. Let $h = \alpha_1\gamma^1 + \alpha_2\gamma^2$. Then, system (1.23) turns into

$$\begin{cases}
-\psi_t + \psi_{xxxx} + \nu\psi_{xx} - \bar y\psi_x = g^0 + \alpha_1\gamma^1(\mathbb{1}_{\mathcal{O}_{1,d}} - \mathbb{1}_{\mathcal{O}_{2,d}}) + h\mathbb{1}_{\mathcal{O}_{2,d}} & (x,t)\in Q,\\
\gamma_t^1 + \gamma_{xxxx}^1 + \nu\gamma_{xx}^1 + (\bar y\gamma^1)_x = g^1 & (x,t)\in Q,\\
h_t + h_{xxxx} + \nu h_{xx} + (\bar y h)_x = \alpha_1 g^1 + \alpha_2 g^2 & (x,t)\in Q,\\
\psi(0,t) = \psi(L,t) = 0 & t\in(0,T),\\
\psi_x(0,t) = \psi_x(L,t) = 0 & t\in(0,T),\\
\gamma^1(0,t) = \frac{1}{\mu_1}\psi_{xxx}(0,t), \quad \gamma_x^1(0,t) = -\frac{1}{\mu_2}\psi_{xx}(0,t) & t\in(0,T),\\
\gamma^1(L,t) = \gamma_x^1(L,t) = 0 & t\in(0,T),\\
h(0,t) = \frac{\alpha_1}{\mu_1}\psi_{xxx}(0,t), \quad h_x(0,t) = -\frac{\alpha_1}{\mu_2}\psi_{xx}(0,t) & t\in(0,T),\\
h(L,t) = \frac{\alpha_2}{\nu_1}\psi_{xxx}(L,t), \quad h_x(L,t) = -\frac{\alpha_2}{\nu_2}\psi_{xx}(L,t) & t\in(0,T),\\
\psi(x,T) = \psi^T(x), \quad \gamma^1(x,0), \quad h(x,0) = 0 & x\in(0,L).
\end{cases} \qquad (3.74)$$

In this way, we use (3.11) for the choice $(\omega,u,\sigma,\xi) = (\omega_2,h,\sigma_2,\xi_2)$, and we get that

$$\begin{aligned}
s^7\lambda^8\iint_Q e^{-2s\sigma_2}\xi_2^7|h|^2\,dxdt &\leqslant C\Bigg(\iint_Q e^{-2s\sigma_2}|\alpha_1 g^1 + \alpha_2 g^2|^2\,dxdt \\
&\quad + s^7\lambda^7\int_0^T e^{-2s\sigma_2^*}(\xi_2^*)^7\left(\frac{\alpha_1^2}{\mu_1^2}|\psi_{xxx}(0,t)|^2 + \frac{\alpha_2^2}{\nu_1^2}|\psi_{xxx}(L,t)|^2\right)dt \\
&\quad + s^5\lambda^5\int_0^T e^{-2s\sigma_2^*}(\xi_2^*)^5\left(\frac{\alpha_1^2}{\mu_2^2}|\psi_{xx}(0,t)|^2 + \frac{\alpha_2^2}{\nu_2^2}|\psi_{xx}(L,t)|^2\right)dt \\
&\quad + s^7\lambda^8\int_0^T\int_{\omega_2} e^{-2s\sigma_2}\xi_2^7|h|^2\,dxdt\Bigg), \qquad (3.75)
\end{aligned}$$

for every $s\geq C(T^{2/3}+T^{1/3})$ and every $\lambda\geq C$.

Now, we use (3.58) for ψ and (3.11) for $(u,\sigma,\xi)=(\gamma_1,\sigma_1,\xi_1)$, and combine both estimates with (3.75), to get

$$\begin{aligned}
&s^{14}\lambda^{15}\int_0^T\!\!\int_0^L e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dxdt+s^7\lambda^8\sum_{i=1}^2\iint_Q\left(e^{-2s\alpha_1}\xi_1^7|\gamma^1|^2+e^{-2s\alpha_2}\xi_2^7|h|^2\right)\,dxdt\\
&\leqslant C\Bigg(s^{14}\lambda^{15}\int_0^T\!\!\int_{\mathcal{O}} e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dx\,dt+s^7\lambda^8\int_0^T\!\!\int_{\omega_1} e^{-2s\alpha_1}\xi_1^7|\gamma^1|^2\,dxdt+s^7\lambda^8\int_0^T\!\!\int_{\omega_2} e^{-2s\sigma_2}\xi_2^7|h|^2\,dxdt\\
&\qquad s^7\lambda^7\int_0^T\!\!\int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_1}\xi_1^7|\gamma^1|^2\,dxdt+s^7\lambda^7\int_0^T\!\!\int_{(0,L)\setminus\mathcal{O}_0} e^{-2s\sigma_2}\xi_2^7|\gamma^2|^2\,dxdt\\
&\qquad+s^5\lambda^5\int_0^T(e^{-2s\alpha_1^*}(\xi_1^*)^5+e^{-2s\alpha_2^*}(\xi_2^*)^5)\left(\frac{s^2\lambda^2}{\mu_1^2}(\xi_1^*)^2|\psi_{xxx}(0,t)|^2+\frac{1}{\mu_2^2}|\psi_{xx}(0,t)|^2\right)\,dt\\
&\qquad+s^5\lambda^5\int_0^T e^{-2s\alpha_2^*}(\xi_2^*)^5\left(\frac{s^2\lambda^2}{\nu_1^2}(\xi_2^*)^2|\psi_{xxx}(L,t)|^2+\frac{1}{\nu_2^2}|\psi_{xx}(L,t)|^2\right)\,dt\\
&\qquad+s^7\lambda^8\iint_Q\left((e^{-2s\sigma_1}\xi_1^7+e^{-2s\sigma_2})|g^0|^2+e^{-2s\sigma_1}|g^1|^2+e^{-2s\sigma_2}|g^2|^2\right)\,dxdt\Bigg),
\end{aligned}\tag{3.76}$$

for every $s\geq C((1+\nu^2+\|\bar y\|_\infty^2+\|\bar y_x\|_\infty^2)T^{\frac23}+T^{\frac13})$ and every $\lambda\geq C$.

From the fact that $\eta_1=\eta_2$ in $(0,L)\setminus\mathcal{O}_0$, and that $\gamma_2=\frac{1}{\alpha_2}h-\frac{\alpha_1}{\alpha_2}\gamma_1$, we can absorb the fourth and fifth terms on the right-hand side of (3.76), and the resulting estimate is

$$\begin{aligned}
&s^{14}\lambda^{15}\int_0^T\!\!\int_0^L e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dxdt+s^7\lambda^8\sum_{i=1}^2\iint_Q\left(e^{-2s\alpha_1}\xi_1^7|\gamma^1|^2+e^{-2s\alpha_2}\xi_2^7|h|^2\right)\,dxdt\\
&\leqslant C\Bigg(s^{14}\lambda^{15}\int_0^T\!\!\int_{\mathcal{O}} e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dx\,dt+s^7\lambda^8\int_0^T\!\!\int_{\omega_1} e^{-2s\alpha_1}\xi_1^7|\gamma^1|^2\,dxdt+s^7\lambda^8\int_0^T\!\!\int_{\omega_2} e^{-2s\sigma_2}\xi_2^7|h|^2\,dxdt\\
&\qquad+s^5\lambda^5\int_0^T\sum_{i=1}^2 e^{-2s\alpha_i^*}(\xi_i^*)^5\left(\frac{s^2\lambda^2}{\mu_1^2}(\xi_1^*)^2|\psi_{xxx}(0,t)|^2+\frac{1}{\mu_2^2}|\psi_{xx}(0,t)|^2\right)\,dt\\
&\qquad+s^5\lambda^5\int_0^T e^{-2s\alpha_2^*}(\xi_2^*)^5\left(\frac{s^2\lambda^2}{\nu_1^2}(\xi_2^*)^2|\psi_{xxx}(L,t)|^2+\frac{1}{\nu_2^2}|\psi_{xx}(L,t)|^2\right)\,dt\\
&\qquad+s^7\lambda^8\iint_Q\left((e^{-2s\sigma_1}\xi_1^7+e^{-2s\sigma_2})|g^0|^2+e^{-2s\sigma_1}|g^1|^2+e^{-2s\sigma_2}|g^2|^2\right)\,dxdt\Bigg),
\end{aligned}\tag{3.77}$$

for every $s\geq C((1+\nu^2+\|\bar y\|_\infty^2+\|\bar y_x\|_\infty^2)T^{\frac23}+T^{\frac13})$ and every $\lambda\geq C$.

Using that

$$\begin{cases}
-\psi_t+\psi_{xxxx}+\nu\psi_{xx}-\bar y\psi_x=g^0+\alpha_1\gamma^1 & (x,t)\in\tilde\omega_1\times(0,T),\\
-\psi_t+\psi_{xxxx}+\nu\psi_{xx}-\bar y\psi_x=g^0+h & (x,t)\in\tilde\omega_2\times(0,T),\\
\gamma_t^1+\gamma_{xxxx}^1+\nu\gamma_{xx}^1+(\bar y\gamma^1)_x=g^1 & (x,t)\in\tilde\omega_1\times(0,T),\\
h_t+h_{xxxx}+\nu h_{xx}+(\bar y h)_x=\sum_{i=1}^2\alpha_i g^i & (x,t)\in\tilde\omega_2\times(0,T),
\end{cases}$$

and Lemma 3.4, we can proceed in a similar way as in (3.41), to get

$$\begin{aligned}
&s^7\lambda^8\int_0^T\!\!\int_{\omega_1} e^{-2s\sigma_1}\xi_1^7|\gamma^1|^2\,dxdt+s^7\lambda^8\int_0^T\!\!\int_{\omega_2} e^{-2s\sigma_2}\xi_2^7|h|^2\,dxdt\\
&\qquad\leqslant \epsilon\Big(s^7\lambda^8\int_0^T\!\!\int_0^L e^{-2s\sigma_1}\xi_1^7|\gamma^1|^2\,dxdt+s^7\lambda^8\int_0^T\!\!\int_0^L e^{-2s\sigma_2}\xi_2^7|h|^2\,dxdt\Big)\\
&\qquad+C\left(s^9\lambda^{10}\int_0^T\!\!\int_{\tilde\omega}\sum_{i=1}^2 e^{-2s\sigma_i}\xi_i^9\Big(s^6\lambda^6\xi_i^6|\psi|^2+s^4\lambda^4\xi_i^4|\psi_x|^2+s^2\lambda^2\xi_i^2|\psi_{xx}|^2+|\psi_{xxx}|^2\Big)\,dxdt\right.\\
&\qquad\qquad\left. s^7\lambda^8\int_0^T\!\!\int_Q\sum_{i=1}^2 e^{-2s\sigma_i}\xi_i^7(|g^0|^2+|g^1|^2+|g^2|^2)\,dxdt\right),
\end{aligned}\tag{3.78}$$

for $\epsilon>0$ sufficiently small.

Now, we use (3.78) in (3.77)

$$\begin{aligned}
&s^{14}\lambda^{15}\int_0^T\!\!\int_0^L e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dxdt+s^7\lambda^8\sum_{i=1}^2\iint_Q\left(e^{-2s\alpha_1}\xi_1^7|\gamma^1|^2+e^{-2s\alpha_2}\xi_2^7|h|^2\right)\,dxdt\\
&\qquad\leqslant C\left(s^{15}\lambda^{16}\int_0^T\!\!\int_{\mathcal{O}}\sum_{i=1}^2 e^{-2s\sigma_i}\xi_i^{15}|\psi|^2\,dx\,dt\right.\\
&\qquad+s^9\lambda^{10}\int_0^T\!\!\int_{\tilde\omega}\sum_{i=1}^2 e^{-2s\sigma_i}\xi_i^9\Big(s^6\lambda^6\xi_i^6|\psi|^2+s^4\lambda^4\xi_i^4|\psi_x|^2+s^2\lambda^2\xi_i^2|\psi_{xx}|^2+|\psi_{xxx}|^2\Big)\,dxdt\\
&\qquad+s^5\lambda^5\int_0^T(e^{-2s\alpha_1^*}(\xi_1^*)^5+e^{-2s\alpha_2^*}(\xi_2^*)^5)\left(\frac{s^2\lambda^2}{\mu_1^2}(\xi_1^*)^2|\psi_{xxx}(0,t)|^2+\frac{1}{\mu_2^2}|\psi_{xx}(0,t)|^2\right)\,dt\\
&\qquad+s^5\lambda^5\int_0^T e^{-2s\alpha_2^*}(\xi_2^*)^5\left(\frac{s^2\lambda^2}{\nu_1^2}(\xi_2^*)^2|\psi_{xxx}(L,t)|^2+\frac{1}{\nu_2^2}|\psi_{xx}(L,t)|^2\right)\,dt\\
&\qquad\left.+s^7\lambda^8\iint_Q\Big(\sum_{i=1}^2 e^{-2s\sigma_i}\xi_i^7\Big)\Big(|g^0|^2+|g^1|^2+|g^2|^2\Big)\,dxdt\right),
\end{aligned}\tag{3.79}$$

for every $s\geq C((1+\nu^2+\|\bar y\|_\infty^2+\|\bar y_x\|_\infty^2)T^{\frac{2}{3}}+T^{\frac{1}{3}})$ and every $\lambda\geq C$.

