Existence of controls insensitizing the rotational of the solution of the Navier-Stokes system having a vanishing component

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Abstract

In this paper we study an insensitizing control problem for the Navier-Stokes system. The novelty is that we insensitize the rotational of the solution using controls with one component fixed at zero. This problem can be formulated as a null controllability problem for a nonlinear cascade system for which we follow the usual duality approach. First, we prove a suitable Carleman inequality for a system coupling two Stokes like equations, which leads to the null controllability at any positive time. Finally, we deduce a local null controllability result for the cascade system by a local inverse argument.
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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3) be a bounded simply-connected open set whose boundary $\partial\Omega$ is regular enough. Let $T > 0$ and let $\omega \subset \Omega$ be a (small) nonempty open subset which will usually be referred as *control domain*. We are going to use the notation $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$. Let us also introduce another open set $\mathcal{O} \subset \Omega$ which is called the *observation set*.

Let us remember the definition of some usual spaces in the context of incompressible fluids:

$$V = \{y \in H_0^1(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega\}$$

and

$$H = \{y \in L^2(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega,\ y \cdot \mathrm{n} = 0 \text{ on } \partial\Omega\}.$$

To be more precise about the investigated problem, we present the following control system with incomplete data:

$$\begin{cases} y_t - \Delta y + (y,\nabla)y + \nabla p = f + v\mathbb{1}_\omega, \quad \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 + \tau \hat{y}^0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here, $y(x,t) = (y_i(x,t))_{1\le i\le N}$ is the velocity of the particles of an incompressible fluid, $v = (v_j)_{j=1}^N$ is a distributed control localized in ω, $f(x,t) = (f_i(x,t))_{1\le i\le N} \in L^2(Q)^N$ is a given, externally applied force, and we have denoted

$$\big((y^1,\nabla)y^2\big)_i = \sum_{j=1}^N y_j^1 \partial_j y_i^2, \quad i = 1,\dots,N.$$

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The initial state $y(0)$ is partially unknown in the following sense: we suppose that $y^0 \in H$ is known, $\hat{y}^0 \in H$ is unknown with $\|\hat{y}^0\|_{L^2(\Omega)^N} = 1$ and that τ is a small unknown real number.

The objective of this study is to prove the existence of controls that insensitize some functional J_τ depending on the velocity field y. In other words, we have to find a control v such that the influence of the unknown data $\tau\hat{y}^0$ is not perceptible for our functional:

$$\left.\frac{\partial J_\tau(y)}{\partial \tau}\right|_{\tau=0} = 0 \text{ for all } \hat{y}^0 \in L^2(\Omega)^N \text{ such that } \left\|\hat{y}^0\right\|_{L^2(\Omega)^N} = 1, \tag{1.2}$$

In the important work [30], J.-L. Lions considers this kind of problem and introduces many related questions. One of these questions, in non-classical terms, was the existence of insensitizing controls for the Navier-Stokes equations.

In this work the idea of using a new *observation functional* arose, which was related to the notion of *curl*. As a motivation the rotor or rotational physically measures the rotation in the movement of a fluid, in this case (1.2) means that a small perturbation in the initial condition does not alter (desensitize) the rotation in the movement of the fluid. For example, the *curl* is involved when an airplane suffers turbulence during a flight because it can measure the chaos that is generated behind the wings of the airplanes, or when turbulence is generated at the rudder of a ship, see [23, 41, 33]. In the literature the usual observation functional is given by the square of the local L^2-norm of the state variable y (see [6, 29, 36]). Here, the functional is given by the square of the local L^2-norm of the rotational of the state variable y, that is:

$$J_\tau = \iint\limits_Q |\nabla \times y|^2 \chi \mathrm{d}x\, \mathrm{d}t, \tag{1.3}$$

where $\chi : \Omega \to \mathbb{R}$ is a bounded function such that $\mathrm{supp}(\chi) \Subset \mathcal{O}$, $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in an open set $\mathcal{O}_0$ with $\mathcal{O}_0 \Subset \mathcal{O}$. Actually, for some technical reasons described below, we will assume that

$$\chi = \mathbb{1}_{\mathcal{O}} \quad \text{if } N = 2, \tag{1.4}$$

where $\mathbb{1}_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$, and

$$\chi \in C_0^\infty(\mathcal{O}) \quad \text{if } N = 3. \tag{1.5}$$

The relevance of taking χ as (1.4) or (1.5) is explained in Remark 4.2.

The first results of existence of insensitizing controls were obtained for the heat equation in [6, 39]. Both papers are concerned as functional the local L^2-norm of the state.

An important topic in the control theory is controllability with controls having some vanishing components, which can be an interesting problem from an applications point of view. The first studies were obtained in [17] for the local exact controllability to the trajectories of the Navier-Stokes and Boussinesq system when the closure of the control set ω intersects the boundary of Ω. Then, this geometric assumption was removed for the Stokes system in [12], for the local null controllability of the Navier-Stokes system in [9], and for the Boussinesq system in [7]. Recently, the local null controllability of the three dimensional Navier-Stokes system with a control having two vanishing components has been obtained in [13].

Now, related studies with insensitivity for fluids, the first result was obtained in [31], where the author prove the existence of ϵ-insensitizing controls of the form $(v_1, v_2, 0)$ for the 3D-Stokes system. Later, the existence of insensitizing controls for the Stokes system is demonstraded in [19] and for the Navier-Stokes system in [21]. Finally, in [11], the existence of insensitizing controls for the Navier-Stokes system with one vanishing component was established. The present paper can be considered as a continuation of this last study. The main goal is to establish the existence of insensitizing controls for the Navier-Stokes system (1.1) having one vanishing component, that is, $v_i \equiv 0$ for any given $i \in \{1, \dots, N\}$. Notice that if $N = 2$, this means $v \equiv 0$. Also, here we are going to use a functional not usual in the literature.

The special form of the observation functional J_τ allows us to transform our insensitizing problem as a controllability problem of a cascade system (for more details, see [6], for instance). In particular, condition (1.2) is equivalent to $z(0) = 0$ in Ω, where z together with w solves the following coupled system:

$$\begin{cases} w_t - \Delta w + (w, \nabla) w + \nabla p_w = f + v\mathbb{1}_\omega, \quad \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + (z, \nabla^t) w - (w, \nabla) z + \nabla p_z = \nabla \times ((\nabla \times w)\chi), \quad \nabla \cdot z = 0 & \text{in } Q, \\ w = 0, \quad z = 0 & \text{on } \Sigma, \\ w(0) = y^0, \quad z(T) = 0 & \text{in } \Omega. \end{cases} \tag{1.6}$$

Here, (w, p_w) is the solution of system (1.1) for $\tau = 0$, the equation of z corresponds to a formal adjoint of the equation satisfied by the derivate of y with respect to τ at $\tau = 0$ and we have denoted

$$\big((z, \nabla^t)w\big)_i = \sum_{j=1}^{N} z_j \partial_i w_j, \quad i = 1, \dots, N.$$

In effect, differentiating y solution of (1.1) with respect to τ and evaluating it at $\tau = 0$, condition (1.2) reads

$$\iint_Q y^\tau \cdot \nabla \times ((\nabla \times w)\chi) \mathrm{d}x\, \mathrm{d}t = 0, \quad \forall \hat{y}^0 \in L^2(\Omega)^N \text{ such that } \|\hat{y}^0\|_{L^2(\Omega)^N} = 1, \tag{1.7}$$

where y^τ is the derivate of y solution of (1.1) at $\tau = 0$. Then, y^τ solves

$$\begin{cases} y^\tau_t - \Delta y^\tau + (y^\tau, \nabla)y + (y, \nabla)y^\tau + \nabla p^\tau = 0 & \text{in } Q, \\ \nabla \cdot y^\tau = 0 & \text{in } Q, \\ y^\tau = 0 & \text{on } \Sigma, \\ y^\tau(0) = \hat{y}^0 & \text{in } \Omega. \end{cases} \tag{1.8}$$

Hence, substituting $\nabla \times ((\nabla \times w)\chi)$ by the left-hand side of the equation of z in (1.6) and integrating by parts we obtain

$$\int_\Omega z(0)\hat{y}^0 \mathrm{d}x = \iint_Q y^\tau \cdot \nabla \times ((\nabla \times w)\chi) \mathrm{d}x\, \mathrm{d}t, \quad \forall \hat{y}^0 \in L^2(\Omega)^N \text{ such that } \|\hat{y}^0\|_{L^2(\Omega)^N} = 1. \tag{1.9}$$

Combining (1.7) with (1.9), we deduce that $z(0) = 0$ in Ω.

We are going to prove the following controllability result for system (1.6):

Theorem 1.1. *Let $i \in \{1, \dots, N\}$, $m \geq 14$ be a real number, and $\chi : \Omega \to \mathbb{R}$ given by (1.4) if $N = 2$, or (1.5) if $N = 3$. Assume $\omega \cap \mathcal{O} \neq \emptyset$ and $y^0 \equiv 0$. Then, there exist $\delta > 0$ and $\hat{C} > 0$ depending on Ω, ω, $\mathcal{O}$ and T such that for any $f \in L^2(Q)^N$ satisfying $\left\| e^{\hat{C}/t^m} f \right\|_{L^2(Q)^N} < \delta$, there exists a control $v \in L^2(\omega \times (0,T))^N$ with $v_i \equiv 0$ and a corresponding solution (w, z) of (1.6) satisfying $z(0) = 0$ in Ω.*

Remark 1.1. *Besides, respect to insensitizing the functional J_τ one can lead the state w to 0 at time $t = T$ just by assuming an extra condition on f at time $t = T$*

$$\left\| e^{\frac{\hat{C}}{t^m (T-t)^m}} f \right\|_{L^2(Q)^N} < +\infty, \tag{1.10}$$

for a constant $\hat{C}$ probably different to the one shown in Theorem 1.1.

Remark 1.2. *The condition $y^0 = 0$ in the main theorem is due to the fact that the first equation in (1.6) is forward and the second one is backward in time. Other works related with insensitizing controls in the parabolic case, including linear equations, assume this condition on the initial data. A study of the possible initial conditions which can be insensitized is made for the heat equation in [40]. This work shows that the answer is not obvious.*

Remark 1.3. *Notice that if $\omega \cap \mathcal{O} \neq \emptyset$, it is always possible to choose $\mathcal{O}_0 \Subset \mathcal{O}$ such that $\mathcal{O}_0 \cap \omega \neq \emptyset$. Thus, from now on, the open set $\mathcal{O}_0$ from (1.3) will be fixed like this.*

As a consequence of Theorem 1.1, we obtain the following result:

Corollary 1.1. *There are insensitizing controls v for the functional J_τ given by (1.3).*

To prove Theorem 1.1 we follow a standard approach introduced in [18] (see also, [9, 17, 25]). We first deduce a null controllability result for the linear system:

$$\begin{cases} w_t - \Delta w + \nabla p_w = f^w + v \mathbb{1}_\omega + f, \quad \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + \nabla p_z = f^z + \nabla \times ((\nabla \times w)\chi), \quad \nabla \cdot z = 0 & \text{in } Q, \\ w = 0, \quad z = 0 & \text{on } \Sigma, \\ w(0) = y^0, \quad z(T) = 0 & \text{in } \Omega, \end{cases} \tag{1.11}$$

where f^w and f^z are going to be taken to decrease exponentially to zero at $t = 0$.

The main tool to prove this controllability result for system (1.11), and the second main result of this study, is a suitable Carleman estimate for the solutions of its adjoint system, namely,

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = g^\varphi + \nabla \times ((\nabla \times \psi)\chi), \quad \nabla \cdot \varphi = 0 & \text{in } Q, \\ \psi_t - \Delta\psi + \nabla h = g^\psi, \quad \nabla \cdot \psi = 0 & \text{in } Q, \\ \varphi = 0, \quad \psi = 0 & \text{on } \Sigma, \\ \varphi(T) = 0, \quad \psi(0) = \psi^0 & \text{in } \Omega, \end{cases} \tag{1.12}$$

where $\psi^0 \in H$, g^φ and g^ψ are going to be taken with different regularity properties that will be detailed later on. In fact, this Carleman inequality is of the form

$$\iint\limits_Q \tilde{\rho}_1^2(t)\left(|\varphi|^2 + |\psi|^2\right) \mathrm{d}x\mathrm{d}t \le C \left(\left\| \tilde{\rho}_2(t)\left(g^\varphi, g^\psi\right) \right\|_X^2 + \sum_{\substack{j=1 \\ j\neq i}}^{N} \iint\limits_{\omega\times(0,T)} \tilde{\rho}_3^2(t)|\varphi_j|^2 \mathrm{d}x\mathrm{d}t \right), \tag{1.13}$$

where $\tilde{\rho}_k(t)$, $k \in \{1, 2, 3\}$, are positive weight functions, $j \in \{1, \dots, N\} \setminus \{i\}$, $C > 0$ only depends on Ω, ω, $\mathcal{O}$ and T and X is a suitable Banach space. This estimate is stated in Proposition 3.1.

This paper is organized as follows. In Section 2, we state the main results that we are going to use in the following sections. In Section 3, we prove a Carleman inequality for the adjoint system (1.12). In Section 4, we show the null controllability of the linearized cascade system (1.11). Finally, in Section 5, we deal with the null controllability for the nonlinear cascade system (1.6).

2 Technical results and notations

In this section we introduce some notation and all the technical results needed in this work.

2.1 Some notations

We denote by $Y_0 := L^2(0, T; H)$. For $n \in \mathbb{Z}^+$, we define the space Y_n as follows:

$$Y_n := L^2(0, T; H^{2n}(\Omega)^N \cap V) \cap H^n(0, T; L^2(\Omega)^N),$$

given by the norm

$$\|u\|_{Y_n}^2 := \|u\|_{L^2(0,T;H^{2n}(\Omega)^N)}^2 + \|u\|_{H^n(0,T;L^2(\Omega)^N)}^2.$$

The following subspace is going to be used only in Section 4. For every $n \in \mathbb{Z}^+$, we set

$$Y_{n,0} := \{u \in Y_n : [\mathcal{L}_H^k u]_{|\Sigma} = 0, \ [\mathcal{L}_H^k u](0) = 0, \ k = 0, \dots, n-1\},$$

endowed with the equivalent norm (by Lemma 2.4 with $u_0 \equiv 0$),

$$\|u\|_{Y_{n,0}}^2 := \|\mathcal{L}_H^n u\|_{L^2(Q)^N}^2,$$

Here, $\mathcal{L}_H := \partial_t - \mathcal{P}_L(\Delta)$, where $\mathcal{P}_L$ denotes the Leray projector over the space H, i.e. $\mathcal{P}_L : L^2(Q)^N \mapsto L^2(Q)^N$, $\mathcal{P}_L u := u - \nabla p$, where $\Delta p = \nabla \cdot u$ in Ω and $\nabla p \cdot \overrightarrow{n} = u \cdot \overrightarrow{n}$ on $\partial\Omega$ (see [37], pages 16-18).

Also, we denote by $X_0 := L^2(0, T; V)$ and for $n \in \mathbb{Z}^+$, we define the space X_n as:

$$X_n := L^2(0, T; H^{2n+1}(\Omega)^N \cap V) \cap H^1(0, T; H^{2n-1}(\Omega)^N),$$

given by the norm

$$\|u\|_{X_n}^2 := \|u\|_{L^2(0,T;H^{2n+1}(\Omega)^N)}^2 + \|u\|_{H^1(0,T;H^{2n-1}(\Omega)^N)}^2.$$

2.2 Carleman estimates

Here, we present some Carleman estimates needed to prove estimate (1.13). These inequalities have been proved in previous papers and we give precise references about where to find each one of them. Before we can establish these estimates, let us introduce some classical weight functions. Let ω_0 be a nonempty open subset of $\mathbb{R}^N$ such that $\omega_0 \Subset \omega \cap \mathcal{O}_0$ (see Remark 1.3) and $\eta \in C^2(\overline{\Omega})$ such that

$$|\nabla\eta| > 0 \text{ in } \overline{\Omega \setminus \omega_0}, \quad \eta > 0 \text{ in } \Omega \quad \text{and} \quad \eta \equiv 0 \text{ on } \partial\Omega.$$

The proof of the existence of such a function η is given in [18]. Let also $\ell \in C^\infty([0,T])$ be a positive function in $(0,T)$ satisfying

$$\begin{array}{ll} \ell(t) = t, & \forall t \in [0, T/4], \\ \ell(t) = T - t, & \forall t \in [3T/4, T], \\ \ell(t) \leq \ell(T/2), & \forall t \in [0, T]. \end{array}$$

Then, for all $\lambda \geq 1$ and $m \geq 14$ we consider the following weight functions:

$$\begin{array}{ll} \alpha(x,t) = \dfrac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\ell(t)^m}, & \xi(x,t) = \dfrac{e^{\lambda\eta(x)}}{\ell(t)^m}, \\ \alpha^*(t) = \max\limits_{x\in\overline{\Omega}} \alpha(x,t), & \xi^*(t) = \min\limits_{x\in\overline{\Omega}} \xi(x,t), \\ \hat{\alpha}(t) = \min\limits_{x\in\overline{\Omega}} \alpha(x,t), & \hat{\xi}(t) = \max\limits_{x\in\overline{\Omega}} \xi(x,t). \end{array} \tag{2.1}$$

Notice that from (2.1), we obtain the following properties:

$$|\partial_t^n\alpha|, |\partial_t^n\xi| \leq C\xi^{(1+n/m)}, \quad |\partial_x^\iota\alpha|, |\partial_x^\iota\xi| \leq C\xi^{|\iota|}, \tag{2.2}$$

where n is any nonnegative integer, ι is a N-multi-index and $C > 0$ is a constant only depending on Ω, λ, η and ℓ. This property is also valid for the pairs (α^*, ξ^*) and $(\hat{\alpha}, \hat{\xi})$. The following result is a Carleman inequality for parabolic equations with nonhomogeneous boundary conditions proved in [26]:

Lemma 2.1. *Let $f_0, f_1, \dots, f_N \in L^2(Q)$. There exists a constant $\hat{\lambda}_1 > 0$ such that for any $\lambda \geq \hat{\lambda}_1$ there exists $C > 0$ depending only on λ, Ω, ω_0, η y ℓ such that for every $u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ satisfying*

$$u_t - \Delta u = f_0 + \sum_{j=1}^N \partial_j f_j \text{ in } Q,$$

we have

$$\iint\limits_Q e^{-10s\alpha}(s^{-1}\xi^{-1}|\nabla u|^2 + s\xi|u|^2)\mathrm{d}x\,\mathrm{d}t \leq C\left(s\iint\limits_{\omega_0\times(0,T)} e^{-10s\alpha}\xi|u|^2\mathrm{d}x\,\mathrm{d}t + \left\|s^{-1/4}e^{-5s\alpha}\xi^{-1/4}u\right\|^2_{H^{1/4,1/2}(\Sigma)}\right.$$
$$\left. + \left\|s^{-1/4}e^{-5s\alpha}\xi^{-1/4+1/m}u\right\|^2_{L^2(\Sigma)} + s^{-2}\iint\limits_Q e^{-10s\alpha}\xi^{-2}|f_0|^2\mathrm{d}x\,\mathrm{d}t + \sum_{j=1}^N\iint\limits_Q e^{-10s\alpha}|f_j|^2\mathrm{d}x\,\mathrm{d}t\right),$$

for every $s \geq C$.

