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Boundary controllability of a cascade system coupling fourth- and second-order parabolic equations☆

Nicolás Carreño *, Eduardo Cerpa, Alberto Mercado

Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile

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ABSTRACT

A control system coupling fourth- and second-order parabolic equations is considered in this paper. The main topic is the study of the control properties of this system when we only control the second-order partial differential equation through a boundary condition. Depending on the choice of the diffusion coefficients, we obtain positive and negative results for approximate- and null-controllability. In particular, we prove that for any given positive time T_0, we can find some diffusion coefficients such that the system is null-controllable in time T if $T > T_0$ and is not null-controllable if $T < T_0$.

1. Introduction

Parabolic partial differential equations are used to model several phenomena as population dynamics, chemical processes, phase transitions in fluids, temperature evolution, among others. Once we have an appropriate model to describe a phenomenon, we can wonder if we can influence the evolution of the system. For instance, we can ask if we can drive to a desired target the temperature profile of a rod by means of manipulating some heat source acting at the end of the rod. This and other similar questions concerning a given mathematical model are the kind of questions addressed by control theory. In this work we deal with related issues concerning some parabolic systems. The specific property of steering a system from some initial to a final state in finite time is called exact-controllability. If the final state is the rest, we call it null-controllability. Depending on the localization of the source that we can manipulate, the control property is called boundary- or distributed-controllability if the control acts on the boundary or the interior of the domain, respectively.

Concerning one-dimensional parabolic equations, the first boundary null-controllability result was proved for the heat equation by Fattorini and Russell in [1] using the moment method. Later on, Lebeau and Robbiano in [2] (using local Carleman estimates) and Fursikov and Imanuvilov in [3] (using global Carleman estimates) obtained the distributed null-controllability of the heat equation in higher dimensions. Fourth-order parabolic equations (bilaplacian type) have also been studied recently. Some null-controllability results for the one-dimensional case can be found in [4] (boundary control, moment method), [5] (boundary control, global Carleman estimates) and [6] (distributed control, global Carleman estimates). In higher dimensions, we find [7] (distributed control, global Carleman estimates) and [8] (distributed control, local Carleman estimates).

An interesting extension is the study of control properties for systems of coupled partial differential equations, specially if the number of control inputs is less than the number of equations. In this case, we have to control a part of the system indirectly using the coupling with other equations which are directly under the influence of the control. Regarding control of parabolic systems, many works have been devoted to study different type of couplings of second-order equations (see the survey [9] and the references therein). Some Kalman type conditions naturally appear to characterize good couplings (see [10] and [11]) and a very useful tool to prove distributed controllability of parabolic systems with less controls than equations is the Carleman estimates approach [12]. Unfortunately, it is very hard to use it in the boundary control case. In the framework of second-order operators, other recent results we can cite are [13] (considering non-local terms in the equations), [14,15] (algebraic methods to eliminate controls), [16] (approximate-controllability), and [17] (control and coupling region do not intersect).

Regarding systems involving fourth-order parabolic equations, we find controllability results for the stabilized Kuramoto-Sivashinsky system, which is a non-linear system coupling a bilaplacian equation with a heat equation. Using a Carleman estimates approach, the boundary [18] and distributed null-controllability [19,20] has been obtained. In [19] the case of one distributed control acting on the fourth-order equation is

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* Corresponding author.
E-mail addresses: nicolas.carrenog@usm.cl (N. Carreño), eduardo.cerpa@usm.cl (E. Cerpa), alberto.mercado@usm.cl (A. Mercado).

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considered while in [20] the control acts on the second-order equation.

In this paper, we study the boundary controllability properties of a cascade system coupling a bilaplacian operator to a heat equation, which reads as

$$\begin{cases} u_t(t,x) + u_{xxxx}(t,x) = v(t,x), & t \in (0,T),\ x \in (0,\pi), \\ v_t(t,x) - d v_{xx}(t,x) = 0, & t \in (0,T),\ x \in (0,\pi), \\ u(t,0) = u_{xx}(t,0) = 0, & t \in (0,T), \\ u(t,\pi) = u_{xx}(t,\pi) = 0, & t \in (0,T), \\ v(t,0) = h(t),\ v(t,\pi) = 0, & t \in (0,T), \end{cases} \tag{1.1}$$

where the state is given by (u, v) and the control is h, which only acts on the heat equation. The parameter $d > 0$ is the diffusion of the heat equation and it will play a crucial role in order to define which kind of control properties we are able to obtain.

To study the null-controllability for (1.1), we will prove that this system enters in the general framework introduced in [21], where the authors consider the existence of minimal time of control for parabolic systems. See also [10] and [22] for some results for second-order operators.

Now we state the precise definitions of the properties we are interested in.

Definition 1.1. Let $T > 0$. System (1.1) is said to be approximate-controllable in time T if for any $\varepsilon > 0$ and for any states $(u_0, v_0), (u_T, v_T) \in L^2(0,\pi) \times H^{-1}(0,\pi)$, there exists a control $h \in L^2(0,T)$ such that the solution of (1.1) with initial condition

$$u(0,\cdot) = u_0 \quad \text{and} \quad v(0,\cdot) = v_0$$

satisfies

$$\|u(T,\cdot) - u_T\|_{L^2(0,\pi)} + \|v(T,\cdot) - v_T\|_{H^{-1}(0,\pi)} \le \varepsilon.$$

Definition 1.2. Let $T > 0$. System (1.1) is said to be null-controllable in time T if for any state $(u_0, v_0) \in L^2(0,\pi) \times H^{-1}(0,\pi)$, there exists a control $h \in L^2(0,T)$ such that the solution of (1.1) with initial condition

$$u(0,\cdot) = u_0 \quad \text{and} \quad v(0,\cdot) = v_0$$

satisfies

$$u(T,\cdot) = 0 \quad \text{and} \quad v(T,\cdot) = 0.$$

The main result we obtain concerning approximate-controllability is the following one.