Using Lemma (3.5) to estimate the boundary terms,

$$\begin{aligned}
&s^{14}\lambda^{15}\int_0^T\!\!\int_0^L e^{-2s\sigma_1}\xi_1^{14}|\psi|^2\,dxdt+s^7\lambda^8\sum_{i=1}^2\iint_Q\left(e^{-2s\alpha_1}\xi_1^7|\gamma^1|^2+e^{-2s\alpha_2}\xi_2^7|h|^2\right)\,dxdt\\
&\qquad\leqslant C\left(s^{15}\lambda^{16}\sum_{i=1}^2\int_0^T\!\!\int_{\mathcal{O}} e^{-2s\sigma_i}\xi_i^{15}|\psi|^2\,dx\,dt\right.\\
&\qquad+s^9\lambda^{10}\int_0^T\!\!\int_{\tilde\omega}\sum_{i=1}^2 e^{-2s\sigma_i}\xi_i^9\Big(s^6\lambda^6\xi_i^6|\psi|^2+s^4\lambda^4\xi_i^4|\psi_x|^2+s^2\lambda^2\xi_i^2|\psi_{xx}|^2+|\psi_{xxx}|^2\Big)\,dxdt\\
&+\frac{C}{\mu}s^5\lambda^5\Big(s\lambda\iint_Q e^{-2s\alpha_1^*}(\xi_1^*)^6|\psi_{xxxx}|^2\,dxdt+s^3\lambda^3\iint_Q e^{-2s\alpha_1^*}(\xi_1^*)^8|\psi_{xxx}|^2\,dxdt+\iint_Q e^{-2s\alpha_1^*}(\xi_1^*)^5|\psi_{xx}|^2\,dxdt\Big)\\
&\qquad\left.+s^7\lambda^8\iint_Q\Big(\sum_{i=1}^2 e^{-2s\sigma_i}\xi_i^7\Big)\Big(|g^0|^2+|g^1|^2+|g^2|^2\Big)\,dxdt\right),
\end{aligned}\tag{3.80}$$

for every $s\geq C((1+\nu^2+\|\bar y\|_\infty^2+\|\bar y_x\|_\infty^2)T^{\frac{2}{3}}+T^{\frac{1}{3}})$ and every $\lambda\geq C$, where $\mu=\min\{\mu_1,\mu_2,\nu_1,\nu_2\}$.

Proceeding in a very similar way as in (3.68), we can add terms with high-order derivatives in the left-hand side of (3.77), and we get

$$\begin{aligned}
&\|e^{-s\sigma_1^*}(\xi_1^*)^3\psi\|^2_{L^\infty(0,T);H^2\cap H_0^1(0,L))}+\sum_{i=0}^{4}s^{14-\frac{7i}{4}}\lambda^{15-\frac{7i}{4}}\iint_Q e^{-2s\sigma_1^*}(\xi_1^*)^{14-2i}|\partial_x^i\psi|^2\,dxdt\\
&\quad+s^7\lambda^8\sum_{i=1}^{2}\iint_Q\left(e^{-2s\alpha_1}\xi_1^7|\gamma^1|^2+e^{-2s\alpha_2}\xi_2^7|h|^2\right)\,dxdt\leqslant C\Bigg(s^{15}\lambda^{16}\sum_{i=1}^{2}\int_0^T\!\!\int_{\mathcal{O}}e^{-2s\sigma_i}\xi_i^{15}|\psi|^2\,dx\,dt\\
&\quad+s^9\lambda^{10}\int_0^T\!\!\int_{\tilde\omega}\sum_{i=1}^{2}e^{-2s\sigma_i}\xi_i^9\Big(s^6\lambda^6\xi_i^6|\psi|^2+s^4\lambda^4\xi_i^4|\psi_x|^2+s^2\lambda^2\xi_i^2|\psi_{xx}|^2+|\psi_{xxx}|^2\Big)\,dxdt\\
&+\frac{C}{\mu}s^5\lambda^5\Big(s\lambda\iint_Q e^{-2s\alpha_1^*}(\xi_1^*)^6|\psi_{xxxx}|^2\,dxdt+s^3\lambda^3\iint_Q e^{-2s\alpha_1^*}(\xi_1^*)^8|\psi_{xxx}|^2\,dxdt+\iint_Q e^{-2s\alpha_1^*}(\xi_1^*)^5|\psi_{xx}|^2\,dxdt\Big)\\
&\quad+s^7\lambda^8\iint_Q\Big(\sum_{i=1}^{2}e^{-2s\sigma_i}\xi_i^7\Big)\Big(|g^0|^2+|g^1|^2+|g^2|^2\Big)\,dxdt\Bigg),
\end{aligned}\tag{3.81}$$

for every $s\geq C((1+\nu^2+\|\bar y\|^2_\infty+\|\bar y_x\|^2_\infty)T^{\frac{2}{3}}+T^{\frac{1}{3}})$ and every $\lambda\geq C$. We can take s,λ sufficiently large, such that

$$\begin{aligned}
&\frac{C}{\mu}s^5\lambda^5\Big(s\lambda\iint_Q e^{-2s\alpha_1^*}(\xi_1^*)^6|\psi_{xxxx}|^2\,dxdt+s^3\lambda^3\iint_Q e^{-2s\alpha_1^*}(\xi_1^*)^8|\psi_{xxx}|^2\,dxdt+\iint_Q e^{-2s\alpha_1^*}(\xi_1^*)^5|\psi_{xx}|^2\,dxdt\Big)\\
&\qquad\leqslant\epsilon\sum_{i=0}^{4}s^{14-\frac{7i}{4}}\lambda^{15-\frac{7i}{4}}\iint_Q e^{-2s\sigma_1^*}(\xi_1^*)^{14-2i}|\partial_x^i\psi|^2\,dxdt,
\end{aligned}\tag{3.82}$$

for $\epsilon>0$ sufficiently small. In this way, Combining (3.81) and (3.82)

$$\begin{aligned}
&\|e^{-s\sigma_1^*}(\xi_1^*)^3\psi\|^2_{L^\infty(0,T);H^2\cap H_0^1(0,L))}+\sum_{i=0}^{4}s^{14-\frac{7i}{4}}\lambda^{15-\frac{7i}{4}}\iint_Q e^{-2s\sigma_1^*}(\xi_1^*)^{14-2i}|\partial_x^i\psi|^2\,dxdt\\
&\quad+s^7\lambda^8\sum_{i=1}^{2}\iint_Q\left(e^{-2s\alpha_1}\xi_1^7|\gamma^1|^2+e^{-2s\alpha_2}\xi_2^7|h|^2\right)\,dxdt\leqslant s^{15}\lambda^{16}\sum_{i=1}^{2}\int_0^T\!\!\int_{\mathcal{O}}e^{-2s\sigma_i}\xi_i^{15}|\psi|^2\,dx\,dt\\
&\quad+s^9\lambda^{10}\int_0^T\!\!\int_{\tilde\omega}\sum_{i=1}^{2}e^{-2s\sigma_i}\xi_i^9\Big(s^6\lambda^6\xi_i^6|\psi|^2+s^4\lambda^4\xi_i^4|\psi_x|^2+s^2\lambda^2\xi_i^2|\psi_{xx}|^2+|\psi_{xxx}|^2\Big)\,dxdt\\
&\quad+s^7\lambda^8\iint_Q\Big(\sum_{i=1}^{2}e^{-2s\sigma_i}\xi_i^7\Big)\Big(|g^0|^2+|g^1|^2+|g^2|^2\Big)\,dxdt,
\end{aligned}\tag{3.83}$$

for every $s\geq C((1+\nu^2+\|\bar y\|^2_\infty+\|\bar y_x\|^2_\infty)T^{\frac{2}{3}}+T^{\frac{1}{3}})$, $\lambda\geq C$ and $\epsilon>0$ sufficiently small.

Also, from a localization argument, we can prove that

$$\begin{aligned}
&s^{\frac{21}{2}}\lambda^{\frac{23}{2}}\int_0^T e^{-2s\sigma_1^*}(\xi_1^*)^8\big(|\psi_{xxx}(0,t)|^2+|\psi_{xxx}(L,t)|^2+(\xi_1^*)^2(|\psi_{xx}(0,t)|^2+|\psi_{xx}(L,t)|^2)\big)\,dt\\
&\qquad\leqslant C\sum_{i=1}^{4}s^{14-\frac{7i}{4}}\lambda^{15-\frac{7i}{4}}\iint_Q e^{-2s\sigma_1^*}(\xi_1^*)^{14-2i}|\partial_x^i\psi|^2\,dxdt.
\end{aligned}\tag{3.84}$$

To finish, we use Lemma 3.6, and proceed in a similar way as in (3.71), and we obtain (3.54). We remark that the term of γ_2 in the left-hand side of (3.54) is not difficult to add since $\gamma_2=\frac{1}{\alpha_2}h-\frac{\alpha_1}{\alpha_2}\gamma_1$. □

4 Null Controllability of the Linear System

To prove the null controllability for system (1.22), we first prove an observability estimate of the form (1.24) to the solutions of the adjoint system (1.23). To do so, we introduce some new weight functions, similar to the ones in (3.2) and (3.52), that do not degenerate at $t=0$.

Let $\tau(t) \in C^1([0,T])$ be defined by

$$\tau(t) = \begin{cases} (T/2)^{2/3} & t \in [0, T/2), \\ t^{1/3}(T-t)^{1/3} & t \in [T/2, T], \end{cases}$$

and

$$\beta_i(x,t) := \frac{\exp\big(4\lambda\|\eta_i\|_\infty\big) - \exp\big(\lambda(\|\eta_i\|_\infty + \eta_i(x))\big)}{\tau(t)}, \quad \zeta_i(x,t) := \frac{\exp\big(\lambda(\|\eta_i\|_\infty + \eta_i(x))\big)}{\tau(t)}, \quad i = 1,2, \tag{4.1}$$

where $\lambda > 1$, and consider the following notations:

$$\beta^*(t) = \max_{x\in[0,L]} \beta_i(x,t), \quad \hat{\beta}(t) = \min_{x\in[0,L]} \beta_i(x,t), \quad \zeta^*(t) = \min_{x\in[0,L]} \zeta_i(x,t), \quad \hat{\zeta}(t) = \max_{x\in[0,L]} \zeta_i(x,t), \tag{4.2}$$

for $i = 1,2$. Note that, since $\|\eta_1\|_\infty = \|\eta_2\|_\infty$, the definitions (4.2) does not depend on the index $i = 1,2$.

In this way, we have the following result.