Recall that

$$\|u\|_{H^{\frac14,\frac12}(\Sigma)} = (\|u\|^2_{H^{1/4}(0,T;L^2(\partial\Omega))} + \|u\|^2_{L^2(0,T;H^{1/2}(\partial\Omega))})^{1/2}.$$

The next technical result corresponds to Lemma 3 in [12].

Lemma 2.2. *Let $r \in \mathbb{R}$. There exists $C > 0$ depending only on Ω, ω_0, η y ℓ such that , for every $T > 0$ and every $u \in L^2(0,T,H^1(\Omega))$,*

$$s^2\iint\limits_Q e^{-10s\alpha}\xi^{r+2}|u|^2\mathrm{d}x\,\mathrm{d}t \leq C\left(\iint\limits_Q e^{-10s\alpha}\xi^r|\nabla u|^2\mathrm{d}x\,\mathrm{d}t + s^2\iint\limits_{\omega_0\times(0,T)} e^{-10s\alpha}\xi^{r+2}|u|^2\mathrm{d}x\,\mathrm{d}t\right),$$

for every $s \geq C$.

The following result corresponds to a new Carleman estimate that we are going to prove in Appendix A:

Lemma 2.3. *Let $u_0 \in H$, $f_0 \in L^2(Q)^N$ and $f_1 \in L^2(Q)^{N\times N}$. Then, there exist a constant $C(\Omega,\omega_0,T)>0$ such that for any $i\in\{1,\dots,N\}$, the weak solution $u \in L^2(0,T;V)\cap L^\infty(0,T,L^2(\Omega)^N)\cap H^1(0,T;H^{-1}(\Omega)^N)$ of*

$$\begin{cases} u_t - \Delta u + \nabla h = f_0 + \nabla\cdot f_1, \quad \nabla\cdot u = 0 & \text{in } Q,\\ u = 0 & \text{on } \Sigma,\\ u(0) = u^0 & \text{in } \Omega. \end{cases} \tag{2.3}$$

satisfies

$$s^4 \iint\limits_Q e^{-13s\alpha^*}(\xi^*)^4|u|^2\mathrm{d}x\,\mathrm{d}t + s^3 \iint\limits_Q e^{-13s\alpha^*}(\xi^*)^{3-1/m}|\nabla u|^2\mathrm{d}x\,\mathrm{d}t + s^2 \iint\limits_Q e^{-13s\alpha^*}(\xi^*)^{2-2/m}|h|^2\mathrm{d}x\,\mathrm{d}t$$
$$\le C\left(\iint\limits_Q e^{-11s\alpha^*}|f_0|^2\mathrm{d}x\,\mathrm{d}t + s^7\iint\limits_Q e^{-11s\alpha^*}(\hat\xi)^7|f_1|^2\mathrm{d}x\,\mathrm{d}t + s^7\sum_{\substack{j=1\\ j\neq i}}^N \iint\limits_{\omega\times(0,T)} e^{-2s\hat\alpha-11s\alpha^*}(\hat\xi)^7|u_j|^2\mathrm{d}x\,\mathrm{d}t\right), \tag{2.4}$$

for every $s \ge C$.

2.3 Regularity estimates

Here, we state some regularity results concerning the Stokes equation.

The next result concerns the regularity of the solutions to the Stokes system which can be found in [28] (see also [37]):

Lemma 2.4. *For every $T>0$, every $u^0\in V$ and every $f\in L^2(Q)^N$, there exists a unique solution*

$$u \in L^2(0,T;H^2(\Omega)^N)\cap H^1(0,T;L^2(\Omega)^N)\cap L^\infty(0,T;V)$$

to the Stokes system

$$\begin{cases} u_t - \Delta u + \nabla p = f, \quad \nabla\cdot u = 0 & \text{in } Q,\\ u = 0 & \text{on } \Sigma\\ u(0) = u^0 & \text{in } \Omega, \end{cases} \tag{2.5}$$

for some $p\in L^2(0,T;H^1(\Omega))$, and there exists a constant $C>0$ depending only on Ω such that

$$\|u\|^2_{L^2(0,T;H^2(\Omega)^N)} + \|u\|^2_{H^1(0,T;L^2(\Omega)^N)} + \|u\|^2_{L^\infty(0,T;V)} + \|p\|^2_{L^2(0,T;H^1(\Omega))} \le C\left(\|f\|^2_{L^2(Q)^N} + \left\|u^0\right\|^2_V\right). \tag{2.6}$$

In order to deal with more regular solutions, let us introduce some compatibility conditions. We are going to say that f satisfies the compatibility condition of order r if, for any nonnegative integer $k\le r-1$, we have

$$\nabla p^k(x) = \sum_{i=0}^k (\partial_t^i\Delta^{k-i}f)(0,x), \ x\in\partial\Omega.$$

where $p^0\equiv 0$ and, for $k>0$, p^k is the solution of the Neumann boundary-value problem

$$\begin{cases} \Delta p^k = (\nabla\cdot\partial_t^{k-1})f(0) & \text{in } \Omega,\\ \partial_n p^k = \Delta^k u^0\cdot n + \sum\limits_{i=0}^{k-1}\left((\partial_t^i\Delta^{k-1-i}f)(0)\right)\cdot n - \sum\limits_{i=0}^{k-2}\partial_n\left(\partial_t^i\Delta^{k-2-i}\right)\nabla\cdot f(0) & \text{on } \partial\Omega. \end{cases}$$

One has the following lemma (see, for instance [28, 35, 38]):

Lemma 2.5. *Let $T>0$ and let r be a positive integer. There exists $C>0$ depending only on r and Ω such that, for every $f\in Y_r$ satisfying the compatibility conditions of order r, the solution u of (2.5) satisfies $u\in Y_{r+1}$ and*

$$\|u\|^2_{Y_{r+1}} \le C(\|f\|^2_{Y_r} + \|u^0\|^2_{H^{2r+1}(\Omega)^N\cap V}). \tag{2.7}$$

The following regularity result can be found in [19] where the author does a proof's sketch of the regularity result when the function of the right-hand side, $f \in L^2(0,T;V)$. On the other hand, when $f \in L^2(0,T;H^1(\Omega)^N) \cap H^1(0,T;H^{-1}(\Omega)^N)$ see [38, 28] and [34].

Lemma 2.6. *For every $T > 0$, every $u^0 \in H^2(\Omega)^N$ and every $f \in L^2(0,T;V)$, the unique solution to the Stokes system (2.5) satisfies*

$$u \in L^2(0,T;H^3(\Omega)^N) \cap H^1(0,T;V)$$

and there exists a constant $C > 0$ depending only on Ω such that

$$\|u\|^2_{L^2(0,T;H^3(\Omega)^N)} + \|u\|^2_{H^1(0,T;V)} \leq C\left(\|f\|^2_{L^2(0,T;V)} + \left\|u^0\right\|^2_{H^2(\Omega)^N}\right). \tag{2.8}$$

In order to treat more regular solutions, let us introduce some compatibility conditions. We are going to say that f satisfies the compatibility condition of order r if, for any nonnegative integer $k \leq r-1$, we have

$$\nabla p^k(x) = \sum_{i=0}^{k} (\partial_t^i \Delta^{k-i} f)(0,x), \ \ x \in \partial\Omega.$$

where $p^0 \equiv 0$ and, for $k > 0$, p^k is the solution of the Neumann boundary-value problem

$$\begin{cases} \Delta p^k = 0 & \text{in } \Omega, \\ \partial_n p^k = \Delta^k u^0 \cdot n + \sum\limits_{i=0}^{k-1}\left((\partial_t^i \Delta^{k-1-i} f)(0)\right)\cdot n & \text{on } \partial\Omega. \end{cases}$$

We have the following lemma which is analogous to Lemma 2.5:

Lemma 2.7. *Let $T > 0$ and let r be a positive integer. There exists $C > 0$ depending only on r and Ω such that, for every $f \in X_r$ satisfying the compatibility conditions of order r, the solution u of (2.5) satisfies $u \in X_{r+1}$ and*

$$\|u\|^2_{X_{r+1}} \leq C(\|f\|^2_{X_r} + \|u^0\|^2_{H^{2r+2}(\Omega)^N}). \tag{2.9}$$

3 Carleman estimate for the adjoint system

In this section we are going to prove a new Carleman estimate for the Stokes coupled system:

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = g^\varphi + \nabla\times((\nabla\times\psi)\chi), \quad \nabla\cdot\varphi = 0 & \text{in } Q, \\ \psi_t - \Delta\psi + \nabla h = g^\psi, \quad \nabla\cdot\psi = 0 & \text{in } Q, \\ \varphi = 0, \quad \psi = 0 & \text{on } \Sigma, \\ \varphi(T) = 0, \quad \psi(0) = \psi^0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $g^\varphi \in L^2(Q)^N$, $g^\psi \in X_3$ and $\psi^0 \in H$. One has the following proposition:

Proposition 3.1. *Assume that $\omega \cap \mathcal{O} \neq \emptyset$. Then, there exists a constant $C > 0$ depending only on λ, Ω, ω and ℓ such that for any $i \in \{1,\dots,N\}$, any $g^\varphi \in L^2(Q)^N$, any $g^\psi \in X_3$ and any $\psi^0 \in H$, the solution (φ,ψ) of (3.1) satisfies*

$$\begin{aligned} &s^4 \iint\limits_Q e^{-13s\alpha^*}(\xi^*)^4|\varphi|^2 \mathrm{d}x\,\mathrm{d}t + s^7 \iint\limits_Q e^{-10s\alpha^*}(\xi^*)^7|\nabla\psi|^2 \mathrm{d}x\,\mathrm{d}t \\ &\quad \leq C\left(\sum_{\substack{j=1\\ j\neq i}}^{N} s^{16} \iint\limits_{\omega\times(0,T)} e^{-10s\alpha}\xi^{16+5/m}|\varphi_j|^2 \mathrm{d}x\,\mathrm{d}t + s^{11}\left\|e^{-5s\alpha}\xi^{11/2}g^\varphi\right\|^2_{L^2(Q)^N} + \left\|e^{-9/2s\alpha^*}g^\psi\right\|^2_{X_3}\right), \end{aligned} \tag{3.2}$$

for every $s \geq C$.

For simplicity, we are going to prove Proposition 3.1 with $N = 2$ and $i = 2$ (we can also to take $i = 1$). The same method can be applied to the case $N = 3$.

The idea of the proof of Proposition 3.1 is as follows. First, we prove a Carleman inequality for the system satisfied by ψ. Then, we apply Lemma 2.3 to the system satisfied by φ. Finally, we combine the Carleman inequalities of ψ and φ and we use the coupling of the equation $(1.12)_1$ (the first equation of (1.12)) in the observation set $\mathcal{O} \times (0,T)$ to absorb the local term with $\Delta^2\psi_1$. This will prove the estimate (3.2).

3.1 Carleman estimate for ψ

We prove the Carleman estimate for ψ. We consider the Stokes system

$$\begin{cases} \psi_t - \Delta\psi + \nabla h = g^\psi, \quad \nabla\cdot\psi = 0 & \text{in } Q,\\ \psi = 0 & \text{on } \Sigma,\\ \psi(0) = \psi^0 & \text{in } \Omega, \end{cases} \tag{3.3}$$

where $\psi^0 \in H$ and $g^\psi \in X_3$. We prove the following estimate for the solutions of system (3.3).

Proposition 3.2. *Let $\tilde\omega \subset \Omega$ be a nonempty open set such that $\omega_0 \Subset \tilde\omega$. Then, there exists a constant $C > 0$ depending only on λ, Ω, ω and ℓ such that for any $i \in \{1,\dots,N\}$, any $\psi^0 \in H$ and any $g^\psi \in X_3$, the solution ψ of (3.3) satisfies*

$$\begin{aligned}
&\sum_{\substack{j=1\\ j\neq i}}^{N}\Bigg(s^{-1}\iint_Q e^{-10s\alpha}\xi^{-1}\left|\nabla\nabla\nabla\nabla\Delta^2\psi_j\right|^2 \mathrm{d}x\,\mathrm{d}t + s\iint_Q e^{-10s\alpha}\xi\left|\nabla\nabla\nabla\Delta^2\psi_j\right|^2\mathrm{d}x\,\mathrm{d}t\\
&+s^3\iint_Q e^{-10s\alpha}\xi^3\left|\nabla\nabla\Delta^2\psi_j\right|^2\mathrm{d}x\,\mathrm{d}t + s^5\iint_Q e^{-10s\alpha}\xi^5\left|\nabla\Delta^2\psi_j\right|^2\mathrm{d}x\,\mathrm{d}t + s^7\iint_Q e^{-10s\alpha}\xi^7\left|\Delta^2\psi_j\right|^2\mathrm{d}x\,\mathrm{d}t\Bigg)\\
&\quad + s^7\iint_Q e^{-10s\alpha^*}(\xi^*)^7|\psi|^2\mathrm{d}x\,\mathrm{d}t + s^7\iint_Q e^{-10s\alpha^*}(\xi^*)^7|\nabla\psi|^2\mathrm{d}x\,\mathrm{d}t + s^7\iint_Q e^{-10s\alpha^*}(\xi^*)^7|\Delta\psi|^2\mathrm{d}x\,\mathrm{d}t\\
&\leq C\Bigg(\sum_{\substack{j=1\\ j\neq i}}^{N}s^7\iint_{\tilde\omega\times(0,T)} e^{-10s\alpha}\xi^7|\Delta^2\psi_j|^2\mathrm{d}x\,\mathrm{d}t + \iint_Q e^{-10s\alpha}|\nabla^2\Delta^2 g^\psi|^2\mathrm{d}x\,\mathrm{d}t + \left\|s^{5/2}e^{-5s\alpha^*}(\xi^*)^{5/2-1/m}g^\psi\right\|^2_{X_0}\\
&\quad+\left\|s^{3/2}e^{-5s\alpha^*}(\xi^*)^{3/2-2/m}g^\psi\right\|^2_{X_1} + \left\|s^{1/2}e^{-5s\alpha^*}(\xi^*)^{1/2-3/m}g^\psi\right\|^2_{X_2} + \left\|s^{-1/2}e^{-5s\alpha^*}(\xi^*)^{-1/2-4/m}g^\psi\right\|^2_{X_3}\\
&\qquad\qquad + s^7\iint_Q e^{-10s\alpha^*}(\xi^*)^7|\Delta g^\psi|^2\mathrm{d}x\,\mathrm{d}t\Bigg),
\end{aligned}\tag{3.4}$$

for every $s \geq C$.

Proof. Proposition 3.2 is proved in Appendix B. □

To continue with the proof of Proposition 3.1, we take $\tilde\omega \subset \Omega$ such that $\omega_0 \Subset \tilde\omega \Subset \omega\cap\mathcal{O}_0$. Furthermore, from estimate (3.4), notice that

$$I(\psi) \leq C\left(s^7\iint_{\tilde\omega\times(0,T)} e^{-10s\alpha}\xi^7|\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t + \left\|e^{-9/2s\alpha^*}g^\psi\right\|^2_{X_3}\right), \tag{3.5}$$

for every $s \geq C$, where

$$I(\psi) := s^{-1} \iint\limits_Q e^{-10s\alpha} \xi^{-1} |\nabla\nabla\nabla\nabla\Delta^2 \psi_1|^2 \mathrm{d}x\,\mathrm{d}t + s \iint\limits_Q e^{-10s\alpha} \xi |\nabla\nabla\nabla\Delta^2 \psi_1|^2 \mathrm{d}x\,\mathrm{d}t$$
$$+ s^3 \iint\limits_Q e^{-10s\alpha} \xi^3 |\nabla\nabla\Delta^2 \psi_1|^2 \mathrm{d}x\,\mathrm{d}t + s^5 \iint\limits_Q e^{-10s\alpha} \xi^5 |\nabla\Delta^2 \psi_1|^2 \mathrm{d}x\,\mathrm{d}t + s^7 \iint\limits_Q e^{-10s\alpha} \xi^7 |\Delta^2 \psi_1|^2 \mathrm{d}x\,\mathrm{d}t$$
$$+ s^7 \iint\limits_Q e^{-10s\alpha^*} (\xi^*)^7 |\psi|^2 \mathrm{d}x\,\mathrm{d}t + s^7 \iint\limits_Q e^{-10s\alpha^*} (\xi^*)^7 |\nabla\psi|^2 \mathrm{d}x\,\mathrm{d}t + s^7 \iint\limits_Q e^{-10s\alpha^*} (\xi^*)^7 |\Delta\psi|^2 \mathrm{d}x\,\mathrm{d}t.$$

3.2 Carleman estimate for φ

Now, we deal with the Stokes system:

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = \nabla \times ((\nabla \times \psi)\chi) + g^{\varphi}, \quad \nabla \cdot \varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = 0 & \text{in } \Omega. \end{cases} \tag{3.6}$$

Applying Lemma 2.3 to the system (3.6), we see that

$$s^4 \iint\limits_Q e^{-13s\alpha^*} (\xi^*)^4 |\varphi|^2 \mathrm{d}x\,\mathrm{d}t + s^3 \iint\limits_Q e^{-13s\alpha^*} (\xi^*)^{3-1/m} |\nabla\varphi|^2 \mathrm{d}x\,\mathrm{d}t$$
$$\leq C \left(s^7 \iint\limits_{\mathcal{O}\times(0,T)} e^{-11s\alpha^*} (\hat{\xi})^7 |\nabla \times \psi|^2 \mathrm{d}x\,\mathrm{d}t + s^7 \iint\limits_{\omega\times(0,T)} e^{-2s\hat{\alpha}-11s\alpha^*} (\hat{\xi})^7 |\varphi_1|^2 \mathrm{d}x\,\mathrm{d}t \right.$$
$$\left. + \iint\limits_Q e^{-11s\alpha^*} |g^{\varphi}|^2 \mathrm{d}x\,\mathrm{d}t \right), \tag{3.7}$$

for every $s \geq C$.