Theorem 1.3. *System (1.1) is approximate-controllable if and only if $\sqrt{d}$ is irrational.*

The proof to this theorem is given in Section 2. It is based on a duality approach leading us to study a unique continuation property for the adjoint system.

Regarding null-controllability, the results that we obtain also depend on the coefficient d, and there are cases when the system is null-controllable for all time $T > 0$ or only when T is larger than a given $T_0 > 0$. It turns out that the key property of the coefficient d is how closely $\sqrt{d}$ can be approximated by rational numbers. In the literature we find the following measure of this property.

Definition 1.4. The Liouville–Roth constant of a real number x is the least upper bound of the set of real positive numbers μ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^{\mu}}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

Using this definition we can state our main results on null-controllability.

Theorem 1.5. *If $\sqrt{d}$ is an irrational number with finite Liouville–Roth constant, then system (1.1) is null-controllable in time T for any $T > 0$.*

Theorem 1.6. *For any $T_0 > 0$, there exists $d > 0$ such that system (1.1) is null-controllable in time T if $T > T_0$ and is not null-controllable if $T < T_0$.*

Theorem 1.7. *There exists $d > 0$ such that system (1.1) is not null-controllable in time T for any time of control $T > 0$.*

These theorems will be proved in Section 3. To do that we write our control system (1.1) in the general form used in [21], which reduces the proof to the computation of what is called the *condensation index* of the sequence of eigenvalues of the system (see Remark 3.2). The authors of [21] use the moment method and a duality approach.

Remark 1.8. For the sake of simplicity in the corresponding eigenvalue–eigenfunctions formulae, we study our system on the interval $(0,\pi)$. However, by simply rescaling, our results still hold if the system is posed on any interval $(0, L)$ with $L > 0$.

Remark 1.9. The Liouville–Roth constant is also called *irrationality exponent* or *irrationality measure* (see Appendix E of [23]). It gives a degree of the accuracy of the approximation by rationals of the given real number. It is known that it is 1 for rational numbers, it is no less than 2 for irrational numbers and it is exactly 2 for irrational algebraic numbers. The real numbers having infinite Liouville–Roth constant are called Liouville numbers. It is known that the set of Liouville numbers has null Lebesgue measure. The role of this value in the controllability properties is given in Proposition 3.5 (see also Remark 3.6).

Remark 1.10. The moment method is very useful to study boundary controllability of systems with less controls than equations. Very precise results can be obtained concerning the existence of minimal time of control (see [24]). The drawback is that precise expressions for spectrum data are needed. Thus, this method is hard to be applied when non-constant coefficients appear in the partial differential equations forming the system (see [17]).

2. Approximate-controllability

Let us start by saying a few words on the well-posedness of system (1.1). We notice that, given the cascade structure of the system, we can first solve the heat equation. This is possible in the framework $h \in L^2(0,T)$ and initial condition $v(0,\cdot) = v_0$ with $v_0 \in H^{-1}(0,\pi)$ (see [21]). Once we have the solution $v \in C(0,T; H^{-1}(0,\pi)) \cap L^2(0,T; L^2(0,\pi))$ for the heat equation, we plug it into the right-hand side of the fourth-order equation. Using [25, Proposition 2.1] we obtain that the solution of the fourth-order equation with initial condition $u(0,\cdot) = u_0 \in L^2(0,\pi)$ satisfies $u \in C(0,T; L^2(0,\pi)) \cap L^2(0,T; H^2(0,\pi) \cap H_0^1(0,\pi))$.

With this regularity framework we can start the proof of Theorem 1.3.

Proof of Theorem 1.3. Let us introduce the operator

$$\Lambda : h \in L^2(0,T) \longmapsto (u(T,\cdot), v(T,\cdot)) \in L^2(0,\pi) \times H^{-1}(0,\pi),$$

where (u, v) is the solution to (1.1) with control h and initial data $u(0,\cdot) = 0$ and $v(0,\cdot) = 0$.

Thus, the approximate-controllability of (1.1) is equivalent to the property that the range of the operator Λ is dense in $L^2(0,\pi)\times H^{-1}(0,\pi)$. It is already a classical duality approach to see that the latter is equivalent to the inyectivity of the operator Λ^*. After simple computations, we find

$$\Lambda^*:(\varphi_T,\psi_T)\in L^2(0,\pi)\times H_0^1(0,\pi)\longmapsto \psi_x(\cdot,0)\in L^2(0,T),$$

where (φ,ψ) is the solution of

$$\begin{cases}-\varphi_t(t,x)+\varphi_{xxxx}(t,x)=0, & t\in(0,T),\ x\in(0,\pi),\\ -\psi_t(t,x)-d\psi_{xx}(t,x)=\varphi(t,x), & t\in(0,T),\ x\in(0,\pi),\\ \varphi(t,0)=\varphi_{xx}(t,0)=0, & t\in(0,T),\\ \varphi(t,\pi)=\varphi_{xx}(t,\pi)=0, & t\in(0,T),\\ \psi(t,0)=0,\ \psi(t,\pi)=0, & t\in(0,T),\end{cases} \tag{2.1}$$

with initial data $\varphi(T,\cdot)=\varphi_T$ and $\psi(T,\cdot)=\psi_T$.