Proposition 4.1. *Assume that condition* (1.18) *holds and* $\bar{y} \in L^\infty(0,T; W^{1,\infty}((0,L)))$*. Also, let the functions defined by* (3.51) *and,* s *and* λ *be constants such that Proposition 3.8 is verified. If either* (1.19) *or* (1.20) *hold, then there exists a constant* $C > 0$ *such that every solution* $(\psi, \gamma^1, \gamma^2)$ *of the system* (1.23) *satisfies*

$$\begin{aligned} &\int_0^T e^{-2s\beta^*}(\zeta^*)^8\big(|\psi_{xxx}(0,t)|^2 + |\psi_{xxx}(L,t)|^2 + (\zeta^*)^2(|\psi_{xx}(0,t)|^2 + |\psi_{xx}(L,t)|^2)\big)\,dt \\ &\quad + \|e^{-s\beta^*}(\zeta^*)^3\psi\|^2_{L^\infty(0,T);H^2\cap H^1_0(0,L))} + \int_0^T\!\!\int_0^L e^{-2s\beta^*}(\zeta^*)^7|\psi_{xxxx}|^2\,dx\,dt + \int_0^T\!\!\int_0^L e^{-2s\beta^*}|h|^2\,dx\,dt \\ &\qquad \leqslant C\Big(\sum_{i=0}^2 \int_0^T\!\!\int_0^L e^{-2s\hat{\beta}}\hat{\zeta}^7|g^i|^2\,dx\,dt + \int_0^T\!\!\int_{\mathcal{O}} e^{-2s(8\hat{\beta}-7\beta^*)}\hat{\zeta}^{72}|\psi|^2\,dx\,dt\Big), \end{aligned} \tag{4.3}$$

if (1.19) *holds, and*

$$\begin{aligned} &\int_0^T e^{-2s\beta^*}(\zeta^*)^8\big(|\psi_{xxx}(0,t)|^2 + |\psi_{xxx}(L,t)|^2 + (\zeta^*)^2(|\psi_{xx}(0,t)|^2 + |\psi_{xx}(L,t)|^2)\big)\,dt \\ &+ \|e^{-s\beta^*}(\zeta^*)^3\psi\|^2_{L^\infty(0,T);H^2\cap H^1_0(0,L))} + \int_0^T\!\!\int_0^L e^{-2s\beta^*}(\zeta^*)^7|\psi_{xxxx}|^2\,dx\,dt + \int_0^T\!\!\int_0^L e^{-2s\beta^*}\big(|\gamma^1|^2 + |\gamma^2|^2\big)\,dx\,dt \\ &\qquad \leqslant C\Big(\sum_{i=0}^2 \int_0^T\!\!\int_0^L e^{-2s\hat{\beta}}\hat{\zeta}^7|g^i|^2\,dx\,dt + \int_0^T\!\!\int_{\mathcal{O}} e^{-2s(8\hat{\beta}-7\beta^*)}\hat{\zeta}^{72}|\psi|^2\,dx\,dt\Big), \end{aligned} \tag{4.4}$$

in case (1.20) *holds.*

Proof. The proof of estimates (4.3), (4.4) are quite standard and mainly follows from combinations of Carleman estimates (3.35) or (3.54), Corollary A.11, and the fact that

$$\zeta_1^* \geq \frac{4^{1/13}}{T^{2/3}}.$$

For details, please see [17, Lemma 1], [5, Proposition 4.1] or [6, Proposition 4.3]. □

Now, we proceed to the proof of the null controllability for the system (1.22).

Let us denote by $\mathcal{L}$ the linear operator

$$\mathcal{L}u := u_t + u_{xxxx} + \nu u_{xx} + (\bar{y}u)_x$$

and by $\mathcal{L}^*$ its formal adjoint

$$\mathcal{L}^*u := -u_t + u_{xxxx} + \nu u_{xx} - \bar{y}u_x.$$

Also, consider the functional space

$$\begin{aligned}\mathcal{S} = \big\{(z,\phi_1,\phi_2,f):\quad & e^{s\hat\beta}\hat\xi^{-\frac{7}{2}}z \in L^2((0,L)\times(0,T)),\\
& e^{s\hat\beta}\hat\xi^{-\frac{7}{2}}\phi^i \in L^2((0,L)\times(0,T)),\ i=1,2,\\
& e^{s(8\hat\beta-7\beta^*)}\hat\xi^{-36}z \in C([0,T];H^{-2}(0,L))\cap L^2((0,L)\times(0,T)),\\
& e^{s(8\hat\beta-7\beta^*)}\hat\xi^{-36}\phi^i \in C([0,T];H_0^2(0,L))\cap L^2(0,T;H^4(0,L)), i=1,2,\\
& e^{s(8\hat\beta-7\beta^*)}\hat\xi^{-36}f \in L^2(\mathcal{O}\times(0,T)),\\
& e^{s\beta^*}(\mathcal{L}z - f\mathbb{1}_{\mathcal{O}}) \in L^2(0,T;H^{-2}(0,L)),\\
& e^{s\beta^*}(\mathcal{L}^*\phi^i - \alpha_i z\mathbb{1}_{\mathcal{O}_{i,d}}) \in L^2((0,L)\times(0,T)), i=1,2,\\
& \Big(e^{s\beta^*}\big(z(0,t)+\tfrac{1}{\mu_1}\phi^1_{xxx}(0,t)\big), e^{s\beta^*}\big(z(L,t)-\tfrac{1}{\nu_1}\phi^2_{xxx}(L,t)\big)\Big) \in [L^2(0,T)]^2\\
& \Big(e^{s\beta^*}\big(z_x(0,t)-\tfrac{1}{\mu_2}\phi^1_{xx}(0,t)\big), e^{s\beta^*}\big(z_x(L,t)+\tfrac{1}{\nu_2}\phi^2_{xx}(L,t)\big)\Big) \in [L^2(0,T)]^2\big\},\end{aligned}$$

which is a Banach space endowed with its natural norm.

We have the following result.

Proposition 4.2. *Let the assumptions of Proposition 4.1 be satisfied. Then, for any $z^0 \in L^2(0,L)$, and any triplet (f^0,f^1,f^2) such that*

$$\int_0^T e^{2s\beta^*}\|f^0\|_{H^{-2}(0,L))}\,dt < +\infty, \qquad \int_0^T\!\!\int_0^L e^{2s\beta^*}|f^i|^2\,dx\,dt < +\infty, \quad i=1,2, \tag{4.5}$$

and any (g^1,g^2,h^1,h^2) such that

$$\int_0^T e^{2s\beta^*}(|g^i(t)|^2+|h^i(t)|^2)\,dt < +\infty, \quad i=1,2, \tag{4.6}$$

there exists a control $f \in L^2((0,L)\times(0,T))$ such that the solution (z,ϕ^1,ϕ^2) of system (1.22) satisfies $(z,\phi_1,\phi_2,f)\in\mathcal{S}$. In particular, $z(x,T)=0$ in $(0,L)$.

Proof. We follow a classical strategy (see [5,17], for instance). To fix the ideas, we are going to assume that (1.20) holds. In this case, we are going to use Observability estimate (4.4). The case where (1.19) holds follows in analogous way, by using estimate (4.3) and a simiar argument as below.

Consider the space

$$\begin{aligned}\mathcal{P}_0 = \big\{(p,q^1,q^2)\in C^4([0,L]\times[0,T]):\quad & p(0,t)=p_x(0,t)=p(L,t)=p_x(L,t)=0 \quad \forall t\in(0,T),\\
& q^1(0,t)-\tfrac{1}{\mu_1}p_{xxx}(0,t)=q^1_x(0,t)+\tfrac{1}{\mu_2}p_{xx}(0,t)=0 \quad \forall t\in(0,T),\\
& q^1(L,t)=q^1_x(L,t)=0 \quad \forall t\in(0,T),\\
& q^2(x,0)=q^2_x(0,t)=0 \quad \forall t\in(0,T),\\
& q^2(L,t)-\tfrac{1}{\nu_1}p_{xxx}(L,t)=q^2_x(L,t)+\tfrac{1}{\nu_2}p_{xx}(L,t)=0 \quad \forall t\in(0,T)\big\}.\end{aligned}$$

Now, let $b:\mathcal{P}_0\times\mathcal{P}_0\to\mathbb{R}$ be the bilinear functional

$$\begin{aligned}& b\big((p,q^1,q^2),(\tilde p,\tilde q^1,\tilde q^2)\big)\\
&\quad = \int_0^T\!\!\int_0^L e^{-2s\hat\beta}\hat\zeta^7\big(\mathcal{L}^*p-\alpha_1 q^1\mathbb{1}_{\mathcal{O}_{1,d}}-\alpha_2 q^2\mathbb{1}_{\mathcal{O}_{2,d}}\big)\big(\mathcal{L}^*\tilde p-\alpha_1\tilde q^1\mathbb{1}_{\mathcal{O}_{1,d}}-\alpha_2\tilde q^2\mathbb{1}_{\mathcal{O}_{2,d}}\big)\,dx\,dt\\
&\qquad + \sum_{i=1}^2\int_0^T\!\!\int_0^L e^{-2s\hat\beta}\hat\zeta^7\mathcal{L}q^i\mathcal{L}\tilde q^i\,dxdt + \int_0^T\!\!\int_{\mathcal{O}} e^{-2s(8\hat\beta-7\beta^*)}\hat\zeta^{72}p\tilde p\,dxdt.\end{aligned}$$

Also, define $\ell:\mathcal{P}_0\to\mathbb{R}$ the linear functional

$$\begin{aligned}\ell(\tilde p,\tilde q^1,\tilde q^2) &= \langle f_0,\tilde p\rangle_{L^1(0,T;H^{-2})\,L^\infty(0,T;H_0^2)} + \int_0^T\!\!\int_0^L\big(f^1\tilde q^1+f^2\tilde q^2\big)\,dx\,dt + \langle z^0(x),\tilde p(x,0)\rangle_{H^{-2}\,H_0^2}\\
&\quad - \int_0^T h^2(t)\tilde p_{xx}(L,t)\,dt + \int_0^T h^1(t)\tilde p_{xx}(0,t)\,dt + \int_0^T g^2(t)\tilde p_{xxx}(L,t)\,dt - \int_0^T g^1(t)\tilde p_{xxx}(0,t)\,dt.\end{aligned}$$

Using Proposition 4.1, one can show that $b(\cdot,\cdot)^{1/2}$ defines a norm in $\mathcal{P}_0$. We denote by $\mathcal{P}$ the closure of $\mathcal{P}_0$ with respect to this norm. Furthermore, $\mathcal{P}$ is a Hilbert space with the inner product coming from b and, again from (4.4), ℓ is bounded in $\mathcal{P}$. Therefore, by Lax-Milgram's Theorem, we deduce that the problem: find $(p, q^1, q^2) \in \mathcal{P}$ such that

$$b\big((p, q^1, q^2), (\tilde{p}, \tilde{q}^1, \tilde{q}^2)\big) = \ell(\tilde{p}, \tilde{q}^1, \tilde{q}^2) \quad \forall(\tilde{p}, \tilde{q}^1, \tilde{q}^2) \in \mathcal{P} \tag{4.7}$$

possesses a unique solution that we call $(\hat{p}, \hat{q}^1, \hat{q}^2)$.

Define

$$\begin{cases} \hat{z} = e^{-2s\hat{\beta}}\hat{\zeta}^7\big(\mathcal{L}^*\hat{p} - \alpha_1\hat{q}^1\mathbb{1}_{\mathcal{O}_{1,d}} - \alpha_2\hat{q}^2\mathbb{1}_{\mathcal{O}_{2,d}}\big), \\ \hat{\phi}^i = e^{-2s\hat{\beta}}\zeta^7\mathcal{L}\hat{q}^i, \quad i = 1, 2, \\ \hat{f} = -e^{-2s(8\hat{\beta}-7\beta^*)}\hat{\xi}^{72}\hat{p}\mathbb{1}_{\mathcal{O}}. \end{cases} \tag{4.8}$$

Since $b\big(\hat{p}, \hat{q}^1, \hat{q}^2\big) < +\infty$, we have that

$$\int_0^T\!\!\int_0^L e^{2s\hat{\beta}}\hat{\zeta}^{-7}|\hat{z}|^2\,dx\,dt + \sum_{i=1}^2\int_0^T\!\!\int_0^L e^{2s\hat{\beta}}\zeta^{-7}|\hat{\phi}^i|^2\,dx\,dt + \int_0^T\!\!\int_{\mathcal{O}} e^{2s(8\hat{\beta}-7\beta^*)}\hat{\zeta}^{-72}|\hat{f}|^2\,dx\,dt < +\infty. \tag{4.9}$$

It is not difficult to see that $(\hat{z}, \hat{\phi}^1, \phi^2)$ is in fact the solution in the transposition sense of (1.22) (see Definition A.12) with $f = \hat{f}$. Now, we prove that this solution is indeed more regular.