Notice that, using the fact that $\psi|_{\Sigma} = 0$ we obtain

$$s^7 \iint\limits_{\mathcal{O}\times(0,T)} e^{-11s\alpha^*} (\hat{\xi})^7 |\nabla \times \psi|^2 \mathrm{d}x\,\mathrm{d}t \leq C s^7 \iint\limits_Q e^{-11s\alpha^*} (\hat{\xi})^7 |\nabla\psi|^2 \mathrm{d}x\,\mathrm{d}t$$
$$\leq C s^7 \iint\limits_Q e^{-10s\alpha^*} (\xi^*)^7 |\nabla\psi|^2 \mathrm{d}x\,\mathrm{d}t.$$

Then, using this inequality in (3.7), we have that

$$s^4 \iint\limits_Q e^{-13s\alpha^*} (\xi^*)^4 |\varphi|^2 \mathrm{d}x\,\mathrm{d}t + s^3 \iint\limits_Q e^{-13s\alpha^*} (\xi^*)^{3-1/m} |\nabla\varphi|^2 \mathrm{d}x\,\mathrm{d}t$$
$$\leq C \left(s^7 \iint\limits_Q e^{-10s\alpha^*} (\xi^*)^7 |\nabla\psi|^2 \mathrm{d}x\,\mathrm{d}t + s^7 \iint\limits_{\omega\times(0,T)} e^{-2s\hat{\alpha}-11s\alpha^*} (\hat{\xi})^7 |\varphi_1|^2 \mathrm{d}x\,\mathrm{d}t \right.$$
$$\left. + \iint\limits_Q e^{-11s\alpha^*} |g^{\varphi}|^2 \mathrm{d}x\,\mathrm{d}t \right), \tag{3.8}$$

for every $s \geq C$.

Therefore, the first term of the right-hand side of (3.7) is absorbed by the penultimate term of the left-hand side of (B.15).

3.3 End of the proof of Proposition 3.1

Notice that combining (3.5) with (3.8), we obtain:

$$
\begin{aligned}
&s^4 \iint\limits_{Q} e^{-13s\alpha^*}(\xi^*)^4|\varphi|^2 \mathrm{d}x\,\mathrm{d}t + I(\psi)\\
\leq & C\Bigg(s^7 \iint\limits_{\omega\times(0,T)} e^{-2s\hat{\alpha}-11s\alpha^*}(\hat{\xi})^7|\varphi_1|^2 \mathrm{d}x\,\mathrm{d}t + s^7 \iint_{\tilde{\omega}\times(0,T)} e^{-10s\alpha}\xi^7|\Delta^2\psi_1|^2 \mathrm{d}x\,\mathrm{d}t\\
&+ \left\| e^{-9/2s\alpha^*} g^{\psi}\right\|^2_{X_3} + + \iint\limits_{Q} e^{-11s\alpha^*}|g^{\varphi}|^2 \mathrm{d}x\,\mathrm{d}t \Bigg),
\end{aligned} \tag{3.9}
$$

for every $s \geq C$.

To conclude the proof of Proposition 3.1, we estimate the local term $\Delta^2\psi_1$ in terms of local integrals of φ_1 of the left-hand side of (3.9).

We start by looking at the equation satisfied by φ_1 in $\mathcal{O}_0 \times (0,T)$, and applying the Laplacian, we find

$$
\Delta^2\psi_1 = (\Delta\varphi_1)_t + \Delta^2\varphi_1 + \Delta g_1^{\varphi} - \partial_1\nabla\cdot g^{\varphi} \text{ in } \mathcal{O}_0\times(0,T), \tag{3.10}
$$

where we have used that $\Delta\pi = \nabla\cdot g^{\varphi}$ in $\mathcal{O}_0\times(0,T)$.

Now, let $\theta \in C_c^8(\omega\cap\mathcal{O}_0)$ be a nonnegative function such that $\theta\equiv 1$ in $\tilde{\omega}$ with $\tilde{\omega} \Subset \omega\cap\mathcal{O}_0$. Using (3.10), and since $\tilde{\omega}\subset\mathcal{O}_0$, we have:

$$
\begin{aligned}
J :=& s^7 \iint\limits_{\tilde{\omega}\times(0,T)} e^{-10s\alpha}\xi^7|\Delta^2\psi_1|^2 \mathrm{d}x\,\mathrm{d}t.\\
\leq & s^7 \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \theta e^{-10s\alpha}\xi^7|\Delta^2\psi_1|^2 \mathrm{d}x\,\mathrm{d}t.\\
=& s^7 \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \theta e^{-10s\alpha}\xi^7\Delta^2\psi_1\Big((\Delta\varphi_1)_t + \Delta^2\varphi_1 + \Delta g_1^{\varphi} - \partial_1\nabla\cdot g^{\varphi}\Big)\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

After integrating by parts in space and time, we obtain:

$$
\begin{aligned}
J \leq & - \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \theta(s^7e^{-10s\alpha}\xi^7)_t\Delta^2\psi_1\Delta\varphi_1 \mathrm{d}x\,\mathrm{d}t\\
&+ \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \Big(\Delta(\theta s^7e^{-10s\alpha}\xi^7)\Delta^2\psi_1 + 2\nabla(\theta s^7e^{-10s\alpha}\xi^7)\cdot\nabla\Delta^2\psi_1\Big)\Delta\varphi_1 \mathrm{d}x\,\mathrm{d}t\\
&+ \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \theta s^7e^{-10s\alpha}\xi^7\Big(-(\Delta^2\psi_1)_t + \Delta(\Delta^2\psi_1)\Big)\Delta\varphi_1 \mathrm{d}x\,\mathrm{d}t\\
&+ \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \Delta(\theta s^7e^{-10s\alpha}\xi^7\Delta^2\psi_1)g_1^{\varphi} \mathrm{d}x\,\mathrm{d}t - \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \nabla\left(\partial_1(\theta s^7e^{-10s\alpha}\xi^7\Delta^2\psi_1)\right)\cdot g^{\varphi} \mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.11}
$$

Now, we use the equation satisfied by ψ_1:

$$
\Delta^2\psi_1 = (\Delta\psi_1)_t - \Delta g_1^{\psi} \text{ in } \mathcal{O}_0\times(0,T),
$$

where we have used the fact that $\Delta h = 0$ in $\mathcal{O}_0 \times (0,T)$. Therefore,

$$
\begin{aligned}
J \leq & - \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \theta(s^7 e^{-10s\alpha}\xi^7)_t \Delta^2\psi_1 \Delta\varphi_1 \mathrm{d}x\,\mathrm{d}t \\
& + \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \Big(\Delta(\theta s^7 e^{-10s\alpha}\xi^7)\Delta^2\psi_1 + 2\nabla(\theta s^7 e^{-10s\alpha}\xi^7)\cdot\nabla\Delta^2\psi_1\Big)\Delta\varphi_1 \mathrm{d}x\,\mathrm{d}t \\
& - \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \theta s^7 e^{-10s\alpha}\xi^7 \Delta^2 g_1^{\psi} \Delta\varphi_1 \mathrm{d}x\,\mathrm{d}t + \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \Delta(\theta s^7 e^{-10s\alpha}\xi^7\Delta^2\psi_1) g_1^{\varphi} \mathrm{d}x\,\mathrm{d}t \\
& - \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \nabla\left(\partial_1(\theta s^7 e^{-10s\alpha}\xi^7\Delta^2\psi_1)\right)\cdot g^{\varphi} \mathrm{d}x\,\mathrm{d}t. \\
= & \sum_{k=1}^{5} J_k,
\end{aligned}
\tag{3.12}
$$

for every $s \geq C$.

For J_1, we use integration by parts again, also, we apply the properties of the weight functions shown in (2.2) and, finally, we use Young's inequality, to get:

$$
\begin{aligned}
J_1 = & \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \Delta(\theta(s^7 e^{-10s\alpha}\xi^7)_t)\Delta^2\psi_1\varphi_1 \mathrm{d}x\,\mathrm{d}t + 2\iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \nabla(\theta(s^7 e^{-10s\alpha}\xi^7)_t)\nabla\Delta^2\psi_1\varphi_1 \mathrm{d}x\,\mathrm{d}t \\
& + \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \theta(s^7 e^{-10s\alpha}\xi^7)_t \Delta(\Delta^2\psi_1)\varphi_1 \mathrm{d}x\,\mathrm{d}t. \\
\leq & \epsilon I(\psi) + C(\epsilon) s^{13} \iint\limits_{\omega\times(0,T)} s^{13} e^{-10s\alpha}\xi^{13+2/m}|\varphi_1|^2 \mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.13}
$$

for every $s \geq C$ and any $\epsilon > 0$.

For J_2, we use integration by parts again, also, we apply the properties of the weight functions shown in (2.2) and, finally, we use Young's inequality, to have:

$$
\begin{aligned}
J_2 = & \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \Delta^2(\theta s^7 e^{-10s\alpha}\xi^7)\Delta^2\psi_1\varphi_1 \mathrm{d}x\,\mathrm{d}t + 2\iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \nabla\Delta(\theta s^7 e^{-10s\alpha}\xi^7)\nabla\Delta^2\psi_1\varphi_1 \mathrm{d}x\,\mathrm{d}t \\
& + \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \Delta(\theta s^7 e^{-10s\alpha}\xi^7)\Delta^3\psi_1\varphi_1 \mathrm{d}x\,\mathrm{d}t + 2\iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \Delta\nabla(\theta s^7 e^{-10s\alpha}\xi^7)\nabla\Delta^2\psi_1\varphi_1 \mathrm{d}x\,\mathrm{d}t \\
& + 4\iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \nabla^2(\theta s^7 e^{-10s\alpha}\xi^7)\nabla^2\Delta^2\psi_1\varphi_1 \mathrm{d}x\,\mathrm{d}t + 2\iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \nabla(\theta s^7 e^{-10s\alpha}\xi^7)\nabla\Delta^3\psi_1\varphi_1 \mathrm{d}x\,\mathrm{d}t \\
\leq & \epsilon I(\psi) + C(\epsilon) s^{15} \iint\limits_{\omega\times(0,T)} e^{-10s\alpha}\xi^{15}|\varphi_1|^2 \mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.14}
$$

for every $s \geq C$ and any $\epsilon > 0$.

For J_3, we use integration by parts again, also, we apply the properties of the weight functions shown in (2.2), to obtain:

$$
\begin{aligned}
J_3 = & \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \Delta(s^7\theta e^{-10s\alpha}\xi^7)\Delta^2 g_1^{\psi}\varphi_1 \mathrm{d}x\,\mathrm{d}t + 2\iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} \nabla(s^7\theta e^{-10s\alpha}\xi^7)\nabla\Delta^2 g_1^{\psi}\varphi_1 \mathrm{d}x\,\mathrm{d}t \\
& + \iint\limits_{\omega\cap\mathcal{O}_0\times(0,T)} s^7\theta e^{-10s\alpha}\xi^7\Delta^3 g_1^{\psi}\varphi_1 \mathrm{d}x\,\mathrm{d}t := \sum_{k=1}^{3} J_{3k}.
\end{aligned}
$$

To estimate J_{3k} with $k \in \{1,2,3\}$, we use Young's inequality. We have

$$J_{31} \leq C \left\| s e^{-5s\alpha} \xi^{1-5/(2m)} g_1^{\psi} \right\|^2_{L^2(0,T;H^4(\Omega)^N)} + C s^{16} \iint\limits_{\omega \times (0,T)} e^{-10s\alpha} \xi^{16+5/m} |\varphi_1|^2 \mathrm{d}x\,\mathrm{d}t, \tag{3.15}$$

$$J_{32} \leq C \left\| s^{1/2} e^{-5s\alpha} \xi^{1/2-3/m} g_1^{\psi} \right\|^2_{L^2(0,T;H^5(\Omega)^N)} + C s^{15} \iint\limits_{\omega \times (0,T)} e^{-10s\alpha} \xi^{15+6/m} |\varphi_1|^2 \mathrm{d}x\,\mathrm{d}t, \tag{3.16}$$

$$J_{33} \leq C \left\| e^{-5s\alpha} \xi^{-7/(2m)} g_1^{\psi} \right\|^2_{L^2(0,T;H^6(\Omega)^N)} + C s^{14} \iint\limits_{\omega \times (0,T)} e^{-10s\alpha} \xi^{14+7/m} |\varphi_1|^2 \mathrm{d}x\,\mathrm{d}t, \tag{3.17}$$

for every $s \geq C$ and any $\epsilon > 0$.

For J_4, we use integration by parts again, also, we apply the properties of the weight functions shown in (2.2) and, finally, we use Young's inequality, to get:

$$\begin{aligned} J_4 = & \iint\limits_{\omega \cap \mathcal{O}_0 \times (0,T)} \Delta(\theta s^7 e^{-10s\alpha} \xi^7) \Delta^2 \psi_1 g_1^{\varphi} \mathrm{d}x\,\mathrm{d}t + 2 \iint\limits_{\omega \cap \mathcal{O}_0 \times (0,T)} \nabla(\theta s^7 e^{-10s\alpha} \xi^7) \nabla \Delta^2 \psi_1 g_1^{\varphi} \mathrm{d}x\,\mathrm{d}t \\ & + \iint\limits_{\omega \cap \mathcal{O}_0 \times (0,T)} \theta s^7 e^{-10s\alpha} \xi^7 \Delta(\Delta^2 \psi_1) g_1^{\varphi} \mathrm{d}x\,\mathrm{d}t. \\ \leq & \epsilon I(\psi) + C(\epsilon) s^{11} \iint\limits_{\omega \times (0,T)} e^{-10s\alpha} \xi^{11} |g_1^{\varphi}|^2 \mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{3.18}$$

for every $s \geq C$ and any $\epsilon > 0$.

For J_5, we use integration by parts again, also, we apply the properties of the weight functions shown in (2.2) and, finally, we use Young's inequality, to obtain:

$$J_5 \leq \epsilon I(\psi) + C(\epsilon) s^{11} \iint\limits_{\omega \times (0,T)} e^{-10s\alpha} \xi^{11} |g^{\varphi}|^2 \mathrm{d}x\,\mathrm{d}t, \tag{3.19}$$

for every $s \geq C$ and any $\epsilon > 0$.

Combining (3.13)–(3.19) and (3.9), together with the fact that

$$s^7 e^{-2s\hat{\alpha} - 11s\alpha^*} (\hat{\xi})^7 \leq C s^{16} e^{-10s\alpha} \xi^{16+5/m} \text{ and } e^{-11s\alpha^*} \leq C s^{11} e^{-10s\alpha} \xi^{11},$$

for every $s \geq C$, we deduce (3.2). This concludes the proof of Proposition 3.1.

4 Null controllability of the linear system

In this section we deal with the null controllability of system:

$$\begin{cases} \mathcal{L}w + \nabla p_w = f^w + v \mathbb{1}_{\omega}, \quad \nabla \cdot w = 0 & \text{in } Q, \\ \mathcal{L}^* z + \nabla p_z = f^z + \nabla \times ((\nabla \times w)\chi), \quad \nabla \cdot z = 0 & \text{in } Q, \\ z = 0, \quad w = 0 & \text{on } \Sigma, \\ w(0) = 0, \quad z(T) = 0 & \text{in } \Omega. \end{cases} \tag{4.1}$$

where

$$\mathcal{L} := \partial_t - \Delta \quad \text{and} \quad \mathcal{L}^* := -\partial_t - \Delta,$$

which is the adjoint operator of $\mathcal{L}$. We look for a control v with $v_i \equiv 0$, for some given $i \in \{1, \ldots, N\}$ such that the associated solution of (4.1) satisfies $z(0) = 0$. To do this, let us first state a Carleman inequality with weight functions not vanishing in $t=T$. We introduce the following weight functions:

$$\begin{aligned}
\beta(x,t) &= \frac{e^{2\lambda\|\eta\|_\infty}-e^{\lambda\eta(x)}}{\tilde{\ell}(t)^m}, & \gamma(x,t) &= \frac{e^{\lambda\eta(x)}}{\tilde{\ell}(t)^m},\\
\beta^*(t) &= \max_{x\in\overline{\Omega}}\beta(x,t), & \gamma^*(t) &= \min_{x\in\overline{\Omega}}\gamma(x,t),\\
\hat{\beta}(t) &= \min_{x\in\overline{\Omega}}\beta(x,t), & \hat{\gamma}(t) &= \max_{x\in\overline{\Omega}}\gamma(x,t).
\end{aligned}$$

where

$$\tilde{\ell}(t) = \begin{cases} \ell(t), & 0\le t\le T/2,\\ \|\ell\|_\infty, & T/2 < t \le T. \end{cases}$$

Here, we will assume more regularity for the function g^ψ to deal with the null controllability of the linear system.

Now, we have to following Lemma:

Lemma 4.1. *Let $i\in\{1,\dots,N\}$ and let s be like in Proposition 3.1. Then, there exists a constant $C>0$ (depending on s and λ) such that for any $g^\varphi\in L^2(Q)^N$, any $g^\psi\in Y_{4,0}$, every solution (φ,ψ) of (3.1) satisfies*

$$\iint\limits_Q e^{-14s\beta^*}\left(|\varphi|^2+|\nabla\psi|^2\right)\mathrm{d}x\,\mathrm{d}t \le C\left(\sum_{\substack{j=1\\ j\neq i}}^N \iint\limits_{\omega\times(0,T)} e^{-9s\beta}|\varphi_j|^2\mathrm{d}x\,\mathrm{d}t + \left\|e^{-4s\beta^*}g^\varphi\right\|_{Y_0}^2 + \left\|e^{-4s\beta^*}g^\psi\right\|_{Y_{4,0}}^2\right). \tag{4.2}$$

To prove estimate (4.2) it suffices to combine (3.2) and classical energy estimates for the Stokes system satisfies by φ and ψ. For simplicity, we omit the proof. For more details on how to get (4.2), see [9, 7] or [21].