In order to study the injectivity of Λ^*, we write

$$\varphi_T=\sum_{k\in\mathbb{N}} b_k\varphi_k(x),\qquad \text{and}\qquad \psi_T=\sum_{k\in\mathbb{N}} a_k\varphi_k(x),$$

where for any $k\in\mathbb{N}$ we denote by φ_k the normalized eigenfunctions associated to the Laplacian operator with homogeneous Dirichlet boundary conditions:

$$\varphi_k(x)=\sqrt{\frac{2}{\pi}}\sin(kx). \tag{2.2}$$

Then, the family of functions φ_k are eigenfunctions of each one of the two differential operators appearing on system (2.1) (and (1.1)), with eigenvalues given by k^4 and dk^2, respectively. Hence, the solution of system (2.1) can be explicitly given. We have

$$\varphi(t,x)=\sum_{k\in\mathbb{N}} b_k e^{-k^4(T-t)}\varphi_k(x), \tag{2.3}$$

and the expression of ψ depends on the possible fact that $k^4=dk^2$ for some k. More precisely, if $k^4-dk^2\neq 0$ for any integer k, we have

$$\begin{aligned}\psi(t,x)=&\sum_{k\in\mathbb{N}} a_k e^{-dk^2(T-t)}\varphi_k(x)\\ &+\left(e^{-dk^2(T-t)}-e^{-k^4(T-t)}\right)\frac{b_k}{k^4-dk^2}\varphi_k(x),\end{aligned} \tag{2.4}$$

and, if $\sqrt{d}=m\in\mathbb{N}$, then

$$\begin{aligned}\psi(t,x)=&\sum_{k\neq m}\Bigg[a_k e^{-dk^2(T-t)}\varphi_k(x)\\ &+\left(e^{-dk^2(T-t)}-e^{-k^4(T-t)}\right)\frac{b_k}{k^4-dk^2}\varphi_k(x)\Bigg]\\ &+b_m(T-t)e^{-m^4(T-t)}\varphi_m(x)+a_m e^{-dm^2(T-t)}\varphi_m(x).\end{aligned}$$

In order to analyze the injectivity of Λ^*, we deal with the cases $\sqrt{d}\notin\mathbb{Q}$ and $\sqrt{d}\in\mathbb{Q}$.

• Case 1: $\sqrt{d}\notin\mathbb{Q}$.

In this case, ψ is given by (2.4), and then

$$\begin{aligned}\psi_x(t,0)=&\sum_{k\in\mathbb{N}}\Bigg[k a_k e^{-dk^2(T-t)}\\ &+\frac{b_k}{k^3-dk}\left(e^{-dk^2(T-t)}-e^{-k^4(T-t)}\right)\Bigg].\end{aligned} \tag{2.5}$$

If $\psi_x(\cdot,0)=0$, from the fact that the family $\{e^{-dk^2(T-t)},e^{-m^4(T-t)}: k,m\in\mathbb{N}\}$ is minimal in $L^2(0,T)$, we get that

$$k a_k+\frac{b_k}{k^3-dk}=0\ \text{ and }\ \frac{b_k}{k^3-dk}=0,$$

for all $k\in\mathbb{N}$. Thus, $a_k=b_k=0$ for all $k\in\mathbb{N}$, and we get the injectivity of Λ^*.

• Case 2: $\sqrt{d}\in\mathbb{Q}$.

Notice that we can write $\sqrt{d}=m^2/k$, with $m,k\in\mathbb{N}$, $m\neq k$ and $k\neq 1$. Let us consider the initial conditions

$$\varphi_T=(m^3-dm)\varphi_m$$

and

$$\psi_T=-\frac{1}{m}\varphi_m+\frac{1}{k}\varphi_k.$$

Then the solution of system (2.1) is given by

$$\varphi(t,x)=(m^3-dm)e^{-m^4(T-t)}\varphi_m(x)$$

and

$$\begin{aligned}\psi(t,x)=&\Bigg[-\frac{1}{m}e^{-dm^2(T-t)}+\left(e^{-dm^2(T-t)}-e^{-m^4(T-t)}\right)\\ &\times\frac{m^3-dm}{m^4-dm^2}\Bigg]\varphi_m(x)+\frac{1}{k}e^{-dk^2(T-t)}\varphi_k(x).\end{aligned}$$

It is not difficult to see that this is a non-trivial solution such that $\psi_x(\cdot,0)=0$. We conclude that the operator Λ^* is not injective in this case.

Recalling that the injectivity of Λ^* is equivalent to the approximate-controllability of system (1.1), we conclude the proof of Theorem 1.3. ■

3. Null-controllability

3.1. Abstract control system

We will state some results for abstract control systems obtained in [21]. Given an unbounded operator $\mathcal{A}$ in the complex Hilbert space $\mathbb{X}$, we consider the system

$$\begin{cases} y' &= \mathcal{A}y+\mathcal{B}h,\\ y(0) &= y_0,\end{cases} \tag{3.1}$$

with $y_0\in\mathbb{X}$ and $h\in L^2(0,T)$, where $\mathcal{B}$ is the control operator. We introduce some assumptions and notation in order to state that result.

We will assume that $\mathbb{X}$ has a Riesz basis given by the eigenfunctions of operator $\mathcal{A}$ denoted by $\{\phi_k\}_{k\in\mathbb{N}}$ with eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$. We denote by $\{\psi_k\}_{k\in\mathbb{N}}$ the corresponding sequence of biorthogonal functions to $\{\phi_k\}_{k\in\mathbb{N}}$. Then $\mathcal{A}$ can be characterized by

$$\mathcal{A}=-\sum_{k=1}^{\infty}\lambda_k(\cdot,\psi_k)\phi_k. \tag{3.2}$$

Furthermore, we denote by $\mathbb{X}_{-1}$ the completion of $\mathbb{X}$ with respect to the norm

$$\|y\|_{-1}:=\left(\sum_{k=1}^{\infty}\frac{|(y,\psi_k)|^2}{|\lambda_k|^2}\right)^{1/2},\quad \text{for each } y\in\mathbb{X}_{-1}.$$