Let $\hat{\rho}(t) = e^{s(8\hat{\beta}-7\beta^*)}\hat{\xi}^{-36}$ and define $z_* := \hat{\rho}\hat{z}$ and $\phi_*^i := \hat{\rho}\hat{\phi}^i$, $i = 1, 2$. Then, $(z_*, \phi_*^1, \phi_*^2)$ solves

$$\begin{cases} \mathcal{L}z_* = f_*^0 + f_*\,\mathbb{1}_{\mathcal{O}} - \dfrac{1}{\mu_1}\phi_*^1\mathbb{1}_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\phi_*^2\mathbb{1}_{\mathcal{O}_2} + \rho_t\hat{z} & (x,t) \in Q, \\ \mathcal{L}^*\phi_*^i = f_*^i + \alpha_i z_*\mathbb{1}_{\mathcal{O}_{i,d}} - \rho_t\hat{\phi}^i \quad i = 1, 2 & (x,t) \in Q, \\ z_*(0,t) = -\frac{1}{\mu_1}\phi_{*xxx}^1(0) + h^1(t), z_*(L,t) = \frac{1}{\nu_1}\phi_{*xxx}^2(L) + h^2(t) & t \in (0,T), \\ z_{*x}(0,t) = \frac{1}{\mu_2}\phi_{*xx}^1(0) + r^1(t), z_{*x}(L,t) = -\frac{1}{\nu_2}\phi_{*xx}^2(L) + r^2(t) & t \in (0,T), \\ \phi_*^i(0,t) = \phi_*^i(L,t) = \phi_{*x}^i(0,t) = \phi_{*x}^i(L,t) = 0 \quad i = 1, 2 & t \in (0,T), \\ z_*(x,0) = e^{s\beta_1^*(0)}\hat{\zeta}_1^{-15/2}(0)z^0(x), \quad \phi_*^i(x,T) = 0 & x \in (0,L), \end{cases}$$

where we have denoted $(f_*^0, f_*^i, f_*) = \hat{\rho}(f^0, f^i, \hat{f})$, $i = 1, 2$. From the fact that $|\rho_t| \leqslant Ce^{s(8\hat{\beta}-7\beta^*)}\hat{\zeta}^{-32}$, (4.5) that $e^{s(8\hat{\beta}-7\beta^*)} \leqslant e^{s\hat{\beta}}$, and from (4.9), we get that all the right-hand side of the previous system are in $L^2(Q)$. Consequently, using Proposition (A.9), we obtain that

$$(z_*, \phi_*^1, \phi_*^2) \in C([0,T]; H^{-2}(0,L)) \cap L^2((0,L)\times(0,T)) \times [C([0,T]; H_0^2(0,L)) \cap L^2(0,T; H^4(0,L))]^2$$

and this completes the proof of Proposition 4.2. □

5 Controllability of the nonlinear system

In this section we complete the proof of Theorem 1.2. The proof is based on a local inversion argument, for which we use the following theorem (see [1]).

Theorem 5.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces and let $\mathcal{F} : \mathcal{B}_1 \to \mathcal{B}_2$ satisfy $\mathcal{F} \in C^1(\mathcal{B}_1; \mathcal{B}_2)$. Assume that $b_1 \in \mathcal{B}_1$, $\mathcal{F}(b_1) = b_2$ and that $\mathcal{F}'(b_1) : \mathcal{B}_1 \to \mathcal{B}_2$ is surjective. Then, there exists $\delta > 0$ such that, for every $b' \in \mathcal{B}_2$ satisfying $\|b' - b_2\|_{\mathcal{B}_2} < \delta$, there exists a solution of the equation*

$$\mathcal{F}(b) = b', \quad b \in \mathcal{B}_1.$$

Now, we proceed to the proof of Theorem 1.2.

Proof. Let us define the spaces

$$\mathcal{B}_1 := \mathcal{S},$$

$$\mathcal{B}_2 := X \times L^2(0,L) \times Y \times Y \times Z \times Z,$$

where $X := \big\{u : e^{s\beta^*}u \in L^2((0,T);H^{-2}(0,L)\big\}$, $Y := \big\{u : e^{s\beta^*}u \in L^2((0,L)\times(0,T))\big\}$, and $Z = \big\{u : e^{s\beta^*}u \in L^2(0,T)\big\}$.

For every $(z,\phi^1,\phi^2,f) \in \mathcal{S}$, let the operator $\mathcal{F} : \mathcal{B}_1 \to \mathcal{B}_2$ be defined by

$$\mathcal{F}(z,\phi^1,\phi^2,f) := \begin{pmatrix} \mathcal{L}z + zz_x - f\mathbb{1}_{\mathcal{O}} \\ z(x,0) \\ \mathcal{L}^*\phi^1 - z\phi^1 - \alpha_1 z\mathbb{1}_{\mathcal{O}_{1,d}} \\ \mathcal{L}^*\phi^2 - z\phi^2 - \alpha_2 z\mathbb{1}_{\mathcal{O}_{2,d}} \\ \big((z+\frac{1}{\mu_1}\phi^1_{xxx})|_{x=0}, (z-\frac{1}{\nu_1}\phi^2_{xxx})|_{x=L}\big) \\ \big((z_x-\frac{1}{\mu_2}\phi^1_{xx})|_{x=0}, (z_x+\frac{1}{\nu_2}\phi^2_{xx})|_{x=L}\big) \end{pmatrix}.$$

To check the hypothesis of Theorem 5.1, the following lemma will be useful.

Lemma 5.2. *The nonlinear maps* $A : L^2(e^{s\hat{\beta}}\hat{\xi}^{-\frac{7}{2}}(0,T);L^2(0,L)) \to X$ *defined by* $A(z) = (|z|^2)_x$, *and* $B : L^2(e^{s\hat{\beta}}\hat{\xi}^{-\frac{7}{2}}(0,T);L^2(0,L)) \times L^2(e^{s(8\hat{\beta}-7\beta^*)}\hat{\xi}^{-36}(0,T);H^4(0,L)) \to Y$ *given by* $B(z,\phi) = z\phi_x$ *are* C^1.

Proof. For any test function υ, we have that

$$\begin{aligned} \langle e^{s\beta^*}\big(A(z)-A(z^*)\big),\upsilon\rangle &= \langle e^{s\beta^*}\big(|z|^2-|z^*|^2\big)_x,\upsilon\rangle \\ &= -\iint_Q e^{2s\hat{\beta}}\hat{\xi}^{-7}\big(|z|^2-|z^*|^2\big)e^{-2s\hat{\beta}}\hat{\xi}^{7}e^{s\beta^*}\upsilon_x\,dxdt \\ &\leqslant C\|\upsilon_x\|_\infty\|e^{s\hat{\beta}}\hat{\xi}^{-\frac{7}{2}}(z+z^*)\|_2\|e^{s\hat{\beta}}\hat{\xi}^{-\frac{7}{2}}(z-z^*)\|_2. \end{aligned} \tag{5.1}$$

This proves that A is continuous.

It is not difficult to see that $A'(z)(h) = 2zh$, and then

$$\begin{aligned} \langle e^{s\beta^*}(A'(z)-A(z^*))(h),\upsilon\rangle &= \langle e^{s\beta^*}\big((z-z^*)h\big)_x,\upsilon\rangle \\ &= -\iint_Q e^{2s\hat{\beta}}\hat{\xi}^{-7}\big((z-z^*)h\big)e^{-2s\hat{\beta}}\hat{\xi}^{7}e^{s\beta^*}\upsilon_x\,dxdt \\ &\leqslant C\|\upsilon_x\|_\infty\|e^{s\hat{\beta}}\hat{\xi}^{-\frac{7}{2}}(z-z^*)\|_2\|e^{s\hat{\beta}}\hat{\xi}^{-\frac{7}{2}}h\|_2. \end{aligned} \tag{5.2}$$

Consequently, A' is also continuous.

To prove that B is also C^1, we proceed in a very similar way.

$$\begin{aligned} &\langle e^{s(8\hat{\beta}-7\beta^*)}\hat{\xi}^{-36}\big(B(z,\phi)-B(z^*,\phi^*)\big),\upsilon\rangle = \langle e^{s(8\hat{\beta}-7\beta^*)}\hat{\xi}^{-36}\big(z\phi_x - z^*\phi^*_x\big)_x,\upsilon\rangle \\ &= -\iint_Q e^{s(8\hat{\beta}-7\beta^*)}\hat{\xi}^{-36}\big(z\phi_x-z^*\phi^*_x\big)\upsilon_x\,dxdt = -\iint_Q e^{s(8\hat{\beta}-7\beta^*)}\hat{\xi}^{-36}\big((z-z^*)\phi_x - z^*(\phi^*_x-\phi)\big)\upsilon_x\,dxdt \\ &= -\iint_Q e^{s(8\hat{\beta}-7\beta^*)}\hat{\xi}^{-36}\big((z-z^*)\phi_x - z^*(\phi^*_x-\phi)\big)\upsilon_x\,dxdt \\ &\leqslant C\|\upsilon_x\|_\infty\big(\|e^{s\hat{\beta}}\hat{\xi}^{-\frac{7}{2}}(z-z^*)\|_2^2\|e^{s(8\hat{\beta}-7\beta^*)}\hat{\xi}^{-36}\phi_x\|_2^2 + \|e^{s\hat{\beta}}\hat{\xi}^{-\frac{7}{2}}z^*\|_2^2\|e^{s(8\hat{\beta}-7\beta^*)}\hat{\xi}^{-36}(\phi^*_x-\phi_x)\|_2^2\big), \end{aligned} \tag{5.3}$$

and so B is continuous. To prove that B is C^1, we just have to prove that $B'(z,\phi)(z^*,\phi^*) = z\phi^*_x + z^*\phi_x$ is also continuous. The computations will be ommited since they are very similar to (5.3). ☐

From the definition of $\mathcal{B}_1$, $\mathcal{B}_2$ and Lemma 5.2, it is fairly simple to check that $\mathcal{F}$ is well defined and of class $C^1(\mathcal{B}_1;\mathcal{B}_2)$. Furthermore,

$$\mathcal{F}'(0,0,0,0)(z,\phi^1,\phi^2,f) = \begin{pmatrix} \mathcal{L}z - f\mathbb{1}_{\mathcal{O}} \\ z(x,0) \\ \mathcal{L}^*\phi^1 - \alpha_1 z\mathbb{1}_{\mathcal{O}_{1,d}} \\ \mathcal{L}^*\phi^2 - \alpha_2 z\mathbb{1}_{\mathcal{O}_{2,d}} \\ \big((z+\frac{1}{\mu_1}\phi^1_{xxx})|_{x=0}, (z+\frac{1}{\nu_1}\phi^2_{xxx})|_{x=L}\big) \\ \big((z_x+\frac{1}{\mu_2}\phi^1_{xx})|_{x=0}, (z_x+\frac{1}{\nu_2}\phi^2_{xx})|_{x=L}\big) \end{pmatrix}$$

is surjective from $\mathcal{B}_1$ to $\mathcal{B}_2$ thanks to Proposition 4.2. From Theorem 5.1 with $b_1 = (0,0,0,0)$, $b_2 = (0,0,0,0,0,0)$ and $b' = (0, z^0, -\alpha_1 z_{1,d} \mathbb{1}_{\mathcal{O}_{1,d}}, -\alpha_2 z_{2,d} \mathbb{1}_{\mathcal{O}_{2,d}}, 0, 0)$ we obtain the existence of a positive number δ such that if

$$\|z^0\|^2_{L^2(0,L)} + \sum_{i=1,2} \int_0^T \int_{\mathcal{O}_{i,d}} e^{2s\beta_1^*} |z_{i,d}|^2 \, dx \, dt < \delta,$$

there exists (z, ϕ^1, ϕ^2, f) solution to system (1.15) belonging to $\mathcal{S}$. In particular, $z(x,T) = 0$ in $(0,L)$ and the proof of Theorem 1.2 is done. ☐

6 Comments and open questions

In this section, we provide several variations of the problem considered in this article, where some of them can be solved straightforwardly while others are open questions. To simplify, we formulate the problems for linear systems only.

6.1 Boundary leader and distributed followers

In this problem, we change leader and followers position and we have a system of the following form:

$$\begin{cases} y_t + y_{xxxx} + \nu y_{xx} = v_1 \mathbb{1}_{\mathcal{O}_1} + v_2 \mathbb{1}_{\mathcal{O}_2} & (x,t) \in Q, \\ y(0,t) = f_1(t), \quad y(L,t) = 0 & t \in (0,T), \\ y_x(0,t) = f_2(t), \quad y_x(L,t) = 0 & t \in (0,T), \\ y(x,0) = y^0(x) & x \in (0,L), \end{cases} \tag{6.1}$$

where (v_1, v_2) are the followers and $f = (f_1, f_2)$ is the leader positioned on the left (or right) boundary. In the same way as before, for each leader fixed, the followers must be chosen as a Nash equilibrium for the cost functionals

$$J_i(f; v_1, v_2) = \frac{\alpha_i}{2} \int_0^T \int_{\mathcal{O}_{i,d}} |y - y_{i,d}|^2 \, dx dt + \frac{\mu_i}{2} \int_0^T \int_{\mathcal{O}_i} |v_i|^2 \, dx dt, \quad i = 1, 2.$$

In this case, the optimality system has the form

$$\begin{cases} y_t + y_{xxxx} + \nu y_{xx} = -\frac{1}{\mu_1} \phi^1 \mathbb{1}_{\mathcal{O}_1} - \frac{1}{\mu_2} \phi^2 \mathbb{1}_{\mathcal{O}_2} & (x,t) \in Q, \\ -\phi^i_t + \phi^i_{xxxx} + \nu \phi^i_{xx} = \alpha_i (y - y_{i,d}) \mathbb{1}_{\mathcal{O}_{i,d}} \quad i = 1,2 & (x,t) \in Q, \\ y(0,t) = f_1(t), \; y(L,t) = 0 & t \in (0,T), \\ y_x(0,t) = f_2(t), \; y_x(L,t) = 0 & t \in (0,T), \\ \phi^i(0,t) = \phi^i(L,t) = \phi^i_x(0,t) = \phi^i_x(L,t) = 0 \quad i = 1,2 & t \in (0,T), \\ y(x,0) = y^0(x), \quad \phi^i(x,T) = 0 \quad i = 1,2 & x \in (0,L), \end{cases} \tag{6.2}$$

and we have to prove the existence of (f_1, f_2) such that

$$y(\cdot, T) = 0. \tag{6.3}$$

This kind of multiobjective control problem was already considered for the linear heat equation (see [4]) and the adaptation to the KS equation follows straightforwardly. Due to that, in what follows, we provide only a sketch of the proof.