Now we are ready to prove the null controllability of system (4.1). The idea is to look for a solution in an appropriate weighted functional space. To this end, we introduce, for $i\in\{1,\dots,N\}$, the spaces

$$\begin{aligned}
E_2^i :=\{&(w,p_w,z,p_z,v) : e^{4s\beta^*}w\in L^2(Q)^2,\quad e^{9s\beta}v\mathbb{1}_\omega\in L^2(Q)^2,\\
&e^{4s\beta^*}(\gamma^*)^{-1-1/m}w\in L^2(0,T;H^2(\Omega)^2)\cap L^\infty(0,T;V),\quad v_i\equiv 0,\\
&e^{4s\beta^*}(\gamma^*)^{-15-15/m}z\in L^2(0,T;V)\cap L^\infty(0,T;H),\quad z(T)\equiv 0,\\
&e^{7s\beta^*}(\mathcal{L}w+\nabla p_w - v\mathbb{1}_\omega)\in L^2(Q)^2,\quad e^{7s\beta^*}(\mathcal{L}^*z+\nabla p_z-\nabla\times((\nabla\times w)\chi))\in L^2(0,T;H^{-1}(\Omega)^2)\},
\end{aligned}$$

and

$$\begin{aligned}
E_3^i :=\{&(w,p_w,z,p_z,v) : e^{4s\beta^*}w\in L^2(Q)^3,\ e^{9s\beta}v\mathbb{1}_\omega\in L^2(Q)^3,\\
&e^{4s\beta^*}(\gamma^*)^{-1-1/m}w\in L^2(0,T;H^2(\Omega)^3)\cap L^\infty(0,T;V),\quad v_i\equiv 0,\\
&e^{4s\beta^*}(\gamma^*)^{-15-15/m}z\in L^2(0,T;H^2(\Omega)^3)\cap L^\infty(0,T;V),\quad z(T)\equiv 0,\\
&e^{7s\beta^*}(\mathcal{L}w+\nabla p_w - v\mathbb{1}_\omega)\in L^2(Q)^3,\quad e^{7s\beta^*}(\mathcal{L}^*z+\nabla p_z-\nabla\times((\nabla\times w)\chi))\in L^2(Q)^3\}.
\end{aligned}$$

It is clear that E_N^i is a Banach space endowed with their natural norms.

Remark 4.1. *In particular, an element $(w,p_w,z,p_z,v)\in E_N^i$ satisfies $w(0)=0$, $z(0)=0$, $v_i\equiv 0$. Moreover,*

$$e^{as\beta^*}(\gamma^*)^c \text{ is bounded},\qquad \forall a>0,\forall c\in\mathbb{R}, \tag{4.3}$$

All the details are given in the Section 5.

Proposition 4.1. *Let $i\in\{1,\dots,N\}$ and $\chi:\Omega\to\mathbb{R}$ given by (1.4) if $N=2$, or (1.5) if $N=3$. Assume the hypothesis of Lemma 4.1 and the following hypothesis on the initial condition and the right-hand side of system (4.1):*

$$\begin{aligned}
&\text{If } N=2: \quad e^{7s\beta^*}(f^w,f^z)\in L^2(Q)^2\times L^2(0,T;H^{-1}(\Omega)^2).\\
&\text{If } N=3: \quad e^{7s\beta^*}(f^w,f^z)\in L^2(Q)^3\times L^2(Q)^3.
\end{aligned} \tag{4.4}$$

Then, we can find a control $v\in L^2(0,T;L^2(\omega)^N)$ such that the associated solution (w,p_w,z,p_z,v) of (4.1) belongs to E_N^i. In particular, $v_i\equiv 0$ and $z(0)=0$ in Ω.

Proof. Following the arguments in [18] and [25], we introduce the space P_0 of functions $(\varphi, \pi, \psi, h) \in \mathcal{C}^\infty(\overline{Q})^{2N+2}$ such that

- $\nabla \cdot \varphi = \nabla \cdot \psi = 0.$
- $\varphi|_\Sigma = \psi|_\Sigma = 0.$
- $\varphi(T) = \psi(0) = 0.$
- $\Delta h|_Q = 0.$
- $(\mathcal{L}_H^k[e^{-4s\beta^*}(\mathcal{L}\psi + \nabla h)])|_\Sigma = 0,\ k = 0, \dots, 3.$
- $(\mathcal{L}_H^k[e^{-4s\beta^*}(\mathcal{L}\psi + \nabla h)])(0) = 0,\ k = 0, \dots, 3.$

We consider the bilinear form

$$
\begin{aligned}
& a((\tilde{\varphi}, \tilde{\pi}, \tilde{\psi}, \tilde{h}), (\varphi, \pi, \psi, h)) \\
:= & \iint\limits_Q e^{-8s\beta^*}(\mathcal{L}^*\tilde{\varphi} + \nabla\tilde{\pi} - \nabla \times ((\nabla \times \tilde{\psi})\chi)) \cdot (\mathcal{L}^*\varphi + \nabla\pi - \nabla \times ((\nabla \times \psi)\chi)) \mathrm{d}x\, \mathrm{d}t \\
& + \iint\limits_Q \mathcal{L}_H^4[e^{-4s\beta^*}(\mathcal{L}\tilde{\psi} + \nabla\tilde{\pi})] \cdot \mathcal{L}_H^4[e^{-4s\beta^*}(\mathcal{L}\psi + \nabla\pi)] \mathrm{d}x\, \mathrm{d}t + \sum_{\substack{j=1 \\ j \neq i}} \iint\limits_{\omega \times (0,T)} e^{-9s\beta^*} \tilde{\varphi}_j \varphi_j \mathrm{d}x\, \mathrm{d}t,
\end{aligned}
$$

and a linear form

$$
\langle G, (\varphi, \pi, \psi, h) \rangle = \iint\limits_Q f^w \cdot \varphi \mathrm{d}x\, \mathrm{d}t + \iint\limits_Q f^z \cdot \psi \mathrm{d}x\, \mathrm{d}t.
$$

Due to (4.2), we have that $a(\cdot, \cdot) : P_0 \times P_0 \mapsto \mathbb{R}$ is a symmetric, definite positive bilinear form on P_0. We denote by P the completion of P_0 for the norm induced by $a(\cdot, \cdot)$. Then $a(\cdot, \cdot)$ is well defined, continuous and definite positive on P. Additionally, thanks to the Carleman estimate (4.2) and the assumptions (4.4), the linear form $(\varphi, \pi, \psi, h) \mapsto \langle G, (\varphi, \pi, \psi, h) \rangle$ is well-defined and continuous on P. Hence, from Lax-Milgram's lemma, we deduce that the variational problem:

$$
\begin{cases}
\text{Find } (\tilde{\varphi}, \tilde{\pi}, \tilde{\psi}, \tilde{h}) \in P \text{ such that} \\
a((\tilde{\varphi}, \tilde{\pi}, \tilde{\psi}, \tilde{h}), (\varphi, \pi, \psi, h)) = \langle G, (\varphi, \pi, \psi, h) \rangle,\ \forall (\varphi, \pi, \psi, h) \in P,
\end{cases}
\tag{4.5}
$$

possesses exactly one solution $(\hat{\varphi}, \hat{\pi}, \hat{\psi}, \hat{h})$.
Let $\hat{v}$ be given by

$$
\begin{cases}
\hat{v}_j := -e^{-9s\beta^*} \hat{\varphi}_j \mathbb{1}_\omega, \\
\hat{v}_i \equiv 0,\ j \neq i \text{ in } Q.
\end{cases}
\tag{4.6}
$$

It is simple from (4.5) and (4.6) that we obtain

$$
\iint\limits_Q (|\tilde{w}|^2 + |\tilde{z}|^2) \mathrm{d}x\, \mathrm{d}t + \sum_{\substack{j=1 \\ j \neq i}}^N \iint\limits_{\omega \times (0,T)} e^{9s\beta^*} |\hat{v}_j|^2 \mathrm{d}x\, \mathrm{d}t < +\infty,
\tag{4.7}
$$

where $\tilde{w}$ and $\tilde{z}$ are given by

$$
\begin{cases}
\tilde{w} & := e^{-4s\beta^*}(\mathcal{L}^*\hat{\varphi} + \nabla\hat{\pi} - \nabla \times ((\nabla \times \hat{\psi})\chi)), \\
\tilde{z} & := \mathcal{L}_H^4[e^{-4s\beta^*}(\mathcal{L}\hat{\psi} + \nabla\hat{h})].
\end{cases}
\tag{4.8}
$$

In particular, $\hat{v} \in L^2(0, T; L^2(\omega)^N)$.

Let $(\hat{w}, \hat{z})$, together with some pressures $(\hat{p}_0, \hat{p}_1)$, the weak solution of (4.1) with $v = \hat{v}$, that is, they solve

$$
\begin{cases}
\mathcal{L}\hat{w} + \nabla\hat{p}_0 = f^w + \hat{v}\mathbb{1}_\omega, \quad \nabla \cdot \hat{w} = 0 & \text{in } Q, \\
\mathcal{L}^*\hat{z} + \nabla\hat{p}_1 = f^z + \nabla \times ((\nabla \times \hat{w})\chi), \quad \nabla \cdot \hat{z} = 0 & \text{in } Q, \\
\hat{w} = \hat{z} = 0 & \text{on } \Sigma, \\
\hat{w}(0) = 0, \quad \hat{z}(T) = 0 & \text{in } \Omega.
\end{cases}
\tag{4.9}
$$

The rest of the proof is dedicated to prove the following exponential decay properties:

$$\begin{array}{ll} e^{4s\beta^*}(\gamma^*)^{-1-1/m}\hat{w} \in L^2(0,T;H^2(\Omega)^N)\cap L^\infty(0,T;V), & \text{if } N=2,3; \\ e^{4s\beta^*}(\gamma^*)^{-15-15/m}\hat{z} \in L^2(0,T;V)\cap L^\infty(0,T;H), & \text{if } N=2; \\ e^{4s\beta^*}(\gamma^*)^{-15-15/m}\hat{z} \in L^2(0,T;H^2(\Omega)^3)\cap L^\infty(0,T;V), & \text{if } N=3; \end{array} \tag{4.10}$$

which will solve the null controllability problem for system (4.1).

First, we are going to prove that $(\tilde{w},\tilde{z})$ given by (4.8) is actually the solution (in the sense of transposition) of

$$\left\{ \begin{array}{ll} e^{-4s\beta^*}\tilde{w} = \hat{w} & \text{in } Q, \\ e^{-4s\beta^*}(\mathcal{L}_H^*)^4\tilde{z} = \hat{z}, \quad \nabla\cdot\tilde{z}=0 & \text{in } Q, \end{array}\right. \tag{4.11}$$

such that

$$\left\{ \begin{array}{lll} (\mathcal{L}_H^*)^\ell\tilde{z} = 0 & \text{on } \Sigma, & \ell=0,\dots,3, \\ (\mathcal{L}_H^*)^\ell\tilde{z}(T) = 0 & \text{in } \Omega, & \ell=0,\dots,3. \end{array}\right. \tag{4.12}$$

Now, from (4.5), (4.6), (4.8) and (4.9), we obtain for every $(\varphi,\pi,\psi,h)\in P_0$

$$\begin{aligned} &\iint_Q \tilde{w}\cdot e^{-4s\beta^*}(\mathcal{L}^*\varphi+\nabla\pi-\nabla\times((\nabla\times\psi)\chi))\mathrm{d}x\,\mathrm{d}t + \iint_Q \tilde{z}\cdot\mathcal{L}_H^4[e^{-4s\beta^*}(\mathcal{L}\psi+\nabla h)]\mathrm{d}x\,\mathrm{d}t \\ &= \iint_Q \varphi\cdot(\mathcal{L}\hat{w}+\nabla\hat{p}_0)\mathrm{d}x\,\mathrm{d}t + \iint_Q \psi\cdot(\mathcal{L}^*\hat{z}+\nabla\hat{p}_1-\nabla\times((\nabla\times\hat{w})\chi))\,\mathrm{d}x\,\mathrm{d}t. \\ &= \iint_Q \hat{w}\cdot(\mathcal{L}^*\varphi+\nabla\pi-\nabla\times((\nabla\times\psi)\chi))\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \hat{z}\cdot(\mathcal{L}\psi+\nabla h)\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

From this last equality, we obtain for all $(h^w,h^z)\in L^2(Q)^{2N}$

$$\iint_Q \tilde{w}\cdot h^w\mathrm{d}x\,\mathrm{d}t + \iint_Q \tilde{z}\cdot h^z\mathrm{d}x\,\mathrm{d}t = \iint_Q \hat{w}\cdot\Phi^w\mathrm{d}x\,\mathrm{d}t + \iint_Q \hat{z}\cdot\Phi^z\mathrm{d}x\,\mathrm{d}t, \tag{4.13}$$

where (Φ^w,Φ^z) is the solution of

$$\left\{ \begin{array}{ll} e^{-4s\beta^*}\Phi^w = h^w & \text{in } Q, \\ \mathcal{L}_H^4[e^{-4s\beta^*}\Phi^z] = h^z, \quad \nabla\cdot\Phi^z=0 & \text{in } Q, \end{array}\right. \tag{4.14}$$

such that

$$\left\{ \begin{array}{lll} \mathcal{L}_H^\ell(e^{-4s\beta^*}\Phi^z) = 0 & \text{on } \Sigma, & \ell=0,\dots,3, \\ \mathcal{L}_H^\ell(e^{-4s\beta^*}\Phi^z)(0) = 0 & \text{in } \Omega, & \ell=0,\dots,3. \end{array}\right. \tag{4.15}$$

It is classical to show that (4.13)–(4.15) is equivalent to (4.11)–(4.12).

Now, let

$$(w_*,p_{0*}) := e^{4s\beta^*}(\gamma^*)^{-1-1/m}(\hat{w},\hat{p}_0), \; f_*^w := e^{4s\beta^*}(\gamma^*)^{-1-1/m}f^w.$$

Then, (w_*,p_{0*}) satisfies

$$\left\{ \begin{array}{ll} \mathcal{L}w_*+\nabla p_{0*} = f_*^w + e^{4s\beta^*}(\gamma^*)^{-1-1/m}\hat{v}\mathbb{1}_\omega + (e^{4s\beta^*}(\gamma^*)^{-1-1/m})_t\hat{w}, \quad \nabla\cdot w_*=0 & \text{in } Q, \\ w_*=0 & \text{on } \Sigma, \\ w_*(0)=0 & \text{in } \Omega, \end{array}\right.$$

From (4.4), (4.7), (4.11), (2.2) and $e^{4s\beta^*}\hat{w}\in L^2(Q)^N$, we have that the right-hand side of this equation belongs to $L^2(Q)^N$. Using Lemma 2.4, we deduce that $w_*\in L^2(0,T;H^2(\Omega)^N)\cap L^\infty(0,T;V)$.

Finally, to complete the proof of (4.10), we will use the following Lemma whose proof is done in Appendix C.

Lemma 4.2. $e^{4s\beta^*}(\gamma^*)^{-14-14/m}\hat{z}\in L^2(Q)^N$.

Now, let

$$(z_*, p_{1*}) := e^{4s\beta^*}(\gamma^*)^{-15-15/m}(\hat{z}, \hat{p}_1), \quad f_*^z := e^{4s\beta^*}(\gamma^*)^{-15-15/m}(f^z + \nabla \times ((\nabla \times \hat{w})\chi)).$$

Then, (z_*, p_{1*}) satisfies

$$\begin{cases} \mathcal{L}^* z_* + \nabla p_{1*} = f_*^z - (e^{4s\beta^*}(\gamma^*)^{-15-15/m})_t \hat{z}, \quad \nabla \cdot z_* = 0 & \text{in } Q, \\ z_* = 0 & \text{on } \Sigma, \\ z_*(T) = 0 & \text{in } \Omega, \end{cases} \tag{4.16}$$

Next, we are going to study the cases $N = 2, 3$. Notice that f_*^z can be written as:

$$f_*^z = e^{-3s\beta^*}(\gamma^*)^{-15-15/m}(e^{7s\beta^*} f^z) + (\gamma^*)^{-14-14/m} \nabla \times ((\nabla \times w_*)\chi)\,.$$

If $N = 2$, we have that $\chi = \mathbb{1}_{\mathcal{O}}$ (recall (1.4)). Then, from (4.4), $(\nabla \times w_*)\mathbb{1}_{\mathcal{O}} \in L^2(Q)^2$, and the fact that $e^{-3s\beta^*}(\gamma^*)^{-15-15/m}$ and $(\gamma^*)^{-14-14/m}$ are bounded, we deduce that $f_*^z \in L^2(0, T; H^{-1}(\Omega)^2)$.

If $N = 3$, since χ is a smooth function (recall (1.5)), we obtain $f_*^z \in L^2(Q)^3$ from (4.4), and $(\nabla \times w_*)\chi \in L^2(0, T; H^1(\Omega)^3)$.

Therefore, from (4.4), (4.7), (4.11), (2.2) and Lemma 4.2, we have that the right-hand side of system (4.16) belongs to $L^2(0, T; H^{-1}(\Omega)^2)$ or $L^2(Q)^3$ in dimension 2 or 3, respectively. Then, in dimension 2, we deduce that $z_* \in L^2(0, T; V) \cap L^\infty(0, T; H)$. Now, in dimension 3, again by Lemma 2.4, we obtain that $z_* \in L^2(0, T; H^2(\Omega)^3) \cap L^\infty(0, T; V)$. This concludes the proof of Proposition 4.1.

Remark 4.2. *Notice that the last part of the the proof of Proposition 4.1 is the first time that the smoothness of χ is used. Everything else up to this point remains to be true if $\chi = \mathbb{1}_{\mathcal{O}}$, even for $N = 3$. The reason to assume that χ is smooth in $N = 3$ is to obtain extra regularity for z in order to deal with the nonlinearities of system 1.6. More details are given in Section 5.*

5 Proof of Theorem 1.1

Recall that we deal with the following null controllability problem: to find controls v verifying $v_i \equiv 0$ such that the solution of the system

$$\begin{cases} \mathcal{L}w + (w, \nabla)w + \nabla p_w = f + v\mathbb{1}_\omega, \quad \nabla \cdot w = 0 & \text{in } Q, \\ \mathcal{L}^* z + (z, \nabla^t)w - (w, \nabla)z + \nabla p_z = \nabla \times ((\nabla \times w)\chi)\,, \quad \nabla \cdot z = 0 & \text{in } Q, \\ w = 0, \quad z = 0 & \text{on } \Sigma, \\ w(0) = y^0, \quad z(T) = 0 & \text{in } \Omega. \end{cases} \tag{5.1}$$

satisfies $z(0) = 0$ in Ω. We proceed using similar arguments to those in [25], (see also [9, 11, 15, 21]). The null controllability result for the linear system given by Proposition 4.1 is going to allow us to apply the following inverse mapping theorem (see [1]):

Theorem 5.1. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two Banach spaces and let $\mathcal{F} : \mathcal{G}_1 \to \mathcal{G}_2$ satisfy $\mathcal{F} \in \mathcal{C}^1(\mathcal{G}_1; \mathcal{G}_2)$. Assume that $g_1 \in \mathcal{G}_1$, $\mathcal{F}(g_1) = g_2$ and that $\mathcal{F}'(g_1) : \mathcal{G}_1 \mapsto \mathcal{G}_2$ is surjective. Then, there exists $\delta > 0$ such that, for every $g' \in \mathcal{G}_2$ satisfying $\|g' - g_2\|_{\mathcal{G}_2} \le \delta$, there exists a solution of the equation*

$$\mathcal{F}(g) = g', \quad g \in \mathcal{G}_1.$$

Let us set the framework to apply Theorem 5.1 to the problem at hand. Let

$$\mathcal{G}_1 := E_N^i, \;\; \mathcal{G}_2 := \begin{cases} L^2(e^{7s\beta^*}(0, T); L^2(\Omega)^2) \times L^2(e^{7s\beta^*}(0, T); H^{-1}(\Omega)^2) & \text{if } N = 2, \\ L^2(e^{7s\beta^*}(0, T); L^2(\Omega)^3) \times L^2(e^{7s\beta^*}(0, T); L^2(\Omega)^3) & \text{if } N = 3, \end{cases}$$

and the operator

$$\begin{aligned} \mathcal{F}(w, p_w, z, p_z, v) :=\Big(&\mathcal{L}w + (w, \nabla)w + \nabla p_w - v\mathbb{1}_\omega, \\ &\mathcal{L}^* z + (z, \nabla^t)w - (w, \nabla)z + \nabla p_z - \nabla \times ((\nabla \times w)\chi)\Big), \end{aligned}$$

for $(w,p_w,z,p_z,v)\in\mathcal{G}_1$. Here, $u\in L^2(e^{7s\beta^*}(0,T);L^2(\Omega)^{2N})$ means $e^{7s\beta^*}u\in L^2(Q)^{2N}$. It only remains to check that the operator $\mathcal{F}$ is of class $\mathcal{C}^1(\mathcal{G}_1;\mathcal{G}_2)$. To do this, we notice that all the terms in $\mathcal{F}$ are linear, except for $(w,\nabla)w$ and $(z,\nabla^t)w-(w,\nabla)z$. Let us check that these terms are continuous from $\mathcal{G}_1\times\mathcal{G}_1$ to $\mathcal{G}_2$.