Also, we say that $\mathcal{B}\in\mathcal{L}(\mathbb{C},\mathbb{X}_{-1})$ is an admissible control operator for the semigroup $\{e^{t\mathcal{A}}\}_{t>0}$ generated by $\mathcal{A}$ if $L_T\in\mathcal{L}(L^2(0,T;\mathbb{C}),\mathbb{X})$, where we have defined

$$L_Tu=\int_0^T e^{(T-s)\mathcal{A}}\mathcal{B}u(s)ds,\quad u\in L^2(0,T;\mathbb{C}).$$

In [21] a characterization of the controllability of system (3.1) is established in terms of the sequence $\{\lambda_k\}$ and the control operator $\mathcal{B}$. The result is stated in terms of a holomorphic complex function $E(z)$ which vanishes at $z=\lambda_k$, for any $k\in\mathbb{N}$. One way to define such a function is given by the infinite product

$$E(z)=\prod_{k\in\mathbb{N}}\left(1-\frac{z^2}{\lambda_k^2}\right),\quad z\in\mathbb{C}. \tag{3.3}$$

The function $E(z)$ converges uniformly and absolutely in compact subsets of $\mathbb{C}$ under the hypothesis of the theorem below, where the controllability properties of system (3.1) are stated.

Theorem 3.1 (*Theorem 2.5 in [21]*). *Assume that $\mathcal{B} \in \mathcal{L}(\mathbb{C}, \mathbb{X}_{-1})$ is an admissible control operator for the semigroup $\{e^{t\mathcal{A}}\}_{t>0}$ generated by $\mathcal{A}$, and $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ is a complex sequence satisfying*

$$\lambda_j \neq \lambda_k, \forall j \neq k; \quad \Re(\lambda_k) \geq \delta|\lambda_k| > 0, \text{ for some } \delta > 0; \text{ and}$$
$$\sum_{k=1}^{\infty} \frac{1}{|\lambda_k|} < \infty. \tag{3.4}$$

Let us suppose in addition that

$$b_k := \mathcal{B}^* \psi_k \neq 0 \quad \forall k \in \mathbb{N}. \tag{3.5}$$

We introduce

$$T_0 := \limsup_{k\to\infty} \left(\frac{\ln \frac{1}{|b_k|}}{\Re(\lambda_k)} + \frac{\ln \frac{1}{|E'(\lambda_k)|}}{\Re(\lambda_k)} \right), \tag{3.6}$$

where E is defined in (3.3). Then:

1. *System (3.1) is null-controllable if $T > T_0$.*
2. *System (3.1) is not null-controllable if $T < T_0$.*

Remark 3.2. The condensation index $c(\Lambda)$ of a sequence $\Lambda = \{\lambda_k\}_{k\in\mathbb{N}}$ is defined by

$$c(\Lambda) := \limsup_{k\to\infty} \frac{\ln \frac{1}{|E'(\lambda_k)|}}{\Re(\lambda_k)}. \tag{3.7}$$

Under the hypothesis of Theorem 3.1, we have $c(\Lambda) \in [0, \infty]$ (see Theorem 4.8 in [21]). Also, if

$$\lim_{k\to\infty} \frac{\ln \frac{1}{|b_k|}}{\Re(\lambda_k)} = 0,$$

it can be easily proved that $T_0 = c(\Lambda)$.

3.2. Null-controllability of system (1.1)

We write the control system (1.1) as a first-order control system of the form (3.1). Let us consider the Dirichlet Laplacian

$$A := -\partial_x^2 : D(A) := H^2(0,\pi) \cap H_0^1(0,\pi) \subset L^2(0,\pi) \longrightarrow L^2(0,\pi),$$

which is a self-adjoint operator. We still denote by A its self-adjoint extension to the spaces $H^{-1}(0,\pi)$ with domain $H_0^1(0,\pi)$, and $(H^2(0,\pi)\cap H_0^1(0,\pi))'$ with domain $L^2(0,\pi)$, respectively.

We define $X = L^2(0,\pi) \times H^{-1}(0,\pi)$, and $\mathcal{A} : D(\mathcal{A}) \subset X \longrightarrow X$ by

$$D(\mathcal{A}) = \{u \in D(A) \; : \; \partial_x^2 u \in D(A)\} \times H_0^1(0,\pi),$$

and

$$\mathcal{A} = \begin{pmatrix} -A^2 & 1 \\ 0 & -dA \end{pmatrix}. \tag{3.8}$$

The operator $\mathcal{B} \in \mathcal{L}(\mathbb{R}, ((H^2(0,\pi)\cap H_0^1(0,\pi))^2)')$ is defined by

$$(\mathcal{B}v)\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = vd\phi_2'(0), \quad \forall v \in \mathbb{R}, \quad \forall \phi_1, \phi_2 \in H^2(0,\pi)\cap H_0^1(0,\pi).$$

Hence, system (1.1) can be equivalently reformulated as system (3.1) with

$$y = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Let us check that our system satisfies hypothesis of Theorem 3.1. We have from the well-posedness of system (1.1) that $\mathcal{B}$ is an admissible control operator for $\mathcal{A}$. On the other hand, the family of eigenfunctions of A in $L^2(0,\pi)$ is given by $\{\varphi_k\}_{k\in\mathbb{N}}$, defined in (2.2). It is well known that $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for $L^2(0,\pi)$, and $\{k\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for $H^{-1}(0,\pi)$.