The corresponding adjoint system is given by

$$\begin{cases} -\psi_t + \psi_{xxxx} + \nu \psi_{xx} = \alpha_1 \gamma^1 \mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2 \gamma^2 \mathbb{1}_{\mathcal{O}_{2,d}} & (x,t) \in Q, \\ \gamma^i_t + \gamma^i_{xxxx} + \nu \gamma^i_{xx} = -\frac{1}{\mu_i} \psi \mathbb{1}_{\mathcal{O}_i} \quad i = 1,2 & (x,t) \in Q, \\ \psi(0,t) = \psi(L,t) = \gamma^i(0,t) = \gamma^i(L,t) = 0 \quad i = 1,2 & t \in (0,T), \\ \psi_x(0,t) = \psi_x(L,t) = \gamma^i_x(0,t) = \gamma^i_x(L,t) = 0 \quad i = 1,2 & t \in (0,T), \\ \psi(x,T) = \psi^T(x), \quad \gamma^i(x,0) = 0 \quad i = 1,2 & x \in (0,L). \end{cases} \tag{6.4}$$

Then, it is well known that the partial null controllability (6.3) is equivalent to an observability inequality of the form

$$\int_0^L |\psi(x,0)|^2\,dx + \int_0^T\int_0^L \rho^{-2}|\gamma^i|^2\,dxdt \leqslant C\int_0^T \left(|\psi_{xx}(0,t)|^2 + |\psi_{xxx}(0,t)|^2\right)\,dt, \tag{6.5}$$

for every solution of (6.4), where ρ is a weight function blowing up at $t = T$. The proof of that follows by the steps:

1. Use a boundary Carleman estimate (see [9, Theorem 3.5]) to the solutions of (6.4), obtaining an inequality of the form:

$$\tilde{I}(\psi,\gamma_1,\gamma_2) \leqslant C\int_0^T \rho_0^{-2}(t)\left(|\psi_{xx}(0,t)|^2 + |\psi_{xxx}(0,t)|^2\right)\,dt \\ + \sum_{i=1}^{2}\int_0^T \rho_1^{-2}(t)\left(|\gamma^i_{xx}(0,t)|^2 + |\gamma^i_{xxx}(0,t)|^2\right), \tag{6.6}$$

where $\tilde{I}$ is a weighted energy similar to (3.6), and ρ_0, ρ_1 are weight functions blowing up at $t = 0$ and $t = T$. So the task is to absorb the norms of the boundary terms $\gamma^i_{xx}(0,t)$ and $\gamma^i_{xxx}(0,t)$, for $i = 1, 2$, on the right-hand side of (6.6).

2. By a localization argument, we can prove that

$$\int_0^T \rho_1^{-2}(t)\left(|\gamma^i_{xx}(0,t)|^2 + |\gamma^i_{xxx}(0,t)|^2\right) \leqslant C\int_0^T \rho_1^{-2}(t)\|\gamma^i(\cdot,t)\|^2_{H_4}\,dt, \quad \text{for } i = 1, 2.$$

3. By energy estimates, we can bound this weighted H^4 norm of γ^i by a L^2 norm of ψ:

$$\int_0^T \rho_1^{-2}(t)\|\gamma^i(\cdot,t)\|^2_{H_4}\,dt \leqslant \frac{C}{\mu_i^2}\int_0^T \rho_1^{-2}(t)\|\psi(\cdot,t)\|^2_{L^2}\,dt,$$

which can finally be absorbed by $\tilde{I}(\psi,\gamma_1,\gamma_2)$ on the left-hand side of (6.6) by taking μ_i large enough.

Clearly, to all that to work, the weight functions appearing in the computations must be chosen wisely. The choices of these functions can be made by adapting the arguments of [4].

6.2 Both leader and followers on the boundary

In this case, the system can be written for instance as:

$$\begin{cases} y_t + y_{xxxx} + \nu y_{xx} = 0 & (x,t) \in Q, \\ y(0,t) = f_1(t), \quad y(L,t) = v_1(t) & t \in (0,T), \\ y_x(0,t) = f_2(t), \quad y_x(L,t) = v_2(t) & t \in (0,T), \\ y(x,0) = y^0(x) & x \in (0,L), \end{cases} \tag{6.7}$$

where (v_1, v_2) are the followers positioned on the right boundary and (f_1, f_2) is the leader positioned on the left boundary. Actually, this particular choice of positioning does not seem to be important. In the same way as before, the followers aim at being a Nash equilibrium for the cost functionals

$$J_i(f; v_1, v_2) := \frac{\alpha_i}{2}\int_0^T\int_{\mathcal{O}_{i,d}} |y - y_{i,d}|^2\,dx\,dt + \frac{\mu_i}{2}\int_0^T |v_i|^2\,dx\,dt, \quad i = 1, 2. \tag{6.8}$$

In this case, the optimality system reads as

$$\begin{cases} y_t + y_{xxxx} + \nu y_{xx} = 0 & & (x,t) \in Q, \\ -\phi^i_t + \phi^i_{xxxx} + \nu\phi^i_{xx} = \alpha_i(y - y_{i,d})\mathbb{1}_{\mathcal{O}_{i,d}} & i = 1, 2 & (x,t) \in Q, \\ y(0,t) = f_1(t),\ y(L,t) = -\frac{1}{\mu_1}\phi^1_{xxx}(L,t) & & t \in (0,T), \\ y_x(0,t) = f_2(t),\ y_x(L,t) = \frac{1}{\mu_2}\phi^2_{xx}(L,t) & & t \in (0,T), \\ \phi^i(0,t) = \phi^i(L,t) = \phi^i_x(0,t) = \phi^i_x(L,t) = 0 & i = 1, 2 & t \in (0,T), \\ y(x,0) = y^0(x), \quad \phi^i(x,T) = 0 \quad i = 1, 2 & & x \in (0,L), \end{cases} \tag{6.9}$$

and we want to find a leader (f_1, f_2) so that $y(T) = 0$.

The corresponding adjoint system is:

$$\begin{cases} -\psi_t + \psi_{xxxx} + \nu\psi_{xx} = \alpha_1\gamma^1 \mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2\gamma^2 \mathbb{1}_{\mathcal{O}_{2,d}} & (x,t) \in Q, \\ \gamma^i_t + \gamma^i_{xxxx} + \nu\gamma^i_{xx} = 0 \quad i = 1,2 & (x,t) \in Q, \\ \psi(0,t) = \psi(L,t) = \psi_x(0,t) = \psi_x(L,t) = 0 & t \in (0,T), \\ \gamma^1(0,t) = \gamma^1_x(0,t) = \gamma^1_x(L,t) = 0, \gamma^1(L,t) = -\frac{1}{\mu_1}\psi_{xxx}(L,t) & t \in (0,T), \\ \gamma^2(0,t) = \gamma^2(L,t) = \gamma^2_x(0,t) = 0, \gamma^2_x(L,t) = \frac{1}{\mu_2}\psi_{xx}(L,t) & t \in (0,T), \\ \psi(x,T) = \psi^T(x), \quad \gamma^i(x,0) = 0 \quad i = 1,2 & x \in (0,L), \end{cases} \tag{6.10}$$

and we have to prove (6.5) for every solution of (6.10). At this point we can see the issue of proving (6.5) in this case. If one tries to follow the same argument of Section 6.1, then we would need a boundary Carleman estimate for a fourth-order operator with nonhomogeneous boundary conditions, and as far as we know, this result cannot be found on the literature so far. In this way, the problem of controlling (6.9) with boundary controls is completely open.

6.3 Other concepts of equilibria

All along the paper, we have used the concept of Nash equilibrium for determining the followers, which is a noncooperative optimization criteria. It is natural to ask if other concepts of equilibrium can be applied to solving other kinds of multiobjective control problem, for instance, the cooperative ones. One classical example is the Pareto optimality criteria which essentially says that no individual or prefence criterion can be better off without making at least one individual or preference criterion worse off or without any loss thereof. In terms of control of PDEs this concept can be applied in the following way. Consider the fourth-order equation with distributed controls:

$$\begin{cases} y_t + y_{xxxx} + \nu y_{xx} = f\mathbb{1}_{\mathcal{O}} + v^1\mathbb{1}_{\mathcal{O}_1} + v^2\mathbb{1}_{\mathcal{O}_2} & (x,t) \in Q, \\ y(0,t) = y(L,t) = 0 & t \in (0,T), \\ y_x(0,t) = y_x(L,t) = 0 & t \in (0,T), \\ y(x,0) = y^0(x) & x \in (0,L), \end{cases} \tag{6.11}$$

where f is the leader, and (v_1, v_2) the followers. For this equation, define the cost functionals

$$J_i(f; v^1, v^2) := \frac{\alpha_i}{2}\int_0^T\!\!\int_{\mathcal{O}_{i,d}} |y - y_{i,d}|^2 \, dx\, dt + \frac{\mu_i}{2}\int_0^T\!\!\int_{\mathcal{O}_i} |v^i|^2 \, dx\, dt, \quad i = 1,2. \tag{6.12}$$

A pair (v^1, v^2) is said to be Pareto optimal if there is no $(\hat{v}^1, \hat{v}^2)$ so that

$$J_i(\hat{v}^1, \hat{v}^2) \leqslant J_i(v^1, v^2), \ i = 1,2$$

with at least one of these inequalities being strict. Then we search for f such that the state y associated to f and (v^1, v^2) satisfies $y(T) = 0$. At this stage, it can be proved by the weighted sum method (see [15]) that Pareto optimals can be found by the minimization of linear combinations of J_i, that is, the solutions of

$$\min_{v_1, v_2}\{(1-\lambda)J_1(v_1, v_2) + \lambda J_2(v_1, v_2)\}, \text{ for } \lambda \in (0,1).$$

Therefore, the optimality system in this case turns to be

$$\begin{cases} y_t + y_{xxxx} + \nu y_{xx} = f\,\mathbb{1}_{\mathcal{O}} - \dfrac{1}{\mu_1(1-\lambda)}((1-\lambda)\phi^1 + \lambda\phi^2)\mathbb{1}_{\mathcal{O}_1} - \dfrac{1}{\mu_2\lambda}((1-\lambda)\phi^1 + \lambda\phi^2)\mathbb{1}_{\mathcal{O}_2} & (x,t) \in Q, \\ -\phi^i_t + \phi^i_{xxxx} + \nu\phi^i_{xx} = \alpha_i(z - z_{i,d})\mathbb{1}_{\mathcal{O}_{i,d}} \quad i = 1,2 & (x,t) \in Q, \\ z(0,t) = z(L,t) = \phi^i(0,t) = \phi^i(L,t) = 0 \quad i = 1,2 & t \in (0,T), \\ z_x(0,t) = z_x(L,t) = \phi^i_x(0,t) = \phi^i_x(L,t) = 0 \quad i = 1,2 & t \in (0,T), \\ z(x,0) = z^0(x), \quad \phi^i(x,T) = 0 \quad i = 1,2 & x \in (0,L). \end{cases} \tag{6.13}$$

Notice that, for each $\lambda \in (0,1)$ fixed, system (6.13) has a similar structure as the one in [6, System (1.18)]. In this way, we expect in this case that by making similar assumptions as (1.18), (1.19) and (1.20), the problem of determining a leader f such that $y(T)=0$ is solvable.

One interesting open question which arises is whether the λ parameter can also be chosen to accomplish specific goals. For instance, one may be interested in the problem of finding λ minimizing the function

$$\lambda \mapsto \min_{\substack{f_\lambda \in L^2(\omega\times(0,T)) \\ y(T)=0}} \int_0^T\int_\omega |f_\lambda|^2\,dxdt.$$

At the present we do not know how to deal with this question, but it may be considered in forthcoming papers.