We will study the cases in dimension 2 and 3, respectively.

If $N=2$: Since $e^{4s\beta^*}(\gamma^*)^{-15-15/m}z\in L^2(0,T;V)\cap L^\infty(0,T;H)$, and $e^{4s\beta^*}(\gamma^*)^{-1-1/m}w\in L^2(0,T;H^2(\Omega)^2)\cap L^\infty(0,T;V)$ for any $(w,p_w,z,p_z,v)\in E_2^i$, then

$$e^{4s\beta^*}(\gamma^*)^{-15-15/m}z\in\ L^4(Q)^2,$$

and

$$e^{4s\beta^*}(\gamma^*)^{-1-1/m}\nabla^t w\in L^4(Q)^4.$$

Therefore, for the term $(z,\nabla^t)w$ we have:

$$\begin{aligned}&\left\|e^{7s\beta^*}(z,\nabla^t)w\right\|_{L^2(0,T;H^{-1}(\Omega)^2)}\\ \leq C&\left\|(e^{4s\beta^*}(\gamma^*)^{-15-15/m}(z,\nabla^t)e^{4s\beta^*}(\gamma^*)^{-1-1/m}w\right\|_{L^2(Q)^2}\\ \leq C&\left\|e^{4s\beta^*}(\gamma^*)^{-15-15/m}z\right\|_{L^4(Q)^2}\left\|e^{4s\beta^*}(\gamma^*)^{-1-1/m}\nabla^t w\right\|_{L^4(Q)^4},\end{aligned}$$

where we have used that $7<8$ and (4.3).

Now, we denote:

$$(\nabla\cdot(w\otimes z))_i=\sum_{j=1}^N\partial_j(z_jw_i),\quad j=1,\dots,N.$$

Observe that, using $\nabla\cdot w=0$ in Q, the term $(w,\nabla)z$ can be treated as follows:

$$\begin{aligned}&\left\|e^{7s\beta^*}(w,\nabla)z\right\|_{L^2(0,T;H^{-1}(\Omega)^2)}\\ \leq&C\left\|e^{8s\beta^*}(\gamma^*)^{-15-15/m}(w,\nabla)z\right\|_{L^2(0,T;H^{-1}(\Omega)^2)}\\ =&C\left\|\nabla\cdot(e^{4s\beta^*}(\gamma^*)^{-1-1/m}w\otimes e^{4s\beta^*}(\gamma^*)^{-15-15/m}z)\right\|_{L^2(0,T;H^{-1}(\Omega)^2)}\\ \leq&C\left\|(e^{4s\beta^*}(\gamma^*)^{-1-1/m}w)\otimes(e^{4s\beta^*}(\gamma^*)^{-15-15/m}z)\right\|_{L^2(Q)^2}\\ \leq&C\left\|e^{4s\beta^*}(\gamma^*)^{-1-1/m}w\right\|_{L^4(Q)^2}\cdot\left\|e^{4s\beta^*}(\gamma^*)^{-15-15/m}z\right\|_{L^4(Q)^2}\end{aligned}$$

then, the continuity follows since $7<8$ and thanks to (4.3). The term $(w,\nabla)w$ is treated analogously.

If $N=3$: Since $e^{4s\beta^*}(\gamma^*)^{-1-1/m}w\in L^2(0,T;H^2(\Omega)^3)\cap L^\infty(0,T;V)$ for any $(w,p_w,z,p_z,v)\in E_3^i$, and $H^2(\Omega)^3\subset\ L^\infty(\Omega)^3$, we have that:

$$\begin{aligned}&\left\|e^{7s\beta^*}(w,\nabla)w\right\|_{L^2(Q)^3}\\ \leq C&\left\|e^{8s\beta^*}(\gamma^*)^{-2-2/m}(w,\nabla)w\right\|_{L^2(Q)^3}\\ \leq C&\left\|(e^{4s\beta^*}(\gamma^*)^{-1-1/m}w,\nabla)e^{4s\beta^*}(\gamma^*)^{-1-1/m}w\right\|_{L^2(Q)^3}\\ \leq C&\left\|e^{4s\beta^*}(\gamma^*)^{-1-1/m}w\right\|_{L^2(0,T;L^\infty(\Omega)^3)}\left\|e^{4s\beta^*}(\gamma^*)^{-1-1/m}w\right\|_{L^\infty(0,T;V)},\end{aligned}$$

and the continuity follows since $7<8$ and due to (4.3). Since $e^{4s\beta^*}(\gamma^*)^{-15-15/m}z\in L^2(0,T;H^2(\Omega)^3)\cap L^\infty(0,T;V)$, the terms $(w,\nabla)z$ and $(z,\nabla^t)w$ are treated similarly.

It is readily seen that $\mathcal{F}'(0):\mathcal{G}_1\to\mathcal{G}_2$ is given by

$$\mathcal{F}'(0)(w,p_w,z,p_z,v):=\left(\mathcal{L}w+\nabla p_w-v1\!\!1_\omega,\mathcal{L}^*z+\nabla p_z-\nabla\times((\nabla\times w)\chi)\right),$$

for all $(w, p_w, z, p_z, v) \in \mathcal{G}_1$. It follows that this functional is surjective in view of the null controllability result for the linear system given by Proposition 4.1.

Now, we are in condition to apply Theorem 5.1. By taking $g_1 = 0$ and $g_2 = 0$, it gives the existence of $\delta > 0$ such that, if $\|e^{C/t^m} f\|_{L^2(Q)^N} \leq \delta$, for some $C > 0$, then we can find $(w, p_w, z, p_z, v) \in \mathcal{G}_1$ solution of (5.1). In particular, $v_i \equiv 0$ and $z(0) \equiv 0$ in Ω. Therefore, the proof of Theorem 1.1 is complete.

6 Some final comments

In this section, we will give some final comments about other control problems or related models.

- *Possible relations to hierarchical control.*

  Hierarchical control problems can be found for the heat equation in [3, 2, 4], and for the Stokes system in [22]. The main similarity between the insensitizing and the hierarchical control problem is that both problems can be formulated as a control problem associated to a system of equations with a reduced number of controls. However, in the case of insensitizing controls, the resulting system has a cascade structure (see (1.6)), which is not the case, for instance, following a Stackelberg-Nash strategy for hierarchical control. Our approach is strongly based on the cascade structure, so it is not clear that solving a hierarchical control problem for the Navier-Stokes system would be a direct consequence of our results.

- *Insensitizing control problem for the Navier-Stokes system with Navier-slip boundary conditions.*

  Although there are local null controllability results with Navier-slip boundary conditions using $N-1$ scalar controls (see [20]), the insensitizing control problem with a reduced amount of scalar controls is an open problem, even the square of the L^2-norm of the state as an observation functional. If we try to follow the strategy in [20], we find that we would need to use the same equation (first equation of in (1.12)) to estimate both the pressure and the local term of ψ, which would need a different approach than the one made here and in [20].

- *The Boussinesq system reducing scalar controls.*

  In this case, the control problem is the following:

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = f + v \mathbb{1}_\omega + \theta e_N, \quad \nabla \cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta \theta + y \cdot \nabla \theta = f_0 + v_0 \mathbb{1}_\omega & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0 + \tau \hat{y}_0, \quad \theta(0) = \theta^0 + \tau \hat{\theta}_0 & \text{in } \Omega. \end{cases}$$

  Here,

$$e_N = \begin{cases} (0,1) & \text{if } N = 2, \\ (0,0,1) & \text{if } N = 3, \end{cases}$$

  stands for the gravity vector field, $y(x,t)$ represents the velocity of the particles of an incompressible fluid, $\theta = \theta(x,t)$ their temperature, $(v_0, v) = (v_0, v_1, \ldots, v_N)$ stands for the control which acts over the set ω, $(f, f_0) \in L^2(Q)^{N+1}$ is a given externally applied force and the initial state, $(y(0), \theta(0))$ is partially unknown, i.e., y^0 and θ^0 are known, while $\tau, \hat{y}_0$, and $\hat{\theta}_0$ are unknown. We want to insensitize as observation functional the sum of the square of the L^2-norm of the curl y with the square of the L^2-norm of the gradient of θ. In this study, the author tries to control with two components fixed at zero. This work is in preparation, see [32], where the author uses the ideas of [10, 8].

- *The primitive equations of ocean.*

  Considering $N = 2$, this control problem can be written as:

$$\begin{cases} \partial_t v - A \Delta v + \gamma v + (f_0 + \beta x_2) v^\perp + \dfrac{1}{\rho_0} \nabla p = \mathcal{T} + h \mathbb{1}_\omega & \text{in } Q, \\ \nabla \cdot v = 0 & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(0) = v_0 + \tau \hat{v}_0 & \text{in } \Omega. \end{cases}$$

Here, $v = v(x,t)$ and $p = p(x,t)$ are the velocity filed and the pressure of the fluid. In this model, A is the horizontal *eddy viscosity* coefficient, γ is the bottom *friction* coefficient, ρ_0 is the fluid density and $(f_0 + \beta x_2)v^\perp$ is the Coriolis term, with $v^\perp = (-v_2, v_1)$. In the right hand side, $\mathbb{1}_\omega$ denotes the characteristic function of ω and $\mathcal{T}$ is a given source. The term $\tau \hat{v}_0$, where $\tau \in \mathbb{R}$, represents a small unknown perturbation of the initial condition v_0 and $h = h(x,t)$ is a control term to be determined.

Note that if we try to study desensitizing control of this system with a reduced number of components of the control, our strategy to obtain the Carleman estimate does not seem to work, because the equations have mixed components due to the term $v^\perp = (-v_2, v_1)$ that appears in the first equation. However, we can mention the work [14], where the ε-insensitizing control for this type of system is achieved.

- *The magnetohydrodynamic system.*

  The controlled MHD equations (with boundary and initial conditions) we deal with are the following:

$$\begin{cases} \partial_t y - \nu \Delta y + (y \cdot \nabla) y + \nabla p + \nabla \left(\dfrac{1}{2} B^2 \right) - (B \cdot \nabla) B = f + \chi_\omega u & \text{in } Q, \\ \partial_t B + \eta \nabla \times (\nabla \times B) + (y \cdot \nabla) B - (B \cdot \nabla) y = P(\chi_\omega v) & \text{in } Q, \\ \nabla \cdot y = 0, \quad \nabla \cdot B = 0 & \text{in } Q, \\ y = 0, \quad B \cdot n = 0, \quad (\nabla \times B) \times n = 0 & \text{on } \Sigma, \\ y(0) = y_0, \quad B(0) = B_0 & \text{in } \Omega. \end{cases}$$

  Here, $y = (y_1, y_2, y_3) : \Omega \times [0,T] \to \mathbb{R}^3$ is the velocity vector field, $p : \Omega \times [0,T] \to \mathbb{R}$ is the (scalar) pressure, and $B = (B_1, B_2, B_3) : \Omega \times [0,T] \to \mathbb{R}^3$ is the magnetic field. The vector functions $u = (u_1, u_2, u_3) : \Omega \times [0,T] \to \mathbb{R}^3$ and $v = (v_1, v_2, v_3) : \Omega \times [0,T] \to \mathbb{R}^3$ are the controls, and χ_ω is the characteristic function of ω. We denote the variables of the functions $y, p, B, u,$ and v by $x = (x_1, x_2, x_3)$ (belonging to Ω) and t. The vector function $f : (f_1, f_2, f_3) : \Omega \to \mathbb{R}^3$ is the know density of the external forces, and the vector fields $y_0 : \Omega \to \mathbb{R}^3$ and $B_0 : \Omega \to \mathbb{R}^3$ are the given initial velocity and magnetic fields, respectively. The operator P is the Leray projector.

  Although there are controllability results for this system, see [24, 5], a possible problem could be control to the trajectories with reduced scalar controls. However, notice that the simpler problem of the controllability to the trajectories of the Navier-Stokes system with $N-1$ scalar controls is still open. On the other hand, following our approach, if we linearize around zero, we find a decoupled system, which means that both control u and v would have to be active. Nonetheless, it seems plausible to believe that an insensitizing control result for this system could obtained if both control u and v are allowed to be active, even if only some of their components.

A Proof of Lemma 2.3

Before we begin, we present a Carleman estimate which is proved in [9], which we are going to use in its proof, and it is as follow:

Lemma A.1. *There exists a constant λ_0, such that, for any $\lambda > \lambda_0$ there exist two constants $C(\lambda) > 0$ and $s_0(\lambda) > 0$ such that for any $i \in \{1, \dots, N\}$, any $g \in L^2(Q)^N$ and any $u_0 \in H$, the solution of*

$$\begin{cases} u_t - \Delta u + \nabla p = g, \quad \nabla \cdot u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{A.1}$$

satisfies

$$s^4 \iint\limits_Q e^{-13 s \alpha^*} (\xi^*)^4 |u|^2 \mathrm{d}x \, \mathrm{d}t \le C \left(\iint\limits_Q e^{-11 s \alpha^*} |g|^2 \mathrm{d}x \, \mathrm{d}t + s^7 \sum_{\substack{j=1 \\ j \ne i}}^N \iint\limits_{\omega \times (0,T)} e^{-2 s \hat{\alpha} - 11 s \alpha^*} (\hat{\xi})^7 |u_j|^2 \mathrm{d}x \, \mathrm{d}t \right),$$

for every $s \ge s_0$.

We are going to develop here the *duality method* introduced in [27] in the context of the heat equation. The same argument has already been used in the context of the heat equation with nonhomogeneous Robin boundary conditions in [15] and in the context of the heat equation with right-hand side belonging to $L^2(0,T;H^{-2}(\Omega))\cap H^{-1}(0,T;L^2(\Omega))$, wich only permits to talk about solutions in $L^2(Q)$; this is explained with detail in [16].

Proof. First, we view u as a solution by transposition of (2.3). This means that u is the unique function in $L^2(Q)^N$ satisfying

$$\iint\limits_Q ug\mathrm{d}x\,\mathrm{d}t = \iint\limits_Q f_0\phi\mathrm{d}x\,\mathrm{d}t - \iint\limits_Q f_1\cdot\nabla\phi\mathrm{d}x\,\mathrm{d}t + \int\limits_\Omega u^0\phi(0)\mathrm{d}x, \quad \forall g\in L^2(Q)^N, \tag{A.2}$$

where we have denoted by $\phi\in L^2(0,T;H^2(\Omega)^N\cap V)\cap H^1(0,T;L^2(\Omega)^N)$, together with p_ϕ, the (strong) solution of the following problem:

$$\begin{cases} -\phi_t-\Delta\phi+\nabla p_\phi = g, \quad \nabla\cdot\phi=0 & \text{in } Q,\\ \phi=0 & \text{on } \Sigma,\\ \phi(T)=0 & \text{in } \Omega.\end{cases} \tag{A.3}$$

Let us first get an estimate of the lower order term in the left-hand side of (2.4), i.e.

$$s^4\iint\limits_Q e^{-13s\alpha^*}(\xi^*)^4|u|^2\mathrm{d}x\,\mathrm{d}t. \tag{A.4}$$

Let us introduce the space

$$Z_0=\{(\phi,p_\phi)\in C^2(\overline{Q})\times C^1(\overline{Q}) : \phi=0 \text{ on } \Sigma \text{ and } \nabla\cdot\phi=0 \text{ in } \Omega\}$$

and the norm $\|\cdot\|_Z$, with

$$\|(\varrho,p_\varrho)\|_Z^2=\iint\limits_Q e^{-11s\alpha^*}|\varrho_t-\Delta\varrho+\nabla p_\varrho|^2\mathrm{d}x\,\mathrm{d}t + s^7\iint\limits_{\omega\times(0,T)} e^{-2s\hat\alpha-11s\alpha^*}(\hat\xi)^7|(\varrho_1,0)|^2\mathrm{d}x\,\mathrm{d}t,$$

for all $(\varrho,p_\varrho)\in Z_0$. Due to Lemma A.1, $\|\cdot\|_Z$ is indeed a norm in Z_0. Let Z be the completion of Z_0 for the norm $\|\cdot\|_Z$. Then Z is a Hilbert space for the scalar product $(\cdot,\cdot)_Z$, with

$$\begin{aligned}((\sigma,p_\sigma),(\gamma,p_\gamma))_Z &= \iint\limits_Q e^{-11s\alpha^*}(\sigma_t-\Delta\sigma+\nabla p_\sigma)(\gamma_t-\Delta\gamma+\nabla p_\gamma)\mathrm{d}x\,\mathrm{d}t\\ &\quad + s^7\iint\limits_{\omega\times(0,T)} e^{-2s\hat\alpha-11s\alpha^*}(\hat\xi)^7\sigma_1\gamma_1\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Then, using Lax Milgram's Lemma there is a unique solution $(\bar\sigma,\bar p_{\bar\sigma})\in Z$ such that

$$((\bar\sigma,\bar p_{\bar\sigma}),(\sigma,p_\sigma))_Z = l(\sigma,p_\sigma), \quad \forall(\sigma,p_\sigma)\in Z, \tag{A.5}$$

where

$$l(\sigma,p_\sigma)=s^4\iint\limits_Q e^{-13s\alpha^*}(\xi^*)^4u\sigma\mathrm{d}x\,\mathrm{d}t.$$

By virtue of Lemma A.1, one can easily check that $l\in Z'$.