We can explicitly compute the eigenfunctions of the operator $\mathcal{A}$. Indeed, if we set

$$\Phi_{1,k} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \varphi_k, \quad \Phi_{2,k} = k \begin{pmatrix} \frac{1}{k^4 - dk^2} \\ 1 \end{pmatrix} \varphi_k, \tag{3.9}$$

then, it can be shown that $\left\{\Phi_{j,k} \,/\, j = 1,2 \text{ and } k \in \mathbb{N}\right\}$ is a Riesz basis of X, and its biorthogonal basis is given by

$$\Psi_{1,k} = \begin{pmatrix} 1 \\ \frac{1}{dk^2 - k^4} \end{pmatrix} \varphi_k, \quad \Psi_{2,k} = k \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k. \tag{3.10}$$

From the definition of $\mathcal{B}$ we get that $\mathcal{B}^* \in ((H^2(0,\pi)\cap H_0^1(0,\pi))^2)'$ is given by

$$\mathcal{B}^* \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = -d\phi_2'(0),$$

and hence we have that

$$\mathcal{B}^*\Psi_{1,k} = -\frac{2}{\pi} \frac{dk}{dk^2 - k^4}$$

and

$$\mathcal{B}^*\Psi_{2,k} = -\frac{2}{\pi} dk^2.$$

We directly get that

$$\lim_{k\to\infty} \frac{\ln |\frac{1}{\mathcal{B}^*\Psi_{1,k}}|}{k^4} = \lim_{k\to\infty} \frac{\ln |\frac{1}{\mathcal{B}^*\Psi_{2,k}}|}{dk^2} = 0.$$

Hence, from Theorem 3.1 and Remark 3.2, we have the following result.

Proposition 3.3. *If $c(\Lambda_d)$ is the condensation index of $\Lambda_d = \{dk^2, k^4\}_{k\in\mathbb{N}}$, then we have:*

1. *System (1.1) is null-controllable if $T > c(\Lambda_d)$.*
2. *System (1.1) is not null-controllable if $T < c(\Lambda_d)$.*

Given that our sequence Λ_d has two branches of eigenvalues, we will use the following characterization of $c(\Lambda_d)$.

Proposition 3.4. *We have $c(\Lambda_d) = \max\{c_1, c_2\}$, where*

$$c_1 := \limsup_{k\to\infty} \frac{-\ln|\sin(\pi \sqrt[4]{d}\sqrt{k})|}{dk^2} \quad \text{and}$$
$$c_2 := \limsup_{k\to\infty} \frac{-\ln\left|\sin\left(\frac{\pi k^2}{\sqrt{d}}\right)\right|}{k^4}. \tag{3.11}$$

Proof. To begin, observe that the product (3.3) corresponding to the sequence $\Lambda_d = \{dk^2, k^4\}_{k\in\mathbb{N}}$ is given by

$$E(z) = \prod_{k\in\mathbb{N}} \left(1 - \frac{z^2}{d^2k^4}\right)\left(1 - \frac{z^2}{k^8}\right), \quad z \in \mathbb{C}.$$

According to definition (3.7), the condensation index associated to the sequence $\Lambda_d = \{dk^2, k^4\}_{k\in\mathbb{N}}$ is given by

$$c(\Lambda_d) = \max\left\{ \limsup_{k\to\infty} \frac{-\ln|E'(dk^2)|}{dk^2}, \limsup_{k\to\infty} \frac{-\ln|E'(k^4)|}{k^4} \right\}. \tag{3.12}$$

Notice that using the identities

$$\sin(\pi z) = \pi z \prod_{k\in\mathbb{N}} \left(1 - \frac{z^2}{k^2}\right), \quad z \in \mathbb{C},$$

and

$$\sinh(\pi z) = \pi z \prod_{k\in\mathbb{N}} \left(1 + \frac{z^2}{k^2}\right), \quad z \in \mathbb{C},$$

we find that

$$E(z) = -i\frac{d}{\pi^6} \sin\left(\pi \frac{\sqrt{z}}{\sqrt{d}}\right) \sin(\pi \sqrt[4]{z}) F(z), \quad z \in \mathbb{C},$$

where

$$F(z) = \frac{\sinh\left(\pi \frac{\sqrt{z}}{\sqrt{d}}\right) \sinh(\pi \sqrt[4]{z}) \sin(\pi \sqrt{i}\sqrt[4]{z}) \sinh(\pi \sqrt{i}\sqrt[4]{z})}{z^2}.$$

It is not difficult to check that

$$E'(z) = -i\frac{d}{\pi^6} \Bigg[\left(\frac{\pi}{2\sqrt{d}\sqrt{z}} \cos\left(\pi \frac{\sqrt{z}}{\sqrt{d}}\right) \sin(\pi \sqrt[4]{z}) + \frac{\pi}{4\sqrt[4]{z^3}} \sin\left(\pi \frac{\sqrt{z}}{\sqrt{d}}\right) \cos(\pi \sqrt[4]{z}) \right) F(z) + \sin\left(\pi \frac{\sqrt{z}}{\sqrt{d}}\right) \sin(\pi \sqrt[4]{z}) F'(z) \Bigg].$$

Hence we obtain

$$|E'(dk^2)| = |\sin(\pi \sqrt[4]{d}\sqrt{k})| \frac{|F(dk^2)|}{2\pi^5 k}, \tag{3.13}$$

and

$$|E'(k^4)| = \frac{d}{4\pi^5 k^3} \left| \sin\left(\pi \frac{k^2}{\sqrt{d}}\right) \right| |F(k^4)|. \tag{3.14}$$

The idea now is to find lower and upper bounds for $|F(dk^2)|$ and $|F(k^4)|$. Notice that

$$|F(dk^2)| = \frac{1}{d^2k^4} \sinh(\pi k) \sinh(\pi \sqrt[4]{d}\sqrt{k}) \times |\sin(\pi \sqrt{i}\sqrt[4]{d}\sqrt{k})| \, |\sinh(\pi \sqrt{i}\sqrt[4]{d}\sqrt{k})|,$$

and

$$\sinh(\pi) \sinh(\pi \sqrt[4]{d}) \leq \sinh(\pi k) \sinh(\pi \sqrt[4]{d}\sqrt{k}) \leq \frac{e^{\pi k}}{2} \frac{e^{\pi \sqrt[4]{d}\sqrt{k}}}{2}.$$