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A Well-Posedness Results

We start studying a general systems of the form

$$\begin{cases} u_t + u_{xxxx} + \nu u_{xx} + (Au)_x = F & (x,t)\in Q, \\ u(0,t)=f_0(t), u(L,t)=f_1(t) & t\in(0,T), \\ u_x(0,t)=g_0(t), u_x(L,t)=g_1(t) & t\in(0,T), \\ u(x,0)=u_0(x) & x\in(0,L). \end{cases} \tag{A.1}$$

The following result is well known, and the proof can be found in [9], or in pg. 24 of [6].

Lemma A.1. *The operator $A_\nu : D(A_\nu)\subset L^2(0,L)\mapsto L^2(0,L)$, $D(A_\nu)=H^4\cap H_0^2(0,L)$, such that $A_\nu u = -u_{xxxx}-\nu u_{xx}$ is the generator of a strongly continuous semigroup $T_\nu(t): L^2(0,L)\mapsto L^2(0,L)$ such that*

$$\|T_\nu(t)\|_{\mathcal{L}(L^2(0,L))} \leqslant e^{\frac{\nu^2}{4}t}. \tag{A.2}$$

In particular, T_0 is a strongly continuous semigroup of contraction.

An important consequence is the following

Corollary A.2. *For any $u_0\in D(A_\nu)$ and $F\in C^1([0,T];L^2(0,L))$, system*

$$\begin{cases} u_t + u_{xxxx} + \nu u_{xx} = F & (x,t)\in Q, \\ u(0,t)=u(L,t)=u_x(0,t)=u_x(L,t)=0 & t\in(0,T), \\ u(x,0)=u_0(x) & x\in(0,L), \end{cases} \tag{A.3}$$

possesses a unique strong solution $u\in C([0,T];D(A_\nu))\cap C^1([0,T];L^2(0,L))$ (see [31]).

We have the following result which gives a notion of solutions in a framework of less regular data.

Corollary A.3. *For $u_0\in L^2(0,L)$ and $F\in L^1(0,T;L^2(0,L))$, the function*

$$u(t) = T_\nu(t)u_0 + \int_0^t T_\nu(t-s)F(s)\,ds,\ t\in[0,T], \tag{A.4}$$

is such that $u\in C([0,T];L^2(0,L))$.

It is important to remind that (A.4) is what we call *mild solution* to (A.3).

Now, consider the space

$$G = L^2((0,L)\times(0,T)) + L^1(0,T;H_0^2(0,L)).$$

For $u_0 \in H_0^2(0,L)$ and $F \in G$, we can use a standard density argument from regular solutions, and classical energy estimates, and we prove that the function u satisfying (A.4) belongs to $C([0,T];H_0^2(0,L)) \cap L^2(0,T;H^4(0,L))$, moreover there exists a constant $C_0 := C\exp(C\nu^2 T)$ such that

$$\|u(\cdot,t)\|^2_{H_0^2(0,L)} + \int_0^T \|u_{xxxx}(\cdot,s)\|^2_{L^2(0,L)}\,ds \leqslant C_0\big(\|u_0\|^2_{H_0^2} + \|F_1\|^2_{L^2((0,L)\times(0,T))} + \|F_2\|_{L^1(0,T;H_0^2(0,L))}\big), \tag{A.5}$$

for every $t\in[0,T]$, where $F = F_1 + F_2$.

Another important result is the following

Proposition A.4. *Let $u_0 \in H_0^2(0,L)$ and $F \in G$. If A and B are $L^2(Q)$ functions, then the equation*

$$\begin{cases} u_t + u_{xxxx} + \nu u_{xx} + A(x,t)u + B(x,t)u_x = F & (x,t)\in Q,\\ u(0,t) = u(L,t) = u_x(0,t) = u_x(L,t) = 0 & t\in(0,T),\\ u(x,0) = u_0(x) & x\in(0,L), \end{cases} \tag{A.6}$$

possesses a unique mild solution, in the sense that $u \in C([0,T];H_0^2(0,L)) \cap L^2(0,T;H^4(0,L))$ and

$$u(t) = T_\nu(t)u_0 + \int_0^t T_\nu(t-s)\big(F(s) - A(s)u(s) - B(s)u_x(s)\big)\,ds,\ t\in[0,T]. \tag{A.7}$$

Moreover, by writting $F = F_1 + F_2$, there exists a constant $C_1 := C\exp(C(\nu^2T + \|A\|_2^2 + \|B\|_2^2))$, such that the solutions of (A.6) *satisfy*

$$\begin{aligned}\|u(\cdot,t)\|^2_{H_0^2(0,L)} &+ \int_0^T \|u_{xxxx}(\cdot,s)\|^2_{L^2(0,L)}\,ds\\ &\leqslant C_1\big(\|u_0\|^2_{H_0^2} + \|F_1\|^2_{L^2((0,L)\times(0,T))} + \|F_2\|_{L^1(0,T;H_0^2(0,L))}\big).\end{aligned} \tag{A.8}$$

Proof. For $k>0$, by making the change of variable $w(r,\zeta) = u(kr,k^4\zeta)$ with $0\leqslant r\leqslant \frac{L}{k}$ and $0\leqslant \zeta \leqslant \frac{T}{k^4}$, we can see that u is solution of (A.6) if and only if w is a solution of

$$\begin{cases} w_\zeta + w_{rrrr} + k^2\nu w_{rr} + k^4A(kr,k^4\zeta)w + k^3B(kr,k^4\zeta)w_r = k^4F(kr,k^4\zeta) & (r,\zeta)\in(0,L/k)\times(0,T/k^4),\\ w(0,\zeta) = w(L/k,\zeta) = w_r(0,s) = w_r(L/k,\zeta) = 0 & \zeta\in(0,T/k^4),\\ w(r,0) = u_0(kr) & r\in(0,L/k). \end{cases} \tag{A.9}$$

Indeed, this comes from the fact that for regular functions u

$$w_\zeta(r,\zeta) + w_{rrrr}(r,\zeta) + k^2\nu w_{rr}(r,\zeta) = k^4\big(u_t(kr,k^4\zeta) + u_{xxxx}(kr,k^4\zeta) + \nu u_{xx}(kr,k^4\zeta)\big). \tag{A.10}$$

In this way, for a given function $\mathcal{F}$, by defining $\mathcal{F}^k(r) = \mathcal{F}(kr)$, we get

$$T_{k^2\nu}(\zeta)u_0^k(r) = T_\nu(k^4\zeta)u_0(kr), \quad \text{for every} \quad r\in[0,L/k],\ \zeta\in[0,T/k^4], \tag{A.11}$$

for every $u_0\in H_0^2(0,L)$. More generaly, u satisfies (A.7) if and only if w satisfies

$$w(\zeta) = T_{k^2\nu}(\zeta)u_0^k + k^4\int_0^\zeta T_{k^2\nu}(\zeta-\tilde{s})\big(F^k(k^4\tilde{s}) - A^k(k^4\tilde{s})w(\tilde{s}) - \frac{1}{k}B^k(k^4\tilde{s})w_x(\tilde{s})\big)\,d\tilde{s}, \tag{A.12}$$

for every $0\leqslant\zeta\leqslant\frac{T}{k^4}$.

Now we proceed by a fixed point argument. Define $\Lambda : L^\infty(0,T;H_0^2(0,L)) \to L^\infty(0,T;H_0^2(0,L))$, such that $\Lambda\hat{w} = w$, where

$$w(\zeta) = T_{k^2\nu}(\zeta)u_0^k + k^4\int_0^\zeta T_{k^2\nu}(\zeta-\tilde{s})\big(F^k(k^4\tilde{s}) - A^k(k^4\tilde{s})\hat{w}(\tilde{s}) - \frac{1}{k}B^k(k^4\tilde{s})\hat{w}_x(\tilde{s})\big)\,d\tilde{s}. \tag{A.13}$$

Since $\|k^4A(k\cdot,k^4\cdot)\|_2^2 = k^3\|A(\cdot,\cdot)\|_2^2$ and $\|k^3B(k\cdot,k^4\cdot)\|_2^2 = k\|B(\cdot,\cdot)\|_2^2$, we have from (A.5) that

$$\|\Lambda\hat{w}^1 - \Lambda\hat{w}^2\|_{L^\infty(0,T;H_0^2(0,L))} \leqslant C_0(k^{\frac{3}{2}}\|A\|_2 + k^{\frac{1}{2}}\|B\|_2)\|\hat{w}^1 - \hat{w}^2\|_{L^\infty(0,T;H_0^2(0,L))}. \tag{A.14}$$

It is important to remark that the constant C_0 is invariant by the given change of variable, since $(k^2\nu)^2(T/k^4) = \nu^2T$. Now, we can take k sufficiently small to Λ be a contraction. By Banach Fixed Point Theorem, problem (A.13) possesses a unique solution $w \in L^\infty(0,T/k^4;H_0^2(0,L/k))$. To conclude, we just have to go back the change of variable, that is, we take $w(r,\zeta) = u(kr,k^4\zeta)$, for $r \in [0,L/k]$, $\zeta \in [0,T/k^4]$, and use (A.13), to get that

$$\begin{aligned} u^k(k^4\zeta) &= T_\nu(k^4\zeta)u_0^k + k^4\int_0^\zeta T_{k^2\nu}(\zeta-\tilde{s})\big(F^k(k^4\tilde{s}) - A^k(k^4\tilde{s})u^k(k^4\tilde{s}) - B^k(k^4\tilde{s})u_x^k(k^4\tilde{s})\big)\,d\tilde{s} \\ &= T_\nu(k^4\zeta)u_0^k + k^4\int_0^\zeta T_\nu(k^4(\zeta-\tilde{s}))\big(F(k^4\tilde{s}) - A(k^4\tilde{s})u(k^4\tilde{s}) - B(k^4\tilde{s})u_x(k^4\tilde{s})\big)\,d\tilde{s}. \end{aligned} \tag{A.15}$$

Above, we have used the fact that $A^ku^k = (Au)^k$ and $B^ku_x^k = (Bu_x)^k$. Finally, we take $t = k^4\zeta$, $x = kr$ and we make the change of variable $k^4\tilde{s} = s$ in the integral, to obtain (A.7).

Estimate (A.8) follows from multiplying the first equation of (A.6) by u_{xxxx} and combining integration by parts with the boundary conditions. Since this proceedure is quite standard, we will omit it. ☐

Remark A.5. *By lifting the boundary conditions, the result of Proposition A.4 can also be applied to equation* (A.1) *when the data is regular enough. An example of this classical approach can be found, for instance, in the book [12, pages 39-42].*

Now, we define the concept of transposition solution to (A.1).

Definition A.6. *Let $u_0 \in H^{-2}(0,L)$, $F \in L^1(0,T;H^{-2}(0,L))$ and f_0, f_1, g_0, g_1 functions in $L^2(0,T)$. We say that u is a solution of* (A.1) *in the transposition sense, if $u \in L^2((0,L)\times(0,T))$ satisfies*

$$\begin{aligned} \int_0^L\int_0^T u(x,t)f(x,t)\,dxdt &= \langle u_0(\cdot),\phi(\cdot,0)\rangle_{H^{-2}\,H_0^2} + \int_0^T \langle F(\cdot,t),\phi(\cdot,t)\rangle_{H^{-2}\,H_0^2}\,dt \\ &+ \int_0^T f_1(t)\phi_{xxx}(L,t)\,dt - \int_0^T f_0(t)\phi_{xxx}(0,t)\,dt - \int_0^T g_1(t)\phi_{xx}(L,t)\,dt + \int_0^T g_0(t)\phi_{xx}(0,t)\,dt, \end{aligned} \tag{A.16}$$

for every $f \in L^2((0,L)\times(0,T))$, where $\phi \in C([0,T];H_0^2(0,L)) \cap L^2(0,T;H^4(0,L))$, is the solution of

$$\begin{cases} -\phi_t + \phi_{xxxx} + \nu\phi_{xx} - A\phi_x = f & (x,t) \in Q, \\ \phi(0,t) = \phi(L,t) = \phi_x(0,t) = \phi_x(L,t) = 0 & t \in (0,T), \\ \phi(x,T) = 0 & x \in (0,L). \end{cases} \tag{A.17}$$

We remark that, despite the fact that system (A.17) is backward in time, the existence of solution to it is justified in a complete analogous way as for (A.6). In fact, by a simple change of variable of the form $t \mapsto T-t$ it is possible to convert system (A.17) into another one very similar to (A.6).