We define:

$$\begin{cases}\hat\phi &= e^{-11s\alpha^*}(\bar\sigma_t-\Delta\bar\sigma+\nabla\bar p_{\bar\sigma}),\\ \hat v &= -s^7e^{-2s\hat\alpha-11s\alpha^*}(\bar\sigma_1,0).\end{cases} \tag{A.6}$$

Recall that $\bar{\sigma} = (\bar{\sigma}_1, \bar{\sigma}_2)$. Then, $(\hat{\phi}, \hat{v})$ is solution of (A.3) and such that

$$\|(\bar{\sigma}, \bar{p}_{\bar{\sigma}})\|_Z^2 = ((\bar{\sigma}, \bar{p}_{\bar{\sigma}}), (\bar{\sigma}, \bar{p}_{\bar{\sigma}}))_Z = l(\bar{\sigma}, \bar{p}_{\bar{\sigma}}).$$

Let us now take $g = s^4 e^{-13s\alpha^*}(\xi^*)^4 u + \hat{v}\mathbb{1}_\omega$ in (A.2). This gives

$$s^4 \iint\limits_Q e^{-13s\alpha^*}(\xi^*)^4 |u|^2 \mathrm{d}x\,\mathrm{d}t = \iint\limits_Q f_0 \hat{\phi} \mathrm{d}x\,\mathrm{d}t - \iint\limits_Q f_1 \cdot \nabla\hat{\phi} \mathrm{d}x\,\mathrm{d}t - \iint\limits_{\omega\times(0,T)} u\hat{v} \mathrm{d}x\,\mathrm{d}t, \tag{A.7}$$

(recall that $\hat{v}$ and $\hat{\phi}$ are given by (A.6)).

From (A.5), we obtain

$$\|(\bar{\sigma}, \bar{p}_{\bar{\sigma}})\|_Z^2 \leq \|l\|_{Z'} \|(\bar{\sigma}, \bar{p}_{\bar{\sigma}})\|_Z.$$

Consequently,

$$\|(\bar{\sigma}, \bar{p}_{\bar{\sigma}})\|_Z^2 = \iint\limits_Q e^{11s\alpha^*}|\hat{\phi}|^2 \mathrm{d}x\,\mathrm{d}t + s^{-7} \iint\limits_{\omega\times(0,T)} e^{2s\hat{\alpha}+11s\alpha^*}|\hat{v}|^2 \mathrm{d}x\,\mathrm{d}t \leq Cs^4 \iint\limits_Q e^{-13s\alpha^*}(\xi^*)^4|u|^2 \mathrm{d}x\,\mathrm{d}t, \tag{A.8}$$

for $s \geq C = C(\Omega, \omega, T) > 0$, since

$$\|l\|_{Z'} \leq s^2 \left(\iint\limits_Q e^{-13s\alpha^*}(\xi^*)^4|u|^2 \mathrm{d}x\,\mathrm{d}t \right)^{1/2}.$$

Now, we multiply the equation satisfied by $\hat{\phi}$ by $s^{-7}e^{2s\hat{\alpha}+11s\alpha^*}(\hat{\xi})^{-7}\hat{\phi}$ and we integrate in Q. After integration by parts, we get:

$$\begin{aligned} s^{-7} \iint\limits_Q e^{2s\hat{\alpha}+11s\alpha^*}(\hat{\xi})^{-7}|\nabla\hat{\phi}|^2 \mathrm{d}x\,\mathrm{d}t &= \frac{s^{-7}}{2} \iint\limits_Q \frac{\partial}{\partial t}(e^{2s\hat{\alpha}+11s\alpha^*}(\hat{\xi})^{-7})|\hat{\phi}|^2 \mathrm{d}x\,\mathrm{d}t \\ +s^{-3} \iint\limits_Q e^{2s\hat{\alpha}+6s\alpha^*}(\hat{\xi})^{-3} u\hat{\phi} \mathrm{d}x\,\mathrm{d}t &+ s^{-7} \iint\limits_{\omega\times(0,T)} e^{2s\hat{\alpha}+11s\alpha^*}(\hat{\xi})^{-7}\hat{v}\hat{\phi} \mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{A.9}$$

Using Holder's inequality and Young's inequality in the last two terms of the right-hand side of (A.9), we have

$$\begin{aligned} s^{-7} \iint\limits_Q e^{2s\hat{\alpha}+11s\alpha^*}(\hat{\xi})^{-7}|\nabla\hat{\phi}|^2 \mathrm{d}x\,\mathrm{d}t \leq & C \Bigg(\iint\limits_Q e^{11s\alpha^*}|\hat{\phi}|^2 \mathrm{d}x\,\mathrm{d}t \\ & + s^{-7} \iint\limits_{\omega\times(0,T)} e^{2s\hat{\alpha}+11s\alpha^*}|\hat{v}|^2 \mathrm{d}x\,\mathrm{d}t + s^4 \iint\limits_Q e^{-13s\alpha^*}(\xi^*)^4|u|^2 \mathrm{d}x\,\mathrm{d}t \Bigg), \end{aligned}$$

where we have taken $s \geq C$. This inequality, together with (A.8), provides

$$\begin{aligned} & \iint\limits_Q e^{11s\alpha^*}|\hat{\phi}|^2 \mathrm{d}x\,\mathrm{d}t + s^{-7} \iint\limits_Q e^{2s\hat{\alpha}+11s\alpha^*}(\hat{\xi})^{-7}|\nabla\hat{\phi}|^2 \mathrm{d}x\,\mathrm{d}t \\ & + s^{-7} \iint\limits_{\omega\times(0,T)} e^{2s\hat{\alpha}+11s\alpha^*}|\hat{v}|^2 \mathrm{d}x\,\mathrm{d}t \leq Cs^4 \iint\limits_Q e^{-13s\alpha^*}(\xi^*)^4|u|^2 \mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{A.10}$$

A combination of this inequality with (A.7) yields the following estimate:

$$s^4 \iint\limits_Q e^{-13s\alpha^*}(\xi^*)^4|u|^2 \mathrm{d}x\,\mathrm{d}t \leq C \left(\iint\limits_Q e^{-11s\alpha^*}|f_0|^2 \mathrm{d}x\,\mathrm{d}t + s^7 \iint\limits_Q e^{-11s\alpha^*}(\hat{\xi})^7|f_1|^2 \mathrm{d}x\,\mathrm{d}t + s^7 \sum_{\substack{j=1\\ j\neq i}}^N \iint\limits_{\omega\times(0,T)} e^{-2s\hat{\alpha}-11s\alpha^*}(\hat{\xi})^7|u_j|^2 \mathrm{d}x\,\mathrm{d}t \right), \tag{A.11}$$

Let us now show that the term associated with ∇u can also be bounded in the same way. To this end, we multiply the equation of u by

$$s^3 e^{-13s\alpha^*}(\xi^*)^{3-1/m}$$

and we obtain

$$\frac{s^3}{2}\iint\limits_Q e^{-13s\alpha^*}(\xi^*)^{3-1/m}\frac{\partial}{\partial t}|u|^2 \mathrm{d}x\,\mathrm{d}t + s^3\iint\limits_Q e^{-13s\alpha^*}(\xi^*)^{3-1/m}|\nabla u|^2 \mathrm{d}x\,\mathrm{d}t$$
$$= s^3 \iint\limits_Q e^{-13s\alpha^*}(\xi^*)^{3-1/m} f_0 u \mathrm{d}x\,\mathrm{d}t - s^3 \iint\limits_Q e^{-13s\alpha^*}(\xi^*)^{3-1/m} f_1\cdot\nabla u \mathrm{d}x\,\mathrm{d}t. \tag{A.12}$$

Now, integrating by parts with respect to t in the first integral of the left-hand side of (A.12) and using that

$$(e^{-13s\alpha^*}(\xi^*)^{3-1/m})_t \leq Cse^{-13s\alpha^*}(\xi^*)^4, \quad s\geq C,$$

we have at this moment,

$$s^4 \iint\limits_Q e^{-13s\alpha^*}(\xi^*)^4|u|^2 \mathrm{d}x\,\mathrm{d}t + s^3 \iint\limits_Q e^{-13s\alpha^*}(\xi^*)^{3-1/m}|\nabla u|^2 \mathrm{d}x\,\mathrm{d}t \leq C \left(\iint\limits_Q e^{-11s\alpha^*}|f_0|^2 \mathrm{d}x\,\mathrm{d}t + s^7 \iint\limits_Q e^{-11s\alpha^*}(\hat{\xi})^7|f_1|^2 \mathrm{d}x\,\mathrm{d}t + s^7 \sum_{\substack{j=1\\ j\neq i}}^N \iint\limits_{\omega\times(0,T)} e^{-2s\hat{\alpha}-11s\alpha^*}(\hat{\xi})^7|u_j|^2 \mathrm{d}x\,\mathrm{d}t \right), \tag{A.13}$$

On the other hand, we consider

$$\tilde{u} := se^{-13/2s\alpha^*}(\xi^*)^{1-1/m}u := \rho_4(t)u, \quad \tilde{h} := se^{-13/2s\alpha^*}(\xi^*)^{1-1/m}h := \rho_4(t)h.$$

Then, $(\tilde{u},\tilde{h})$ satisfies the following system:

$$\begin{cases} \tilde{u}_t - \Delta\tilde{u} + \nabla\tilde{h} = -(\rho_4)_t u + \rho_4 f_0 + \rho_4\nabla\cdot f_1, \quad \nabla\cdot\tilde{u} = 0 & \text{in } Q, \\ \tilde{u} = 0 & \text{on } \Sigma, \\ \tilde{u}(0) = 0 & \text{in } \Omega. \end{cases} \tag{A.14}$$

Applying certain regularity result and using $L^2(\Omega)^N \subset H^{-1}(\Omega)^N$, we have that

$$\|\tilde{u}\|^2_{L^2(0,T;H^1(\Omega)^N)} + \|\tilde{h}\|^2_{L^2(Q)} \leq C\left(\|\rho_4 f_0\|^2_{L^2(Q)^N} + \|\rho_4 f_1\|^2_{L^2(Q)^N} + \|(\rho_4)_t u\|^2_{L^2(Q)^N}\right).$$

for some $C>0$. Then, since $|(\rho_4)_t| \leq Cs^2 e^{-13/2s\alpha^*}(\xi^*)^2$, we can add the term associated with the pressure to the left-hand side of (A.13). Finally, we obtain (2.4). □

B Proof of Proposition 3.2

In this occasion, we will prove the Carleman estimate of ψ following a method introduced in [12]. For simplicity of the proof, we will consider the case $N = 2$ and $i = 2$.

Proof. First, we apply the divergence operator to equation associated with ψ to obtain

$$\Delta h = \nabla \cdot g^{\psi} = 0 \text{ in } Q.$$

Then, applying the operator $\nabla\nabla\nabla\Delta^2(\cdot)$ to the equation satisfied be ψ_1, we have:

$$(\nabla\nabla\nabla\Delta^2\psi_1)_t - \Delta(\nabla\nabla\nabla\Delta^2\psi_1) = \nabla\nabla\nabla\Delta^2 g_1^{\psi}.$$

Thus, we can apply Lemma 2.1 to this equation to obtain

$$\begin{aligned}
&\iint\limits_{Q} e^{-10s\alpha}\Big(s^{-1}\xi^{-1}|\nabla\nabla\nabla\nabla\Delta^2\psi_1|^2 + s\xi|\nabla\nabla\nabla\Delta^2\psi_1|^2\Big)\mathrm{d}x\,\mathrm{d}t \\
&\quad\le C\Bigg(\left\|s^{-1/4}e^{-5s\alpha}\xi^{-1/4+1/m}\nabla\nabla\nabla\Delta^2\psi_1\right\|^2_{L^2(\Sigma)^8} + \left\|s^{-1/4}e^{-5s\alpha}\xi^{-1/4}\nabla\nabla\nabla\Delta^2\psi_1\right\|^2_{H^{1/4,1/2}(\Sigma)^8} \\
&\qquad + s\iint\limits_{\omega_0\times(0,T)} e^{-10s\alpha}\xi|\nabla\nabla\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t + \iint\limits_{Q} e^{-10s\alpha}|\nabla^2\Delta^2 g_1^{\psi}|^2\mathrm{d}x\,\mathrm{d}t\Bigg), \qquad \text{(B.1)}
\end{aligned}$$

for every $s \ge C$.

We divide the rest of the proof in three steps:

- In Step 1, we estimate globals integrals of ψ_1 y ψ_2 by the left-hand side of (B.1).
- In Step 2, we deal with the boundary terms en (B.1).
- In Step 3, we estimate all the local terms.

In the following, C denotes a constant depending only on λ, Ω, ω, $\mathcal{O}$ y ℓ.

Step 1

Estimate of ψ_1 : Applying successively Lemma 2.2 with $r = 1$ and $u := \nabla\nabla\Delta^2\psi_1$, $r = 3$ and $u = \nabla\Delta^2\psi_1$, $r = 5$ and $u = \Delta^2\psi_1$, and combining with (B.1), we get

$$\begin{aligned}
&s^{-1}\iint\limits_{Q} e^{-10s\alpha}\xi^{-1}|\nabla\nabla\nabla\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t + s\iint\limits_{Q} e^{-10s\alpha}\xi|\nabla\nabla\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t + s^3\iint\limits_{Q} e^{-10s\alpha}\xi^3|\nabla\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t \\
&+ s^5\iint\limits_{Q} e^{-10s\alpha}\xi^5|\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t + s^7\iint\limits_{Q} e^{-10s\alpha}\xi^7|\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t \le C\Bigg(\left\|s^{-1/4}e^{-5s\alpha}\xi^{-1/4+1/m}\nabla\nabla\nabla\Delta^2\psi_1\right\|^2_{L^2(\Sigma)^8} \\
&+ \left\|s^{-1/4}e^{-5s\alpha}\xi^{-1/4}\nabla\nabla\nabla\Delta^2\psi_1\right\|^2_{H^{\frac14,\frac12}(\Sigma)^8} + s\iint\limits_{\omega_0\times(0,T)} e^{-10s\alpha}\xi|\nabla\nabla\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t + s^3\iint\limits_{\omega_0\times(0,T)} e^{-10s\alpha}\xi^3|\nabla\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t \\
&+ s^5\iint\limits_{\omega_0\times(0,T)} e^{-10s\alpha}\xi^5|\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t + s^7\iint\limits_{\omega_0\times(0,T)} e^{-10s\alpha}\xi^7|\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t + \iint\limits_{Q} e^{-10s\alpha}|\nabla^2\Delta^2 g_1^{\psi}|^2\mathrm{d}x\,\mathrm{d}t\Bigg),
\end{aligned} \qquad \text{(B.2)}$$

for every $s \ge C$.

Estimate of ψ_2 : Now, we would like to introduce in the left-hand side a term in $\psi = (\psi_1, \psi_2)$. Actually, we are going to add the term $\left\|s^{7/2}e^{-5s\alpha^*}(\xi^*)^{7/2}\psi\right\|_{L^2(0,T;H^2(\Omega)^N)}$ to the left-hand side of (B.2).
Notice that, since $\nabla\cdot\psi_t = 0$ in Q, we have for all $t \in (0,T)$:

$$\begin{aligned}\int_\Omega |\partial_2(\psi_2)_t(t)|^2 \mathrm{d}x &= \int_\Omega |\partial_1(\psi_1)_t(t)|^2 \mathrm{d}x \\ &\leq \int_\Omega |\nabla(\psi_1)_t(t)|^2 \mathrm{d}x.\end{aligned} \tag{B.3}$$

Since $(\psi_2)_t(t)|_{\partial\Omega} = 0$ and Ω is bounded, also, using (B.3), we have

$$\int_\Omega |(\psi_2)_t(t)|^2 \mathrm{d}x \leq C \int_\Omega |\nabla(\psi_1)_t(t)|^2 \mathrm{d}x.$$

Then, we deduce

$$\|\psi_t(t)\|^2_{L^2(\Omega)^N} \leq C\,\|\nabla(\psi_1)_t(t)\|^2_{L^2(\Omega)^N}\,, \quad \forall t \in (0,T). \tag{B.4}$$

Consider now the following Stokes system,

$$\begin{cases} -\Delta\psi + \nabla h = -\psi_t + g^\psi & \text{in } Q, \\ \nabla\cdot\psi = 0 & \text{in } Q, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{B.5}$$

then, using a regularity result of [37] for the stationary Stokes problem (B.5), together with (B.4), we obtain

$$\|\psi(t)\|^2_{H^2(\Omega)^N} \leq C\left(\|\nabla(\psi_1)_t(t)\|^2_{L^2(\Omega)^N} + \left\|g^\psi(t)\right\|^2_{L^2(\Omega)^N}\right),\ \forall t \in (0,T). \tag{B.6}$$

Now, observe that using the divergence free condition and applying the laplacian operator to the equation associated with ψ_1, we get that $(\Delta\psi_1)_t = \Delta^2\psi_1 + \Delta g_1^\psi$ in Q. On the other hand, since $(\psi_1)_t|_{\partial\Omega} = 0$, we have

$$\|(\psi_1)_t\|^2_{H^2(\Omega)} \leq C\,\|\Delta(\psi_1)_t\|_{L^2(\Omega)}\,. \tag{B.7}$$

Using (B.7) in (B.6), we obtain

$$\|\psi(t)\|^2_{H^2(\Omega)^N} \leq C\left(\|\Delta^2\psi_1(t)\|^2_{L^2(\Omega)} + \|\Delta g^\psi(t)\|^2_{L^2(\Omega)^N}\right),\ \forall t \in (0,T).$$

Finally, since α^* and ξ^* do not depend on the space variable x, we have that

$$\begin{aligned} s^7\int_0^T e^{-10s\alpha^*}(\xi^*)^7\,\|\psi\|^2_{H^2(\Omega)^N}\,\mathrm{d}t \leq &Cs^7\Bigg(\iint_Q e^{-10s\alpha^*}(\xi^*)^7|\Delta^2\psi_1|^2 \mathrm{d}x\,\mathrm{d}t \\ &+ \iint_Q e^{-10s\alpha^*}(\xi^*)^7|\Delta g^\psi|^2 \mathrm{d}x\,\mathrm{d}t\Bigg). \end{aligned} \tag{B.8}$$

Therefore, combining (B.2) and (B.8) we get

$$
\begin{aligned}
&s^{-1}\iint\limits_Q e^{-10s\alpha}\xi^{-1}|\nabla\nabla\nabla\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t+s\iint\limits_Q e^{-10s\alpha}\xi|\nabla\nabla\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t+s^3\iint\limits_Q e^{-10s\alpha}\xi^3|\nabla\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t\\
&\quad+s^5\iint\limits_Q e^{-10s\alpha}\xi^5|\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t+s^7\iint\limits_Q e^{-10s\alpha}\xi^7|\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t+s^7\iint\limits_Q e^{-10s\alpha^*}(\xi^*)^7|\psi|^2\mathrm{d}x\,\mathrm{d}t\\
&+s^7\iint\limits_Q e^{-10s\alpha^*}(\xi^*)^7|\nabla\psi|^2\mathrm{d}x\,\mathrm{d}t+s^7\iint\limits_Q e^{-10s\alpha^*}(\xi^*)^7\left|\Delta\psi\right|^2\mathrm{d}x\,\mathrm{d}t\le C\Bigg(\left\|s^{-1/4}e^{-5s\alpha}\xi^{-1/4+1/m}\nabla\nabla\nabla\Delta^2\psi_1\right\|^2_{L^2(\Sigma)^8}\\
&\quad+\left\|s^{-1/4}e^{-5s\alpha}\xi^{-1/4}\nabla\nabla\nabla\Delta^2\psi_1\right\|^2_{H^{\frac14,\frac12}(\Sigma)^8}+s\iint\limits_{\omega_0\times(0,T)}e^{-10s\alpha}\xi|\nabla\nabla\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t\\
&+s^3\iint\limits_{\omega_0\times(0,T)}e^{-10s\alpha}\xi^3|\nabla\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t+s^5\iint\limits_{\omega_0\times(0,T)}e^{-10s\alpha}\xi^5|\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t+s^7\iint\limits_{\omega_0\times(0,T)}e^{-10s\alpha}\xi^7|\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t\\
&\quad+s^7\iint\limits_Q e^{-10s\alpha}|\nabla\nabla\Delta^2 g^\psi|^2\mathrm{d}x\,\mathrm{d}t+s^7\iint\limits_Q e^{-10s\alpha^*}(\xi^*)^7|\Delta g^\psi|^2\mathrm{d}x\,\mathrm{d}t\Bigg),
\end{aligned}\tag{B.9}
$$

for every $s\ge C$.