For the other terms, we use the identities

$$\sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y),$$
$$\sinh(x + iy) = \sinh(x)\cos(y) + i\cosh(x)\sin(y),$$

which hold for every $x, y \in \mathbb{R}$. We have

$$\left| \sin\left(\frac{\pi \sqrt[4]{d}\sqrt{k}}{\sqrt{2}} + i\frac{\pi \sqrt[4]{d}\sqrt{k}}{\sqrt{2}}\right) \right| = \left| \sinh\left(\frac{\pi \sqrt[4]{d}\sqrt{k}}{\sqrt{2}} + i\frac{\pi \sqrt[4]{d}\sqrt{k}}{\sqrt{2}}\right) \right| = \left(\sin^2\left(\frac{\pi \sqrt[4]{d}\sqrt{k}}{\sqrt{2}}\right) \cosh^2\left(\frac{\pi \sqrt[4]{d}\sqrt{k}}{\sqrt{2}}\right) + \cos^2\left(\frac{\pi \sqrt[4]{d}\sqrt{k}}{\sqrt{2}}\right) \sinh^2\left(\frac{\pi \sqrt[4]{d}\sqrt{k}}{\sqrt{2}}\right) \right)^{1/2},$$

from where, since $\sqrt{i} = 1/\sqrt{2} + i/\sqrt{2}$, we obtain

$$\sinh^2\left(\frac{\pi \sqrt[4]{d}}{\sqrt{2}}\right) \leq \left| \sin\left(\pi \sqrt{i}\sqrt[4]{d}\sqrt{k}\right) \right| \left| \sinh\left(\pi \sqrt{i}\sqrt[4]{d}\sqrt{k}\right) \right| \leq \exp\left(\frac{2\pi \sqrt[4]{d}\sqrt{k}}{\sqrt{2}}\right),$$

and therefore

$$\sinh(\pi) \sinh(\pi \sqrt[4]{d}) \sinh^2\left(\frac{\pi \sqrt[4]{d}}{\sqrt{2}}\right) \leq |F(dk^2)| \leq \frac{e^{\pi k} e^{\pi \sqrt[4]{d}\sqrt{k}} e^{\sqrt{2}\pi \sqrt[4]{d}\sqrt{k}}}{4d^2k^4}. \tag{3.15}$$

Now, we have that

$$|F(k^4)| = \frac{1}{k^8} \sinh\left(\frac{\pi k^2}{\sqrt{d}}\right) \sinh(\pi k) |\sin(\pi \sqrt{i}k)| |\sinh(\pi \sqrt{i}k)|.$$

We can use the same estimates as before to obtain

$$\sinh\left(\frac{\pi}{\sqrt{d}}\right) \sinh(\pi) \sinh^2\left(\frac{\pi}{\sqrt{2}}\right) \leq |F(k^4)| \leq \frac{e^{\frac{\pi k^2}{\sqrt{d}}} e^{\pi k} e^{\sqrt{2}\pi k}}{4k^8}. \tag{3.16}$$

In order to finish the proof, it suffices to combine estimates (3.15) and (3.16) with (3.13) and (3.14) in the expression (3.12). ■

3.3. *Proof of Theorem 1.5*

Taking into account Proposition 3.3, it is enough to prove the following result.

Proposition 3.5. *If $\sqrt{d}$ is an irrational number with finite Liouville–Roth constant, then the condensation index of $\Lambda_d = \{dk^2, k^4\}_{k\in\mathbb{N}}$ is $c(\Lambda_d) = 0$.*

Proof. From Proposition 3.4, it suffices to show that $c_1 = c_2 = 0$. From Definition 1.4, we directly get that, if the irrational number x has a Liouville–Roth constant μ and $n > \mu$, then there exists a constant C such that

$$\left| x - \frac{p}{q} \right| \geq \frac{C}{q^n}, \tag{3.17}$$

for any integers p and q with $q > 0$.

Let us first prove that $c_1 = 0$. For each $k \in \mathbb{N}$, let $h_k \in \mathbb{N}$ be such that

$$|\sqrt{k}\sqrt[4]{d} - h_k| \leq 1/2. \tag{3.18}$$

From (3.17) applied to $x = \sqrt{d}$ with $p = h_k^2$ and $q = k$, there exist $n \in \mathbb{N}$ and $C > 0$ such that

$$\left| \sqrt{d} - \frac{h_k^2}{k} \right| \geq \frac{C}{k^n}. \tag{3.19}$$

Hence, from (3.18) and (3.19) we have

$$\frac{1}{2} \geq |\sqrt{k}\sqrt[4]{d} - h_k| = \left| \frac{k\sqrt{d} - h_k^2}{\sqrt{k}\sqrt[4]{d} + h_k} \right| \geq \frac{C}{k^{n-1}(\sqrt{k}\sqrt[4]{d} + h_k)}. \tag{3.20}$$

It is not difficult to check from (3.20) that

$$\begin{aligned} |\sin(\pi(\sqrt{k}\sqrt[4]{d}))| &= |\sin(\pi(\sqrt{k}\sqrt[4]{d} - h_k))| \\ &= \sin(|\pi(\sqrt{k}\sqrt[4]{d} - h_k)|) \\ &\geq \sin\left(\frac{C\pi}{k^{n-1}(\sqrt{k}\sqrt[4]{d} + h_k)}\right), \end{aligned} \tag{3.21}$$

where the inequality comes from the fact that $\sin x$ is increasing in the interval $[0, \pi/2]$. Then

$$\frac{-\ln |\sin(\pi \sqrt{k}\sqrt[4]{d})|}{dk^2} \leq \frac{-\ln \sin(C\pi k^{1-n}(\sqrt{k}\sqrt[4]{d} + h_k)^{-1})}{dk^2}. \tag{3.22}$$

From (3.18), we check that

$$\limsup_{k\to\infty} \frac{-\ln\sin(C\pi k^{1-n}(\sqrt{k}\sqrt[4]{d}+h_k)^{-1})}{dk^2} = \limsup_{k\to\infty} \frac{-\ln|C\pi k^{1-n}(\sqrt{k}\sqrt[4]{d}+h_k)^{-1}|}{dk^2} = 0.$$

Then, thanks to this last identity, (3.22) and (3.11), we conclude that $c_1 = 0$.