Proposition A.7. *Let $u_0 \in H^{-2}(0,L)$, $F \in L^1(0,T;H^{-2}(0,L))$ and f_0, f_1, g_0, g_1 functions in $L^2(0,L)$. There exist a unique solution in the transposition sense to* (A.1) *in the sense of Definition A.6. Moreover, the solution satisfies*

$$\begin{aligned} \int_0^L\int_0^T |u(x,t)|^2\,dxdt \leqslant Ce^{C(T\nu^2+\|A\|_2^2)}\Big(\|u_0\|_{H^{-2}(0,L)}^2 &+ \int_0^T \|F(\cdot,t)\|_{H^{-2}(0,L)}\,dt \\ &+ \int_0^T |f_0(t)|^2 + |f_1(t)|^2 + |g_0(t)|^2 + |g_1(t)|^2\,dt\Big). \end{aligned} \tag{A.18}$$

Proof. The proof follows by a standard procedure and it is based on the fact that, if we see the right-hand side of (A.16) as the operator

$$\begin{aligned}\mathcal{L}(f) = \langle u_0(\cdot), \phi(\cdot,0)\rangle_{H^{-2}\,H_0^2} + \int_0^T \langle F(\cdot,t), \phi(\cdot,t)\rangle_{H^{-2}\,H_0^2}\,dt \\ + \int_0^T f_1(t)\phi_{xxx}(L,t)\,dt - \int_0^T f_0(t)\phi_{xxx}(0,t)\,dt - \int_0^T g_1(t)\phi_{xx}(L,t)\,dt + \int_0^T g_0(t)\phi_{xx}(0,t)\,dt.\end{aligned}$$

Since, from Proposition A.4, $\phi \in C([0,T];H_0^2(0,L)) \cap L^2(0,T;H^4(0,L))$, $\mathcal{L}$ is a continuous from $L^2((0,L)\times(0,T))$ to $\mathbb{R}$. Then, the result follows from the Riesz Representation Theorem.

Estimate (A.5) follows from Holder's estimate, estimate (A.8), and the fact that

$$\begin{aligned}|\mathcal{L}(f)| \leqslant Ce^{C(T\nu^2+\|A\|_2^2)}\|f\|^2_{L^2((0,T)\times(0,L))}\Bigg(\int_0^T \|F(\cdot,t)\|_{H^{-2}(0,L)}\,dt \\ + \int_0^T |f_0(t)|^2 + |f_1(t)|^2 + |g_0(t)|^2 + |g_1(t)|^2\,dt\Bigg), \quad \text{(A.19)}\end{aligned}$$

for every $f \in L^2((0,L)\times(0,T))$. ☐

Remark A.8. *Notice that from Proposition A.4, the operator* $\mathcal{L}$ *is still continuous if we see it defined over* $L^1(0,T;H_0^2(0,L))$. *This means that the tranposition solution of* (A.1) *belongs to* $L^\infty(0,T;H^{-2}(0,L))$ *with the estimate*

$$\begin{aligned}\|u\|^2_{L^\infty(0,T;H^{-2}(0,L))} \leqslant Ce^{C(T\nu^2+\|A\|_2^2)}\Big(\|u_0\|^2_{H^{-2}(0,L)} + \int_0^T \|F(\cdot,t)\|_{H^{-2}(0,L)}\,dt \\ + \int_0^T |f_0(t)|^2 + |f_1(t)|^2 + |g_0(t)|^2 + |g_1(t)|^2\,dt\Big).\end{aligned}$$

From this estimate, we can actually prove that $u \in C([0,T];H^{-2}(0,L))$ *using an usual density argument, since for regular data we have the continuity in time (see Remark A.5).*

Now, we are ready to prove that system (1.22) possesses solutions. We have the following

Proposition A.9. *For any* $z_0 \in H^{-2}(0,L)$, $\bar{y} \in L^2(0,T;W^{1,2}(0,L))$, *and* $f \in L^2((0,L)\times(0,T))$, *and* f^0, f^1, f^2 *in* $L^2(0,T;H^{-2}(0,L))$. *If* $\mu = \min(\mu_1,\mu_2,\nu_1,\nu_2)$ *is sufficiently large, then there is a unique*

$$(z,\phi^1,\phi^2) \in C([0,T];H^{-2}(0,L)) \cap L^2((0,L)\times(0,T)) \times [C([0,T];H_0^2(0,L)) \cap L^2(0,T;H^4(0,L))]^2 \quad \text{(A.20)}$$

solution of (1.22), *that satisfies*

$$\begin{aligned}\|z\|^2_{L^\infty(0,T;H^{-2}(0,L))} + \iint_Q |z|^2\,dxdt + \int_0^L |\phi^i_{xx}(t)|^2\,dx + \iint_Q |\phi^i_{xxxx}|^2\,dxdt \\ \leqslant C(T,\nu,\|\bar{y}\|_2,\|\bar{y}_x\|_2)\left(\|f^0\|^2_{L^2(0,T;H^{-2}(0,L))} + \|f\|^2_{L^2((0,L)\times(0,T)))} + \|z^0\|^2_{H^{-2}(0,L)}\right. \\ \left. + \sum_{i=1}^2 \|f^i\|^2_{L^2(0,T;H^{-2}(0,L))}\right). \quad \text{(A.21)}\end{aligned}$$

Proof. Let $\hat{z}$ in $L^2(Q)$ and consider the following system

$$\begin{cases} z_t + z_{xxxx} + \nu z_{xx} + (\bar{y}z)_x = f^0 + f\,\mathbb{1}_{\mathcal{O}} & (x,t)\in Q, \\ -\hat{\phi}^i_t + \hat{\phi}^i_{xxxx} + \nu\hat{\phi}^i_{xx} - \bar{y}\hat{\phi}^i_x = f^i + \alpha_i\hat{z}\mathbb{1}_{\mathcal{O}_{i,d}} \quad i=1,2 & (x,t)\in Q, \\ z(0,t) = -\frac{1}{\mu_1}\hat{\phi}^1_{xxx}(0), z(L,t) = \frac{1}{\nu_1}\hat{\phi}^2_{xxx}(L) & t\in(0,T), \\ z_x(0,t) = \frac{1}{\mu_2}\hat{\phi}^1_{xx}(0), z_x(L,t) = -\frac{1}{\nu_2}\hat{\phi}^2_{xx}(L) & t\in(0,T), \\ \hat{\phi}^i(0,t) = \hat{\phi}^i(L,t) = \hat{\phi}^i_x(0,t) = \hat{\phi}^i_x(L,t) = 0 \quad i=1,2 & t\in(0,T), \\ z(x,0) = z^0(x), \quad \hat{\phi}^i(x,T) = 0 \quad i=1,2 & x\in(0,L). \end{cases} \quad \text{(A.22)}$$

From Proposition A.4, there exists $\hat{\phi}^i \in C([0,T]; H_0^2(0,L)) \cap L^2(0,T; H^4(0,L))$ such that

$$\begin{cases} -\hat{\phi}^i_t + \hat{\phi}^i_{xxxx} + \nu \hat{\phi}^i_{xx} - \bar{y}\hat{\phi}^i_x = f^i + \alpha_i \hat{z} 1\!\!1_{\mathcal{O}_{i,d}} & (x,t) \in Q, \\ \hat{\phi}^i(0,t) = \hat{\phi}^i(L,t) = \hat{\phi}^i_x(0,t) = \hat{\phi}^i_x(L,t) = 0 & t \in (0,T), \\ \hat{\phi}^i(x,T) = 0 & x \in (0,L), \end{cases}$$

and

$$\int_0^L |\hat{\phi}^i_{xx}(t)|^2 \, dx + \iint_Q |\hat{\phi}^i_{xxxx}|^2 \, dxdt \leqslant C(T, \nu, \|\bar{y}\|_2, \|\bar{y}_x\|_2) \left(\|f^i\|^2_{L^2(0,T;H^{-2}(0,L))} + \alpha_i^2 \|\hat{z}\|^2_{L^2((0,L)\times(0,T))} \right), \tag{A.23}$$

for $i = 1, 2$.

Now, from Proposition A.7, there exists $z \in C([0,T]; H^{-2}(0,L)) \cap L^2((0,L) \times (0,T))$ solution in the transposition sense of

$$\begin{cases} z_t + z_{xxxx} + \nu z_{xx} + (\bar{y}z)_x = f^0 + f\, 1\!\!1_{\mathcal{O}} & (x,t) \in Q, \\ z(0,t) = -\frac{1}{\mu_1}\hat{\phi}^1_{xxx}(0), z(L,t) = \frac{1}{\nu_1}\hat{\phi}^2_{xxx}(L), & t \in (0,T), \\ z_x(0,t) = \frac{1}{\mu_2}\hat{\phi}^1_{xx}(0), z_x(L,t) = -\frac{1}{\nu_2}\hat{\phi}^2_{xx}(L), & t \in (0,T), \\ z(x,0) = z^0(x), & x \in (0,L), \end{cases} \tag{A.24}$$

that satisfies

$$\begin{aligned} \|z\|^2_{L^\infty(0,T;H^{-2}(0,L))} + \iint_Q |z|^2 \, dxdt \leqslant C(T, \nu, \|\bar{y}\|_2, \|\bar{y}_x\|_2) \Bigg(& \|f^0\|^2_{L^2(0,T;H^{-2}(0,L))} \\ & + \|f\|^2_{L^2((0,L)\times(0,T)))} + \|z^0\|^2_{H^{-2}(0,L)} \\ + \frac{1}{\mu^2} \int_0^T |\hat{\phi}^1_{xxx}(0,t)|^2 + |\hat{\phi}^2_{xxx}(L,t)|^2 & + |\hat{\phi}^1_{xx}(0,t)|^2 + |\hat{\phi}^2_{xx}(L,t)|^2 \Bigg) \\ \leqslant C(T, \nu, \|\bar{y}\|_2, \|\bar{y}_x\|_2) \left(\|f^0\|^2_{L^2(0,T;H^{-2}(0,L))} + \|f\|^2_{L^2((0,L)\times(0,T)))} + \|z^0\|^2_{H^{-2}(0,L)} + \frac{1}{\mu^2} \sum_{i=1}^{2} \|\hat{\phi}^i\|^2_{L^2(0,T;H^4(0,L))} \right). \end{aligned} \tag{A.25}$$

We combine (A.8) and (A.18), and we get that

$$\begin{aligned} \|z\|^2_{L^\infty(0,T;H^{-2}(0,L))} + \iint_Q |z|^2 \, dxdt \leqslant C(T, \nu, \|\bar{y}\|_2, \|\bar{y}_x\|_2) \Bigg(& \|f^0\|^2_{L^2(0,T;H^{-2}(0,L))} \\ + \|f\|^2_{L^2((0,L)\times(0,T)))} + \|z^0\|^2_{H^{-2}(0,L)} + \sum_{i=1}^{2} \|f^i\|^2_{L^2(0,T;H^{-2}(0,L))} & + \frac{1}{\mu} \|\hat{z}\|^2_{L^2((0,L)\times(0,T))} \Bigg). \end{aligned} \tag{A.26}$$

Then, by defining $\Lambda^1 : L^2((0,L) \times (0,T)) \to L^2((0,L) \times (0,T))$, such that $\Lambda^1(\hat{z}) = z$, we have from (A.26) and by taking μ sufficiently large that Λ^1 is a contraction. The conclusion follow from the Banach Fixed Point Theorem. If (z, ϕ^1, ϕ^2) is the fixed point, then

$$\begin{aligned} \|z\|^2_{L^\infty(0,T;H^{-2}(0,L))} + \iint_Q |z|^2 \, dxdt \leqslant C(T, \nu, \|\bar{y}\|_2, \|\bar{y}_x\|_2) \Bigg(& \|f^0\|^2_{L^2(0,T;H^{-2}(0,L))} \\ + \|f\|^2_{L^2((0,L)\times(0,T)))} + \|z^0\|^2_{H^{-2}(0,L)} & + \sum_{i=1}^{2} \|f^i\|^2_{L^2(0,T;H^{-2}(0,L))} \Bigg). \end{aligned} \tag{A.27}$$

for μ sufficiently large. Since (A.23) also holds for the fixed point, we can combine (A.27) with (A.23), and we obtain (A.21). □

When we are dealing with the adjoint system (1.23), we have the following

Proposition A.10. *Let* $\psi^T \in H_0^2(0,L)$, $\bar{y} \in L^2(0,T;W^{1\,2}(0,L))$, *and* $(g^0,g^1,g^2) \in [L^2(Q)]^3$. *Then, problem* (1.23) *possesses a unique solution* (ψ,γ^1,γ^2) *in the space*

$$C([0,T];H_0^2(0,L)) \cap L^2(0,T;H^4(0,L)) \times [L^2(Q)]^2,$$

such that

$$\int_0^L |\psi_{xx}(t)|^2\,dt + \iint_Q |\psi_{xxxx}|^2\,dxdt + \sum_{i=1}^{2} \iint_Q |\gamma^i|^2\,dxdt \leqslant C\left(\int_0^L |\psi^T(x)|\,dx + \sum_{i=0}^{2} \iint_Q |g^i|^2\,dxdt\right). \tag{A.28}$$