Step 2

In this step we treat the boundary terms in (B.9). We begin with the first one. Notice that the minimum of the weight functions $e^{s\alpha}$ and ξ is reached at the boundary $\partial\Omega$, where $\alpha=\alpha^*$ and $\xi=\xi^*$ do not depend on x. Since $m\ge 14$, and using Young inequality, we obtain

$$
\begin{aligned}
&\left\|e^{-5s\alpha^*}(\xi^*)^{-1/4+1/m}\nabla\nabla\nabla\Delta^2\psi_1\right\|^2_{L^2(\Sigma)^8}\\
\le&C\left\|e^{-5s\alpha^*}\nabla\nabla\nabla\Delta^2\psi_1\right\|^2_{L^2(\Sigma)^8}\\
\le&C\left(\left\|s^{1/2}e^{-5s\alpha^*}(\xi^*)^{1/2}\nabla\nabla\nabla\Delta^2\psi_1\right\|_{L^2(Q)^8}\cdot\left\|s^{-1/2}e^{-5s\alpha^*}(\xi^*)^{-1/2}\nabla\nabla\nabla\nabla\Delta^2\psi_1\right\|_{L^2(Q)^{16}}\right.\\
&\left.+\left\|e^{-5s\alpha^*}(\xi^*)^{1/2}\nabla\nabla\nabla\Delta^2\psi_1\right\|^2_{L^2(Q)^8}\right)\\
\le&C\left(s^{-1}\iint\limits_Q e^{-10s\alpha}\xi^{-1}|\nabla\nabla\nabla\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t+s\iint\limits_Q e^{-10s\alpha}\xi|\nabla\nabla\nabla\Delta^2\psi_1|^2\mathrm{d}x\,\mathrm{d}t\right).
\end{aligned}
$$

Therefore, this boundary term can be absorbed by left-hand side of (B.9) for $s\ge C$.

Now, we deal the second boundary term in the right-hand side of (B.9). To this end, we use regularity estimates.

First, notice that $\|s^{7/2}e^{-5s\alpha^*}(\xi^*)^{7/2}\psi\|^2_{L^2(0,T;V)}=\|s^{7/2}e^{-5s\alpha^*}(\xi^*)^{7/2}\psi\|^2_{X_0}$ is in the left-hand side of (B.9).
Let us define

$$(\psi^1,h_1):=s^{5/2}e^{-5s\alpha^*}(\xi^*)^{5/2-1/m}(\psi,h)=:\xi_1(t)(\psi,h).$$

Then, from (3.3), (ψ^1,h_1) is the solution of Stokes system:

$$
\begin{cases}
\psi^1_t-\Delta\psi^1+\nabla h_1=(\xi_1)_t\psi+\xi_1 g^\psi, \quad \nabla\cdot\psi^1=0 & \text{in } Q,\\
\psi^1=0 & \text{on } \Sigma,\\
\psi^1(0)=0 & \text{in } \Omega,
\end{cases}\tag{B.10}
$$

Using Lemma 2.6 for the last system, we have

$$\|\psi^1\|^2_{X_1}\le C\left(\|(\xi_1)_t\psi\|^2_{X_0}+\|\xi_1 g^\psi\|^2_{X_0}\right).$$

From (2.1), we see that

$$|(\xi_1)_t| \le Cs^{7/2}e^{-5s\alpha^*}(\xi^*)^{7/2},$$

for every $s \ge C$. Thus, we obtain

$$\left\|\psi^1\right\|_{X_1}^2 \le C\left(\left\|s^{7/2}e^{-5s\alpha^*}(\xi^*)^{7/2}\psi\right\|_{X_0}^2 + \left\|\xi_1 g^\psi\right\|_{X_0}^2\right).$$

Next, we introduce:

$$(\psi^2, h_2) := s^{3/2}e^{-5s\alpha^*}(\xi^*)^{3/2-2/m}(\psi, h) =: \xi_2(t)(\psi, h).$$

Now, from (3.3), (ψ^2, h_2) is the solution of Stokes system:

$$\begin{cases} \psi^2_t - \Delta\psi^2 + \nabla h_2 = (\xi_2)_t\psi + \xi_2 g^\psi, \quad \nabla\cdot\psi^2 = 0 & \text{in } Q,\\ \psi^2 = 0 & \text{on } \Sigma,\\ \psi^2(0) = 0 & \text{in } \Omega, \end{cases} \tag{B.11}$$

Using Lemma 2.6 for the last system, we find:

$$\left\|\psi^2\right\|_{X_2}^2 \le C\left(\|(\xi_2)_t\psi\|_{X_1}^2 + \|\xi_2 g^\psi\|_{X_1}^2\right).$$

Using the estimate

$$|(\xi_2)_t| \le Cs^{5/2}e^{-5s\alpha^*}(\xi^*)^{5/2-1/m},$$

for every $s \ge C$. Thus, we obtain

$$\begin{aligned} \left\|\psi^2\right\|_{X_2}^2 \le& C\left(\left\|s^{5/2}e^{-5s\alpha^*}(\xi^*)^{5/2-1/m}\psi\right\|_{X_1}^2 + \left\|\xi_2 g^\psi\right\|_{X_1}^2\right).\\ \le& C\left(\left\|s^{7/2}e^{-5s\alpha^*}(\xi^*)^{7/2}\psi\right\|_{X_0}^2 + \left\|\xi_1 g^\psi\right\|_{X_0}^2 + \left\|\xi_2 g^\psi\right\|_{X_1}^2\right). \end{aligned}$$

Next, we define:

$$(\psi^3, h_3) := s^{1/2}e^{-5s\alpha^*}(\xi^*)^{1/2-3/m}(\psi, h) =: \xi_3(t)(\psi, h).$$

Then, from (3.3), (ψ^3, h_3) is the solution of Stokes system:

$$\begin{cases} \psi^3_t - \Delta\psi^3 + \nabla h_3 = (\xi_3)_t\psi + \xi_3 g^\psi, \quad \nabla\cdot\psi^3 = 0 & \text{in } Q,\\ \psi^3 = 0 & \text{on } \Sigma,\\ \psi^3(0) = 0 & \text{in } \Omega, \end{cases} \tag{B.12}$$

Using Lemma 2.6 for the last system, we get:

$$\left\|\psi^3\right\|_{X_3}^2 \le C\left(\|(\xi_3)_t\psi\|_{X_2}^2 + \left\|\xi_3 g^\psi\right\|_{X_2}^2\right).$$

Using the estimate

$$|(\xi_3)_t| \le Cs^{3/2}e^{-5s\alpha^*}(\xi^*)^{3/2-2/m},$$

for every $s \ge C$. Thus, we obtain

$$\begin{aligned} \left\|\psi^3\right\|_{X_3}^2 \le& C\left(\left\|\psi^2\right\|_{X_2}^2 + \left\|\xi_3 g^\psi\right\|_{X_2}^2\right).\\ \le& C\left(\left\|s^{7/2}e^{-5s\alpha^*}(\xi^*)^{7/2}\psi\right\|_{X_0}^2 + \left\|\xi_1 g^\psi\right\|_{X_0}^2 + \left\|\xi_2 g^\psi\right\|_{X_1}^2 + \left\|\xi_3 g^\psi\right\|_{X_2}^2\right). \end{aligned}$$

Next, we introduce:

$$(\psi^4, h_4) := s^{-1/2}e^{-5s\alpha^*}(\xi^*)^{-1/2-4/m}(\psi, h) =: \xi_4(t)(\psi, h).$$

Then, from (3.3), (ψ^4, h_4) is the solution of Stokes system:

$$\left\{\begin{array}{ll} \psi^4_t - \Delta\psi^4 + \nabla h_4 = (\xi_4)_t\psi + \xi_4 g^\psi, \quad \nabla\cdot\psi^4 = 0 & \text{in } Q, \\ \psi^4 = 0 & \text{on } \Sigma, \\ \psi^4(0) = 0 & \text{in } \Omega, \end{array}\right. \tag{B.13}$$

Using Lemma 2.6 for the last system, we find:

$$\left\|\psi^4\right\|^2_{X_4} \leq C\left(\|(\xi_4)_t\psi\|^2_{X_3} + \left\|\xi_4 g^\psi\right\|^2_{X_3}\right).$$

Using the estimate

$$|(\xi_4)_t| \leq Cs^{1/2}e^{-5s\alpha^*}(\xi^*)^{1/2-3/m},$$

for every $s \geq C$. Thus, we obtain

$$\begin{aligned} \left\|\psi^4\right\|^2_{X_4} \leq & C\left(\left\|\psi^3\right\|^2_{X_3} + \left\|\xi_4 g^\psi\right\|^2_{X_3}\right). \\ \leq & C\left(\left\|s^{7/2}e^{-5s\alpha^*}(\xi^*)^{7/2}\psi\right\|^2_{X_0} + \left\|\xi_1 g^\psi\right\|^2_{X_0} + \left\|\xi_2 g^\psi\right\|^2_{X_1} + \left\|\xi_3 g^\psi\right\|^2_{X_2} + \left\|\xi_4 g^\psi\right\|^2_{X_3}\right). \end{aligned}$$

Then, using interpolation argument between the spaces X_3 and X_4, we get

$$\begin{aligned} \left\|e^{-5s\alpha^*}(\xi^*)^{-7/(2m)}\psi\right\|^2_{L^2(0,T;H^8(\Omega)^N)\cap H^1(0,T;H^6(\Omega)^N)} \leq & C\left(\left\|s^{1/2}e^{-5s\alpha^*}(\xi^*)^{1/2-3/m}\psi\right\|_{X_3}\right. \\ & \left.\cdot\left\|s^{-1/2}e^{-5s\alpha^*}(\xi^*)^{-1/2-4/m}\psi\right\|_{X_4}\right). \end{aligned}$$

Now, we consider the boundary term

$$\begin{aligned} & s^{-1/2}\left\|e^{-5s\alpha^*}(\xi^*)^{-1/4}\nabla\nabla\nabla\Delta^2\psi_1\right\|^2_{H^{1/4,1/2}(\Sigma)^8} \\ \leq & C\left(s^{-1/2}\left\|e^{-5s\alpha^*}(\xi^*)^{-1/4}\nabla\nabla\nabla\Delta^2\psi_1\right\|^2_{H^1(0,T;H^{-1}(\Omega)^N)}\right. \\ & \left. + s^{-1/2}\left\|e^{-5s\alpha^*}(\xi^*)^{-1/4}\nabla\nabla\nabla\Delta^2\psi_1\right\|^2_{L^2(0,T;H^1(\Omega)^N)}\right). \\ \leq & C\left(s^{-1/2}\left\|e^{-5s\alpha^*}(\xi^*)^{-1/4}\psi_1\right\|^2_{L^2(0,T;H^8(\Omega)^N)}\right. \\ & \left. + s^{-1/2}\left\|e^{-5s\alpha^*}(\xi^*)^{-1/4}\psi_1\right\|^2_{H^1(0,T;H^6(\Omega)^N)}\right). \\ = & Cs^{-1/2}\left\|e^{-5s\alpha^*}(\xi^*)^{-1/4}\psi_1\right\|^2_{L^2(0,T;H^8(\Omega)^N)\cap H^1(0,T;H^6(\Omega)^N)}. \\ \leq & Cs^{-1/2}\left\|e^{-5s\alpha^*}(\xi^*)^{-7/(2m)}\psi\right\|^2_{L^2(0,T;H^8(\Omega)^N)\cap H^1(0,T;H^6(\Omega)^N)}. \end{aligned} \tag{B.14}$$

By taking s large enough in (B.14), the boundary term $s^{-1/2}\left\|e^{-5s\alpha}\xi^{-1/4}\psi\right\|_{H^{1/4,1/2}(\Sigma)^{16}}$ can be absorbed by the term $\left\|e^{-5s\alpha^*}(\xi^*)^{-7/(2m)}\psi\right\|^2_{L^2(0,T;H^8(\Omega)^N)\cap H^1(0,T;H^6(\Omega)^N)}$ and step 2 is finished.

Thus, at this point we have

$$
\begin{aligned}
&s^{-1}\iint\limits_Q e^{-10s\alpha}\xi^{-1}|\nabla\nabla\nabla\nabla\Delta^2\psi_1|^2 \mathrm{d}x\,\mathrm{d}t + s\iint\limits_Q e^{-10s\alpha}\xi|\nabla\nabla\nabla\Delta^2\psi_1|^2 \mathrm{d}x\,\mathrm{d}t\\
&\quad + s^3\iint\limits_Q e^{-10s\alpha}\xi^3|\nabla\nabla\Delta^2\psi_1|^2 \mathrm{d}x\,\mathrm{d}t + s^5\iint\limits_Q e^{-10s\alpha}\xi^5|\nabla\Delta^2\psi_1|^2 \mathrm{d}x\,\mathrm{d}t + s^7\iint\limits_Q e^{-10s\alpha}\xi^7|\Delta^2\psi_1|^2 \mathrm{d}x\,\mathrm{d}t\\
&\quad + s^7\iint\limits_Q e^{-10s\alpha^*}(\xi^*)^7|\psi|^2 \mathrm{d}x\,\mathrm{d}t + s^7\iint\limits_Q e^{-10s\alpha^*}(\xi^*)^7|\nabla\psi|^2 \mathrm{d}x\,\mathrm{d}t + s^7\iint\limits_Q e^{-10s\alpha^*}(\xi^*)^7|\Delta\psi|^2 \mathrm{d}x\,\mathrm{d}t\\
&\qquad\le C\Bigg(s\iint\limits_{\omega_0\times(0,T)} e^{-10s\alpha}\xi|\nabla\nabla\nabla\Delta^2\psi_1|^2 \mathrm{d}x\,\mathrm{d}t + s^3\iint\limits_{\omega_0\times(0,T)} e^{-10s\alpha}\xi^3|\nabla\nabla\Delta^2\psi_1|^2 \mathrm{d}x\,\mathrm{d}t\\
&+ s^5\iint\limits_{\omega_0\times(0,T)} e^{-10s\alpha}\xi^5|\nabla\Delta^2\psi_1|^2 \mathrm{d}x\,\mathrm{d}t + s^7\iint\limits_{\omega_0\times(0,T)} e^{-10s\alpha}\xi^7|\Delta^2\psi_1|^2 \mathrm{d}x\,\mathrm{d}t + s^7\iint\limits_Q e^{-10s\alpha^*}(\xi^*)^7|\Delta g^\psi|^2 \mathrm{d}x\,\mathrm{d}t\\
&\qquad +s^7\iint_Q e^{-10s\alpha}|\nabla\nabla\Delta^2 g^\psi|^2 \mathrm{d}x\,\mathrm{d}t + \|\xi_1 g^\psi\|^2_{X_0} + \|\xi_2 g^\psi\|^2_{X_1} + \|\xi_3 g^\psi\|^2_{X_2} + \|\xi_4 g^\psi\|^2_{X_3}\Bigg),
\end{aligned}\tag{B.15}
$$

for every $s \ge C$.

Step 3

In this step, we estimate the first three terms in the right-hand side of (B.15) in terms of $\Delta^2\psi_1$ and small constants multiplied by the left-hand side of (B.15).

We start by estimating the term on $\nabla\nabla\nabla\Delta^2\psi_1$. Let ω_1 be an open subset satisfying $\omega_0 \Subset \omega_1 \Subset \tilde\omega$ and let $\rho_1 \in C^2_c(\omega_1)$ with $\rho_1 \equiv 1$ in ω_0 and $0 \le \rho_1$. Then, an integration by parts gives

$$
\begin{aligned}
s\iint\limits_{\omega_0\times(0,T)} e^{-10s\alpha}\xi\left|\nabla\nabla\nabla\Delta^2\psi_1\right|^2 \mathrm{d}x\,\mathrm{d}t \le & s\iint\limits_{\omega_1\times(0,T)} \rho_1 e^{-10s\alpha}\xi\left|\nabla\nabla\nabla\Delta^2\psi_1\right|^2 \mathrm{d}x\,\mathrm{d}t.\\
=& - s\iint\limits_{\omega_1\times(0,T)} \rho_1 e^{-10s\alpha}\xi\nabla\nabla\Delta^2\psi_1(\nabla\nabla\nabla\nabla\Delta^2\psi_1)\mathrm{d}x\,\mathrm{d}t\\
&+\frac{s}{2}\iint\limits_{\omega_1\times(0,T)} \Delta(\rho_1 e^{-10s\alpha}\xi)\left|\nabla\nabla\nabla\Delta^2\psi_1\right|^2 \mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Using the Cauchy-Schwarz's inequality for the first term and property (2.2) for the second one, we obtain for every $\epsilon > 0$

$$
\begin{aligned}
s\iint\limits_{\omega_0\times(0,T)} e^{-10s\alpha}\xi\left|\nabla\nabla\nabla\Delta^2\psi_1\right|^2 \mathrm{d}x\,\mathrm{d}t \le & Cs^3\iint\limits_{\omega_1\times(0,T)} e^{-10s\alpha}\xi^3\left|\nabla\nabla\Delta^2\psi_1\right|^2 \mathrm{d}x\,\mathrm{d}t\\
&+\epsilon s^{-1}\iint\limits_Q e^{-10s\alpha}\xi^{-1}\left|\nabla\nabla\nabla\nabla\Delta^2\psi_1\right|^2 \mathrm{d}x\,\mathrm{d}t,
\end{aligned}\tag{B.16}
$$

for every $s \ge C$ (C depending also on ϵ).