In a similar way, we prove that $c_2 = 0$. Indeed, for $k \in \mathbb{N}$, let $h_k \in \mathbb{N}$ be such that

$$\left|\frac{k^2}{\sqrt{d}} - h_k\right| \le \frac{1}{2}.$$

It is not difficult to see that, if x is a Liouville number (this means that it has infinite Liouville–Roth constant), then $1/x$ also is a Liouville number. That implies that $1/\sqrt{d}$ has finite Liouville–Roth constant. Hence, from (3.17) with $x = 1/\sqrt{d}$, $p = h_k$ and $q = k^2$, there exist $n \in \mathbb{N}$ and $C > 0$ such that

$$\left|\frac{1}{\sqrt{d}} - \frac{h_k}{k^2}\right| \ge \frac{C}{k^{2n}}.$$

Then,

$$\frac{1}{2} \ge \left|\frac{k^2}{\sqrt{d}} - h_k\right| \ge \frac{C\pi}{k^{2n-2}},$$

and therefore

$$\left|\sin\left(\pi\left(\frac{k^2}{\sqrt{d}}\right)\right)\right| = \left|\sin\left(\pi\left(\frac{k^2}{\sqrt{d}} - h_k\right)\right)\right| = \sin\left(\left|\pi\left(\frac{k^2}{\sqrt{d}} - h_k\right)\right|\right) \ge \sin\left(\frac{C\pi}{k^{2n-2}}\right). \quad (3.23)$$

Using the same argument as for c_1, we obtain from (3.23) and (3.11) that $c_2 = 0$. This ends the proof of Proposition 3.5. ■

Remark 3.6. The same result stated in Proposition 3.5 is true for the family $\Lambda_d = \{dk^2, k^2\}_{k\in\mathbb{N}}$ appearing in [21], which gives a more explicit criterion for the characterization of possible values of the controllability time T_d in Theorem 2.7 in [21].

3.4. *Proof of Theorem 1.6*

We shall prove that, for any given $\lambda_0 > 0$, there exists $d > 0$ such that $c_1 = c_2 = \lambda_0$. Once this has been achieved, we conclude the proof of Theorem 1.6 thanks to Propositions 3.3 and 3.4.

The following result, coming from approximation of irrational numbers by continued fractions, is the main part of the proof.

Lemma 3.7. *Let λ_0 be any positive real number. There exist an irrational number $d > 0$ and a sequence of rational numbers $\{p_k/q_k\}_{k\in\mathbb{N}}$ such that p_k and q_k are co-prime positive integers, the sequences $\{p_k\}_{k\in\mathbb{N}}$ and $\{q_k\}_{k\in\mathbb{N}}$ are strictly increasing, and*

$$\lim_{k\to\infty} e^{\lambda_0 p_k^4}\left|\sqrt[4]{d} - \frac{p_k}{q_k}\right| = 1. \quad (3.24)$$

Proof. The proof is similar to [21, Lemma 6.22] and therefore omitted here. ■

Let λ_0 be any positive number and let $d > 0$ given by Lemma 3.7. First, we prove that $c_1, c_2 \ge \lambda_0$. From the definition of c_1 in (3.11), and since $\{q_k^2\}_{k\in\mathbb{N}}$ is a subsequence of the natural numbers, we have

$$c_1 \ge \limsup_{k\to\infty} \frac{-\ln|\sin(\pi(q_k\sqrt[4]{d} - p_k))|}{dq_k^4}, \quad (3.25)$$

where $\{p_k\}_{k\in\mathbb{N}}$ and $\{q_k\}_{k\in\mathbb{N}}$ are the sequences given by Lemma 3.7. From (3.24), we see that

$$\lim_{k\to+\infty} \frac{p_k}{q_k} = \sqrt[4]{d}, \quad (3.26)$$

and furthermore, since $\{p_k\}_{k\in\mathbb{N}}$ and $\{q_k\}_{k\in\mathbb{N}}$ are increasing,

$$\lim_{k\to+\infty} (q_k\sqrt[4]{d} - p_k) = 0. \quad (3.27)$$

Then,

$$\limsup_{k\to\infty} \frac{-\ln|\sin(\pi(q_k\sqrt[4]{d} - p_k))|}{dq_k^4} = \limsup_{k\to\infty} \frac{-\ln|\pi(q_k\sqrt[4]{d} - p_k)|}{dq_k^4} = \limsup_{k\to\infty} \frac{-\ln|\pi q_k e^{-\lambda_0 p_k^4}|}{dq_k^4} = \lambda_0,$$

which together with (3.25) gives $c_1 \ge \lambda_0$.