Proof. We start by fixing $\hat{\psi} \in L^2(0,T;H^4(0,L))$. In this way, from Proposition (A.7), there exist $(\hat{\gamma}^1,\hat{\gamma}^2) \in [L^2(Q)]^3$ solution in the transposition sense (see (A.16)) of

$$\begin{cases} \hat{\gamma}^i_t + \hat{\gamma}^i_{xxxx} + \nu\hat{\gamma}^i_{xx} + (\bar{y}\hat{\gamma}^i)_x = g^i & (x,t) \in Q, \\ \hat{\gamma}^1(0,t) = \frac{1}{\mu_1}\hat{\psi}_{xxx}(0,t),\ \hat{\gamma}^1_x(0,t) = -\frac{1}{\mu_2}\hat{\psi}_{xx}(0,t) & t \in (0,T), \\ \hat{\gamma}^1(L,t) = \hat{\gamma}^1_x(L,t) = 0 & t \in (0,T), \\ \hat{\gamma}^2(0,t) = \hat{\gamma}^2_x(0,t) = 0 & t \in (0,T), \\ \hat{\gamma}^2(L,t) = \frac{1}{\nu_1}\hat{\psi}_{xxx}(L,t),\ \hat{\gamma}^2_x(L,t) = -\frac{1}{\mu_2}\hat{\psi}_{xx}(L,t) & t \in (0,T), \\ \hat{\gamma}^1(x,0) = \hat{\gamma}^2(x,0) = 0 & x \in (0,L), \end{cases} \tag{A.29}$$

satisfying

$$\begin{aligned} \sum_{i=1}^{2} \iint_Q |\hat{\gamma}^i|^2\,dxdt \leqslant C(T,\nu,\|\bar{y}\|_2,\|\bar{y}_x\|_2)\Big(&\sum_{i=1}^{2} |g^i|^2 \\ &+ \frac{1}{\mu}\int_0^T \left(|\hat{\psi}_{xx}(0,t)|^2 + |\hat{\psi}_{xx}(L,t)|^2 + |\hat{\psi}_{xxx}(0,t)|^2 + |\hat{\psi}_{xxx}(L,t)|^2\right) dt\Big), \end{aligned} \tag{A.30}$$

where $\mu = \min\{\mu_1,\mu_2,\nu_1,\nu_2\}$. Since $\hat{\psi} \in L^2(0,T;H^4(0,L))$, it is not difficult to see that

$$\int_0^T |\hat{\psi}_{xx}(0,t)|^2 + |\hat{\psi}_{xx}(L,t)|^2 + |\hat{\psi}_{xxx}(0,t)|^2 + |\hat{\psi}_{xxx}(L,t)|^2\,dt \leqslant C\iint_Q |\hat{\psi}_{xxxx}|^2\,dxdt. \tag{A.31}$$

Hence, by combining (A.30) and (A.31), we get that

$$\sum_{i=1}^{2} \iint_Q |\hat{\gamma}^i|^2\,dxdt \leqslant C(T,\nu,\|\bar{y}\|_2,\|\bar{y}_x\|_2)\Big(\sum_{i=1}^{2} |g^i|^2 + \frac{1}{\mu}\iint_Q |\hat{\psi}_{xxxx}|^2\,dxdt\Big). \tag{A.32}$$

From Proposition A.4, there exists a unique $\psi^* \in C([0,T];H_0^2(0,L)) \cap L^2(0,T;H^4(0,L))$, solution of

$$\begin{cases} -\psi^*_t + \psi^*_{xxxx} + \nu\psi^*_{xx} - \bar{y}\psi^*_x = g^0 + \alpha^1\hat{\gamma}^1 \mathbb{1}_{\mathcal{O}_{1,d}} + \alpha^2\hat{\gamma}^2 \mathbb{1}_{\mathcal{O}_{2,d}} & (x,t) \in Q, \\ \psi^*(0,t) = \psi^*(L,t) = \psi^*_x(0,t) = \psi^*_x(L,t) = 0 & t \in (0,T), \\ \psi^*(x,T) = \psi^T & x \in (0,L), \end{cases} \tag{A.33}$$

satisfying

$$\begin{aligned} \int_0^L |\psi^*_{xx}(t)|^2\,dt + \iint_Q |\psi^*_{xxxx}|^2\,dxdt \leqslant C(T,\nu,\|\bar{y}\|_2,\|\bar{y}_x\|_2)\Bigg(&\int_0^L |\psi^T(x)|^2\,dx \\ &+ \iint_Q |g^0|^2\,dxdt + \sum_{i=1}^{2} \iint_Q |\hat{\gamma}^i|^2\,dxdt\Bigg). \end{aligned} \tag{A.34}$$

Now, by combining (A.32) and (A.34), we obtain that

$$\int_0^L |\psi^*_{xx}(t)|^2\,dt + \iint_Q |\psi^*_{xxxx}|^2\,dxdt \leqslant C(T,\nu,\|\bar{y}\|_2,\|\bar{y}_x\|_2)\left(\int_0^L |\psi^T(x)|^2\,dx \right.$$
$$\left. + \sum_{i=0}^{2}\iint_Q |g^i|^2\,dxdt + \frac{1}{\mu}\iint_Q |\hat{\psi}_{xxxx}|^2\,dxdt\right). \quad \text{(A.35)}$$

To finish, we define the application $\Lambda^* : L^2(0,T;H^4(0,L)) \to L^2(0,T;H^4(0,L))$, such that $\Lambda^*(\hat{\psi}) = \psi^*$, where ψ^* is the solution of (A.33). Using (A.35), and the fact that the equations are linear, it is not difficult to see that Λ^* is a contraction for μ sufficiently large, and from Banach Fixed Point Theorem Λ^* possesses a unique fixed point $\psi \in L^2(0,T;H^4(0,L))$, and this implies on the existence of a solution $(\psi,\gamma^1,\gamma^2) \in L^2(0,T;H^4(0,L)) \times [L^2(Q)]^2$. Note that from (A.32) and (A.35) this fixed point satisfies (A.28) for μ sufficiently large. □

A very important consequence is the following

Corollary A.11. *Assume that $\bar{y} \in L^2((0,T);W^{1\,2}(0,L))$. Then, for every $g^0, g^1, g^2 \in L^2((0,L)\times(0,T))$ and every $\psi^T \in H^2_0(0,L)$, the solution (ψ,γ^1,γ^2) of system* (1.23) *satisfies*

$$\int_0^L |\psi_{xx}(t)|^2\,dt + \int_0^{T/2}\!\!\int_0^L |\psi_{xxxx}|^2\,dxdt + \sum_{i=1}^{2}\int_0^{T/2}\!\!\int_0^L |\gamma^i|^2\,dxdt$$
$$\leqslant C\left(\int_{T/2}^{3T/4}\!\!\int_0^L |\psi|^2\,dxdt + \sum_{i=1}^{2}\int_{T/2}^{3T/4}\!\!\int_0^L |\gamma^i|^2\,dxdt + \sum_{i=0}^{2}\int_0^{3T/4}\!\!\int_0^L |g^i|^2\,dxdt\right), \quad \text{(A.36)}$$

for μ suffciently large and C does not depend on μ.

Proof. The proof is standard and follows from the fact that, if $\theta \in C^1[0,T]$ is such that

$$\theta(t) = \begin{cases} 1 & \text{if} \quad 0 < t < T/2, \\ 0 & \text{if} \quad 3T/4 < t < T, \end{cases}$$

then $(\Psi,\Gamma^1,\Gamma^2) = (\theta\psi,\theta\gamma^1,\theta\gamma^2)$ is solution of

$$\begin{cases}
-\Psi_t + \Psi_{xxxx} + \nu\Psi_{xx} - \bar{y}\Psi_x = -\dot{\theta}(t)\psi + \theta(t)g^0 + \alpha_1\Gamma^1 \mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2\Gamma^2\mathbb{1}_{\mathcal{O}_{2,d}} & (x,t)\in Q, \\
\Gamma^i_t + \Gamma^i_{xxxx} + \nu\Gamma^i_{xx} + (\bar{y}\Gamma^i)_x = \dot{\theta}(t)\gamma^i + \theta(t)g^i & (x,t)\in Q, \\
\Psi(0,t) = \Psi(L,t) = 0 & t\in(0,T), \\
\Psi_x(0,t) = \Psi_x(L,t) = 0 & t\in(0,T), \\
\Gamma^1(0,t) = \frac{1}{\mu_1}\Psi_{xxx}(0,t), \quad \Gamma^1_x(0,t) = -\frac{1}{\mu_2}\Psi_{xx}(0,t) & t\in(0,T), \\
\Gamma^1(L,t) = \Gamma^1_x(L,t) = 0 & t\in(0,T), \\
\Gamma^2(0,t) = 0, \quad \Gamma^2_x(0,t) = 0 & t\in(0,T), \\
\Gamma^2(L,t) = \frac{1}{\nu_1}\Psi_{xxx}(L,t), \quad \Gamma^2_x(L,t) = -\frac{1}{\nu_2}\Psi_{xx}(L,t) & t\in(0,T), \\
\Psi(x,T) = 0, \quad \Gamma^i(x,0) = 0 & x\in(0,L).
\end{cases} \quad \text{(A.37)}$$

Then, we just have to apply (A.28) to system (A.37). □

Since we have proved that the adjoint system possesses solutions, we can define the solution in the transposition sense to (1.22).

Definition A.12. *We say that $(z,\phi^1,\phi^2) \in [L^2(Q)]^3$ is a solution of* (1.22) *in the transposition sense if*

$$\int_0^T\!\!\int_0^L \left(zg^0 + \phi^1 g^1 + \phi^2 g^2\right)dxdt = \int_0^T\!\!\int_0^L (f^0 + f\mathbb{1}_{\mathcal{O}})\psi\,dxdt + \sum_{i=1}^{2} f^i\gamma^i\,dxdt + \langle z_0,\psi(0)\rangle_{H^{-2}\,H^2_0}$$
$$-\int_0^T h^2(t)\psi_{xx}(L,t)\,dt + \int_0^T h^1(t)\psi_{xx}(0,t)\,dt + \int_0^T g^2(t)\psi_{xxx}(L,t)\,dt - \int_0^T g^1(t)\psi_{xxx}(0,t)\,dt, \quad \text{(A.38)}$$

for every $(g^0, g^1, g^2) \in [L^2(Q)]^3$*, where* $(\psi, \gamma^1, \gamma^2)$ *is the solution of* (1.23) *(see Proposition A.10) with* $\psi^T = 0$.

In this way, we have the following

Proposition A.13. *If* f *and* $\{f^i\}_{i=0}^2$ *are functions in* $L^2(Q)$*, then problem* (1.22) *possesses a unique solution* $(z, \phi^1, \phi^2) \in [L^2(Q)]^3$ *in the sense of Definition* (A.12).

Proof. The existence of a triplet satisfying (A.38) follows from the fact that $F : [L^2(0, L)]^3 \to \mathbb{R}$

$$F(g^0, g^1, g^2) = \int_0^T\int_0^L (f^0 + f\mathbb{1}_{\mathcal{O}})\psi \, dxdt + \sum_{i=1}^{2} f^i\gamma^i \, dxdt + \langle z_0, \psi(0)\rangle_{H^{-2}\, H_0^2}$$
$$- \int_0^T h^2(t)\psi_{xx}(L, t)\, dt + \int_0^T h^1(t)\psi_{xx}(0, t)\, dt + \int_0^T g^2(t)\psi_{xxx}(L, t)\, dt - \int_0^T g^1(t)\psi_{xxx}(0, t)\, dt,$$

is continuous (see Proposition A.10), and by using Lax Milgram's Theorem.

If (z, ϕ^1, ϕ^2) and $(\hat{z}, \hat{\phi}^1, \hat{\phi}^2)$ are two solutions, then

$$\int_0^T\int_0^L \Big((z - \hat{z})g^0 + (\phi^1 - \hat{\phi}^1)g^1 + (\phi^2 - \hat{\phi}^2)g^2\Big)\, dxdt = 0 \quad \text{for all} \quad (g^0, g^1, g^2) \in [L^2(0, L)]^3.$$

This clearly implies that $(z, \phi^1, \phi^2) = (\hat{z}, \hat{\phi}^1, \hat{\phi}^2)$. ☐

Remark A.14. *It is not difficult to see that the solution given in Proposition* (A.9) *also satisfies the transposition formulation* (A.38). *Since the solutions of* (A.38) *are unique, we obtain that the solution of* (A.38) *possesses the regularity* (A.20).

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