Repeating the same argument we can obtain the estimates of $\nabla\nabla\Delta^2\psi_1$ in terms of $\nabla\Delta^2\psi_1$ and $\nabla\Delta^2\psi_1$ in terms of $\Delta^2\psi_1$, namely

$$
\begin{aligned}
s^3\iint\limits_{\omega_1\times(0,T)} e^{-10s\alpha}\xi^3\left|\nabla\nabla\Delta^2\psi_1\right|^2 \mathrm{d}x\,\mathrm{d}t \le & Cs^5\iint\limits_{\omega_2\times(0,T)} e^{-10s\alpha}\xi^5\left|\nabla\Delta^2\psi_1\right|^2 \mathrm{d}x\,\mathrm{d}t\\
&+\epsilon s\iint\limits_Q e^{-10s\alpha}\xi\left|\nabla\nabla\nabla\Delta^2\psi_1\right|^2 \mathrm{d}x\,\mathrm{d}t,
\end{aligned}\tag{B.17}
$$

$$s^5 \iint\limits_{\omega_2\times(0,T)} e^{-10s\alpha}\xi^5 \left|\nabla\Delta^2\psi_1\right|^2 \mathrm{d}x\,\mathrm{d}t \leq Cs^7 \iint\limits_{\omega_3\times(0,T)} e^{-10s\alpha}\xi^7 \left|\Delta^2\psi_1\right|^2 \mathrm{d}x\,\mathrm{d}t + \epsilon s^3 \iint\limits_{Q} e^{-10s\alpha}\xi^3 \left|\nabla\nabla\Delta^2\psi_1\right|^2 \mathrm{d}x\,\mathrm{d}t, \tag{B.18}$$

for every $s \geq C$ (C depending also on ϵ), where $\omega_1 \Subset \omega_2 \Subset \omega_3 \Subset \tilde{\omega}$.

This estimate, together with (B.15), (B.16) and (B.17), readily gives the desired Carleman inequality (3.4). This concludes the proof of Proposition 3.2. □

C Proof of Lemma 4.2

We are going to prove that $e^{4s\beta^*}(\gamma^*)^{-14-14/m}\hat{z} \in L^2(Q)^N$ using a method introduced in [10], which consists of increasing the regularity of the function $\hat{z}$ through certain weight functions involved in order to then be able to apply a local inverse theorem in a sufficiently regular space (more details, see Section 5).

Proof. Next, we define the following functions:

$$(z_{*,0}, p_{*,0}) := e^{4s\beta^*}(\gamma^*)^{-5-5/m}(\hat{z}, \hat{p}_1),\ \ f^z_{*,0} := e^{4s\beta^*}(\gamma^*)^{-5-5/m}(f^z + \nabla\times((\nabla\times\hat{w})\chi)).$$

Notice that $f^z_{*,0} \in L^2(0,T;H^{-1}(\Omega)^N)$, since this function can be written as:

$$f^z_{*,0} = e^{-3s\beta^*}(\gamma^*)^{-5-5/m}(e^{7s\beta^*}f^z) + (\gamma^*)^{-4-4/m}\nabla\times((\nabla\times w_*)\chi)\,,$$

where $e^{-3s\beta^*}(\gamma^*)^{-5-5/m}$ and $(\gamma^*)^{-4-4/m}$ are bounded; also, using (4.4); in dimension 2, we can consider $\chi = \mathbb{1}_{\mathcal{O}}$, then $(\nabla\times w_*)\mathbb{1}_{\mathcal{O}} \in L^2(Q)^2$, and, on the other hand, in dimension 3, we have that $(\nabla\times w_*)\chi$ belongs to $L^2(0,T;H^1(\Omega)^3)$, and as $H^1(\Omega)^3 \subset L^2(\Omega)^3$, we obtain $(\nabla\times w_*)\chi \in L^2(Q)^3$.

Then, by (4.9) $z_{*,0}$ satisfies

$$\begin{cases} \mathcal{L}^* z_{*,0} + \nabla p_{*,0} = f^z_{*,0} - (e^{4s\beta^*}(\gamma^*)^{-5-5/m})_t\hat{z}, \quad \nabla\cdot z_{*,0} = 0 & \text{in } Q, \\ z_{*,0} = 0 & \text{on } \Sigma, \\ z_{*,0}(T) = 0 & \text{in } \Omega, \end{cases}$$

where the last term in the right-hand side can be written as

$$(e^{4s\beta^*}(\gamma^*)^{-5-5/m})_t\hat{z} = c_4(t)(\mathcal{L}^*_H)^4\tilde{z},$$

where $c_k(t)$ denotes a generic function such that (see (2.2))

$$|c_k^{(\ell)}(t)| \leq C < \infty,\ \forall \ell = 0,\dots,k. \tag{C.1}$$

On the other hand, for any $h \in Y_{3,0}$, we obtain

$$\iint\limits_Q z_{*,0}\cdot h \mathrm{d}x\,\mathrm{d}t = \iint\limits_Q \left\langle f^z_{*,0}, \Phi\right\rangle_{L^2(0,T;H^{-1}(\Omega)^N),L^2(0,T;H^1_0(\Omega)^N)} \mathrm{d}x\,\mathrm{d}t - \iint\limits_Q c_4(t)(\mathcal{L}^*)^4_H\tilde{z}\cdot\Phi \mathrm{d}x\,\mathrm{d}t, \tag{C.2}$$

where Φ solves, together some pressure π_Φ,

$$\begin{cases} \mathcal{L}\Phi + \nabla\pi_\Phi = h, \quad \nabla\cdot\Phi = 0 & \text{in } Q, \\ \Phi = 0 & \text{on } \Sigma, \\ \Phi(0) = 0 & \text{in } \Omega. \end{cases}$$

Using (4.12), we can integrate by parts to obtain

$$\iint\limits_Q z_{*,0}\cdot h \mathrm{d}x\,\mathrm{d}t = \iint\limits_Q \left\langle f^z_{*,0}, \Phi\right\rangle_{L^2(0,T;H^{-1}(\Omega)^N), L^2(0,T;H^1_0(\Omega)^N)} \mathrm{d}x\,\mathrm{d}t - \iint\limits_Q (\mathcal{L}^*_H)^3\tilde{z}\cdot(\mathcal{L}[c_4(t)\Phi] + \nabla(c_4(t)h))\mathrm{d}x\,\mathrm{d}t,$$

$$= \iint\limits_Q \left\langle f^z_{*,0}, \Phi\right\rangle_{L^2(0,T;H^{-1}(\Omega)^N), L^2(0,T;H^1_0(\Omega)^N)} \mathrm{d}x\,\mathrm{d}t$$

$$- \iint\limits_Q \tilde{z}\cdot\Big(c_4^{(4)}(t)\Phi + \mathcal{L}[c_4^{(3)}(t)\Phi] + \mathcal{L}^2[c_4^{(2)}(t)\Phi] + \mathcal{L}^3[c_4^{(1)}(t)\Phi] + \mathcal{L}^3[c_4(t)h]\Big)\mathrm{d}x\,\mathrm{d}t.$$

Notice that here we have relied on the fact that $\mathcal{L}^*_H\tilde{z}$, Φ and h belong to the space H. Since

$$\|\Phi\|_{Y_4} \le C\|h\|_{Y_{3,0}},$$

(see regularity result (2.7)), we obtain from the last equality, together with (C.1),

$$\iint\limits_Q z_{*,0}\cdot h\mathrm{d}x\,\mathrm{d}t \le C\Big[\|f^z_{*,0}\|_{L^2(0,T;H^{-1}(\Omega)^N)} + \|\tilde{z}\|_{L^2(Q)^N}\Big]\|h\|_{Y_{3,0}}, \quad \forall h\in Y_{3,0}. \tag{C.3}$$

Now, let

$$(z_{*,1}, p_{*,1}) := e^{4s\beta^*}(\gamma^*)^{-9-9/m}(\hat{z},\hat{p}_1),\ f^z_{*,1} := e^{4s\beta^*}(\gamma^*)^{-9-9/m}(f^z + \nabla\times((\nabla\times\hat{w})\chi)).$$

Similarly as before, $(z_{*,1}, p_{*,1})$ satisfies

$$\begin{cases} \mathcal{L}^* z_{*,1} + \nabla p_{*,1} = f^z_{*,1} - (e^{4s\beta^*}(\gamma^*)^{-9-9/m})_t\hat{z}, \quad \nabla\cdot z_{*,1} = 0 & \text{in } Q,\\ z_{*,1} = 0 & \text{on } \Sigma,\\ z_{*,1}(T) = 0 & \text{in } \Omega, \end{cases}$$

Thus, for any $h\in Y_{2,0}$, we get

$$\iint\limits_Q z_{*,1}\cdot h\mathrm{d}x\,\mathrm{d}t = \iint\limits_Q \left\langle f^z_{*,1}, \Phi\right\rangle_{L^2(0,T;H^{-1}(\Omega)^N), L^2(0,T;H^1_0(\Omega)^N)} \mathrm{d}x\,\mathrm{d}t - \iint\limits_Q (e^{4s\beta^*}(\gamma^*)^{-9-9/m})_t\hat{z}\cdot\Phi\mathrm{d}x\,\mathrm{d}t.$$

Moreover, since

$$\iint\limits_Q (e^{4s\beta^*}(\gamma^*)^{-9-9/m})_t\hat{z}\cdot\Phi\mathrm{d}x\,\mathrm{d}t = \iint\limits_Q c_3(t)\Phi\cdot z_{*,0}\mathrm{d}x\,\mathrm{d}t.$$

using (C.3) with $c_3(t)\Phi$ instead of h (notice that $c_3(t)\Phi\in Y_{3,0}$), we get the estimate

$$\iint\limits_Q c_3(t)\Phi\cdot z_{*,0}\mathrm{d}x\,\mathrm{d}t \le C\Big[\|f^z_{*,0}\|_{L^2(0,T;H^{-1}(\Omega)^N)} + \|\tilde{z}\|_{L^2(Q)^N}\Big]\|c_3(t)\Phi\|_{Y_{3,0}}.$$

Going back to $z_{*,1}$, we have

$$\iint\limits_Q z_{*,1}\cdot h\mathrm{d}x\,\mathrm{d}t \le C\Big[\,\|f^z_{*,0}\|_{L^2(0,T;H^{-1}(\Omega)^N)} + \|\tilde{z}\|_{L^2(Q)^N}\Big]\|\Phi\|_{Y_{3,0}},$$

where we have used (C.1) and the property $(\gamma^*)^{-9-9/m}\le C(\gamma^*)^{-5-5/m}$. Taking into account that

$$\|\Phi\|_{Y_3}\le C\|h\|_{Y_{2,0}},$$

(see (2.6)) we obtain

$$\iint\limits_Q z_{*,1}\cdot h\mathrm{d}x\,\mathrm{d}t \le C\Big[\|f^z_{*,0}\|_{L^2(0,T;H^{-1}(\Omega)^N)} + \|\tilde{z}\|_{L^2(Q)^N}\Big]\|h\|_{Y_{2,0}}, \quad \forall h\in Y_{2,0}. \tag{C.4}$$

Now, let

$$(z_{*,2}, p_{*,2}) := e^{4s\beta^*}(\gamma^*)^{-12-12/m}(\hat{z}, \hat{p}_1), \;\; f^z_{*,2} := e^{4s\beta^*}(\gamma^*)^{-12-12/m}(f^z + \nabla \times ((\nabla \times \hat{w})\chi)).$$

Analogously, as before, $(z_{*,2}, p_{*,2})$ satisfies

$$\begin{cases} \mathcal{L}^* z_{*,2} + \nabla p_{*,2} = f^z_{*,2} - (e^{4s\beta^*}(\gamma^*)^{-12-12/m})_t \hat{z}, \quad \nabla \cdot z_{*,2} = 0 & \text{in } Q, \\ z_{*,2} = 0 & \text{on } \Sigma, \\ z_{*,2}(T) = 0 & \text{in } \Omega, \end{cases}$$

Thus, for any $h \in Y_{1,0}$, we obtain

$$\iint\limits_Q z_{*,2} \cdot h \mathrm{d}x \, \mathrm{d}t = \iint\limits_Q \left\langle f^z_{*,2}, \Phi \right\rangle_{L^2(0,T;H^{-1}(\Omega)^N), L^2(0,T;H^1_0(\Omega)^N)} \mathrm{d}x \, \mathrm{d}t - \iint\limits_Q (e^{4s\beta^*}(\gamma^*)^{-12-12/m})_t \hat{z} \cdot \Phi \mathrm{d}x \, \mathrm{d}t.$$

Moreover, since

$$\iint\limits_Q (e^{4s\beta^*}(\gamma^*)^{-12-12/m})_t \hat{z} \cdot \Phi \mathrm{d}x \, \mathrm{d}t = \iint\limits_Q c_2(t)\Phi \cdot z_{*,0} \mathrm{d}x \, \mathrm{d}t.$$

using (C.4) with $c_2(t)\Phi$ instead of h (notice that $c_2(t)\Phi \in Y_{2,0}$), we get the estimate

$$\iint\limits_Q c_2(t)\Phi \cdot z_{*,0} \mathrm{d}x \, \mathrm{d}t \leq C \Big[\|f^z_{*,0}\|_{L^2(0,T;H^{-1}(\Omega)^N)} + \|\tilde{z}\|_{L^2(Q)^N} \Big] \|c_2(t)\Phi\|_{Y_{2,0}}.$$

Turning back to $z_{*,2}$, we get

$$\iint\limits_Q z_{*,2} \cdot h \mathrm{d}x \, \mathrm{d}t \leq C \Big[\|f^z_{*,0}\|_{L^2(0,T;H^{-1}(\Omega)^N)} + \|\tilde{z}\|_{L^2(Q)^N} \Big] \|\Phi\|_{Y_{2,0}},$$

where we have used (C.1) and the property $(\gamma^*)^{-12-12/m} \leq C(\gamma^*)^{-9-9/m}$. Taking into account that

$$\|\Phi\|_{Y_2} \leq C\|h\|_{Y_{1,0}},$$

(see (2.6)) we obtain

$$\iint\limits_Q z_{*,2} \cdot h \mathrm{d}x \, \mathrm{d}t \leq C \Big[\|f^z_{*,0}\|_{L^2(0,T;H^{-1}(\Omega)^N)} + \|\tilde{z}\|_{L^2(Q)^N} \Big] \|h\|_{Y_{1,0}}, \quad \forall h \in Y_{1,0}. \tag{C.5}$$

Finally, we set

$$(z_{*,3}, p_{*,3}) := e^{4s\beta^*}(\gamma^*)^{-14-14/m}(\hat{z}, \hat{p}_1), \;\; f^z_{*,3} := e^{4s\beta^*}(\gamma^*)^{-14-14/m}(f^z + \nabla \times ((\nabla \times \hat{w})\chi)).$$

Similarly as before, $(z_{*,3}, p_{*,3})$ satisfies

$$\begin{cases} \mathcal{L}^* z_{*,3} + \nabla p_{*,3} = f^z_{*,3} - (e^{4s\beta^*}(\gamma^*)^{-14-14/m})_t \hat{z}, \quad \nabla \cdot z_{*,3} = 0 & \text{in } Q, \\ z_{*,3} = 0 & \text{on } \Sigma, \\ z_{*,3}(T) = 0 & \text{in } \Omega. \end{cases}$$

Thus, for any $h \in Y_0$, we obtain

$$\iint\limits_Q z_{*,3} \cdot h \mathrm{d}x \, \mathrm{d}t = \iint\limits_Q \left\langle f^z_{*,3}, \Phi \right\rangle_{L^2(0,T;H^{-1}(\Omega)^N), L^2(0,T;H^1_0(\Omega)^N)} \mathrm{d}x \, \mathrm{d}t - \iint\limits_Q (e^{4s\beta^*}(\gamma^*)^{-14-14/m})_t \hat{z} \cdot \Phi \mathrm{d}x \, \mathrm{d}t.$$

Moreover, since

$$\iint\limits_Q (e^{4s\beta^*}(\gamma^*)^{-14-14/m})_t \hat{z} \cdot \Phi \mathrm{d}x \, \mathrm{d}t = \iint\limits_Q c_1(t)\Phi \cdot z_{*,0} \mathrm{d}x \, \mathrm{d}t.$$

using (C.5) with $c_1(t)\Phi$ instead of h (notice that $c_1(t)\Phi \in Y_{1,0}$), we get the estimate

$$\iint\limits_Q c_1(t)\Phi \cdot z_{*,0} \mathrm{d}x\, \mathrm{d}t \leq C\Big[\|f^z_{*,0}\|_{L^2(0,T;H^{-1}(\Omega)^N)} + \|\tilde{z}\|_{L^2(Q)^N}\Big]\|c_1(t)\Phi\|_{Y_{1,0}}.$$

Turning back to $z_{*,3}$, we obtain

$$\iint\limits_Q z_{*,3} \cdot h \mathrm{d}x\, \mathrm{d}t \leq C\Big[\|f^z_{*,0}\|_{L^2(0,T;H^{-1}(\Omega)^N)} + \|\tilde{z}\|_{L^2(Q)^N}\Big]\|\Phi\|_{Y_{1,0}},$$

where we have used (C.1) and the property $(\gamma^*)^{-14-14/m} \leq C(\gamma^*)^{-12-12/m}$. Taking into account that

$$\|\Phi\|_{Y_{1,0}} \leq C\|h\|_{Y_0},$$

(see (2.6)) we obtain

$$\iint\limits_Q z_{*,3} \cdot h \mathrm{d}x\, \mathrm{d}t \leq C\Big[\|f^z_{*,0}\|_{L^2(0,T;H^{-1}(\Omega)^N)} + \|\tilde{z}\|_{L^2(Q)^N}\Big]\|h\|_{Y_0}, \quad \forall h \in Y_0.$$

Thus, we deduce that $z_{*,3} \in L^2(Q)^N$. The proof of Lemma 4.2 is complete. □

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Statements and Declarations

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