Following a similar argument, we have that

$$c_2 \ge \limsup_{k\to\infty} \frac{-\ln\left|\sin\left(\frac{\pi}{\sqrt{d}}({p_k}^2 - q_k^2\sqrt{d})\right)\right|}{p_k^4} = \limsup_{k\to\infty} \frac{-\ln\left|\frac{\pi}{\sqrt{d}}q_k^2(\frac{p_k}{q_k} + \sqrt[4]{d})e^{-\lambda_0 p_k^4}\right|}{p_k^4} = \lambda_0.$$

We now prove $c_1, c_2 \le \lambda_0$. For each $k \in \mathbb{N}$, there exists $h_k \in \mathbb{N}$ such that $|\sqrt{k}\sqrt[4]{d} - h_k| \le 1/2$, and then

$$|\sin(\pi\sqrt{k}\sqrt[4]{d})| = |\sin(\pi(\sqrt{k}\sqrt[4]{d} - h_k))| = \sin(|\pi(\sqrt{k}\sqrt[4]{d} - h_k)|). \quad (3.28)$$

On the other hand, we have already proved that $c(\Lambda_d) \ge \lambda_0 > 0$, and this inequality implies that $\sqrt[4]{d}$ is not an algebraic number. Indeed, if $\sqrt[4]{d}$ were algebraic, we would also have that $\sqrt{d} = (\sqrt[4]{d})^2$ is algebraic, and then its Liouville–Roth constant is 2, which implies, using Theorem 1.5 and Proposition 3.3, that $c(\Lambda_d) = 0$, a contradiction. Accordingly, for each $k \in \mathbb{N}$ we have $|\sqrt{k}\sqrt[4]{d} - h_k| > 0$, and then there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$|q_{n_k}\sqrt[4]{d} - p_{n_k}| < |\sqrt{k}\sqrt[4]{d} - h_k| \quad \text{and} \quad q_{n_k} \ge \sqrt{k}, \quad (3.29)$$

where $\{p_k\}_{k\in\mathbb{N}}$ and $\{q_k\}_{k\in\mathbb{N}}$ are the sequences given by Lemma 3.7. From (3.28) and (3.29) we conclude, using the argument to obtain (3.21)–(3.22) that

$$c_1 \le \limsup_{k\to\infty} \frac{-\ln|\sin(\pi(q_{n_k}\sqrt[4]{d} - p_{n_k}))|}{dq_{n_k}^4} = \limsup_{k\to\infty} \frac{-\ln|\pi(q_{n_k}\sqrt[4]{d} - p_{n_k})|}{dq_{n_k}^4} = \lambda_0.$$

Similarly, taking into account that $(\sqrt{d})^{-1}$ is not an algebraic number, for each $k \in \mathbb{N}$ there exists an increasing sequence $\{g_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$0 < \left|\frac{k^2}{\sqrt{d}} - g_k\right| \le \frac{1}{2},$$

and then there exists an increasing sequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$|q_{n_k}\sqrt[4]{d} - p_{n_k}| < \left|\frac{k^2}{\sqrt{d}} - g_k\right| \quad \text{and} \quad p_{n_k} \ge k. \quad (3.30)$$

Hence

$$c_2 = \limsup_{k\to\infty} \frac{-\ln|\sin(\pi(q_{n_k}\sqrt[4]{d}-p_{n_k}))|}{p_{n_k}^4}$$
$$\leq \limsup_{k\to\infty} \frac{-\ln|\pi(q_{n_k}\sqrt[4]{d}-p_{n_k})|}{p_{n_k}^4} = \lambda_0.$$

Therefore, $c_1, c_2 \leq \lambda_0$, thus $c(\Lambda_d) = \lambda_0$. This ends the proof of Theorem 1.6 taking

$$T_0 := c(\Lambda_d) = \lambda_0.$$

3.5. Proof of Theorem 1.7

From Propositions 3.3 and 3.4, it is enough to prove that $c(\Lambda_d) = +\infty$ for some $d > 0$. Such a parameter will be given by the following result:

Lemma 3.8. *There exist an irrational number $d > 0$ and a sequence of rational numbers $\{p_k/q_k\}_{k\in\mathbb{N}}$ such that p_k and q_k are co-prime positive integers, the sequences $\{p_k\}_{k\in\mathbb{N}}$ and $\{q_k\}_{k\in\mathbb{N}}$ are strictly increasing, and*

$$\lim_{k\to\infty} e^{p_k^5}\left|\sqrt[4]{d}-\frac{p_k}{q_k}\right| = 0. \tag{3.31}$$

Proof. The proof is similar to [21, Lemma 6.22] and therefore omitted here. ■

Let $d > 0$, $\{p_k\}_{k\in\mathbb{N}}$ and $\{q_k\}_{k\in\mathbb{N}}$ be given by Lemma 3.8. We will check that $c_1 = +\infty$, which gives directly that $c(\Lambda_d) = +\infty$.

Since $\{q_k^2\}_{k\in\mathbb{N}}$ is a subsequence of the natural numbers, from (3.11) we have that

$$c_1 \geq \limsup_{k\to\infty} \frac{-\ln|\sin(\pi(q_k\sqrt[4]{d}-p_k))|}{dq_k^4}. \tag{3.32}$$

From (3.31), we can prove that the sequences $\{p_k\}_{k\in\mathbb{N}}$ and $\{q_k\}_{k\in\mathbb{N}}$ satisfy (3.26), (3.27), and, furthermore, that there exist $C > 0$ and $k_0 \in \mathbb{N}$ such that

$$|q_k\sqrt[4]{d}-p_k| \leq Ce^{-p_k^5}, \quad \forall k \geq k_0.$$

Therefore,

$$\limsup_{k\to\infty} \frac{-\ln|\sin(\pi(q_k\sqrt[4]{d}-p_k))|}{dq_k^4} = \limsup_{k\to\infty} \frac{-\ln|\pi(q_k\sqrt[4]{d}-p_k)|}{dq_k^4}$$
$$\geq \limsup_{k\to\infty} \frac{-\ln|\pi Cq_k e^{-p_k^5}|}{dq_k^4}.$$

This last inequality, (3.26) and (3.32) yield

$$c_1 \geq \limsup_{k\to\infty} \frac{p_k^5}{dq_k^4} = +\infty,$$

since $\{p_k\}_{k\in\mathbb{N}}$ is a strictly increasing sequence. This ends the proof of Theorem 1.7.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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