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Uniform null controllability of a linear KdV equation using two controls

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ABSTRACT

In this paper we consider a linear KdV equation posed on a bounded interval. We study the behavior of the cost of null controllability when two boundary controls are employed. By means of suitable Carleman inequalities and a new exponential dissipation estimate, we prove that uniform null controllability with respect to the dispersion coefficient holds, contrary to the case when one control is used at the left end-point of the interval.

1. Introduction

Let $T > 0$, $L > 0$ and $Q := (0, T) \times (0, L)$. We consider the following linear Korteweg–de Vries (KdV) equation:

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } Q, \\ y_{|x=0} = v_0, \quad y_{x\,|x=L} = v_1, \quad y_{xx\,|x=L} = v_2 & \text{in } (0, T), \\ y_{|t=0} = y_0 & \text{in } (0, L), \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is the dispersion coefficient, $M \in \mathbb{R}$ is the transport coefficient, $y_0 \in L^2(0, L)$ is the initial condition and v_0, v_1 and v_2 stand for the controls.

The control of the KdV equation has captured the attention of several researchers over the last twenty years. Most results are related to the boundary conditions

$$y_{|x=0} = u_0, \quad y_{|x=L} = u_1, \quad y_{x\,|x=L} = u_2. \tag{1.2}$$

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We refer to the recent survey [2] for a complete compendium of controllability and stabilization results for (1.2). See also [17,19,18,3,13].

For the boundary conditions in (1.1), introduced by T. Colin and J.-M. Ghidaglia in [6,5], the first controllability result is found in [14], where the null controllability is proved with $M = -1$ and $v_1 = v_2 = 0$. In [4], the authors proved exact controllability results for the remaining cases of active controls.

We are interested in the *uniform null controllability* of (1.1). More precisely, we ask if, given an initial condition y_0 and a time $T > 0$, is it possible to drive the solution of (1.1) to the rest at $t = T$ with controls uniformly bounded with respect to ε as $\varepsilon \to 0$?

For boundary conditions (1.2), positive results are proved for the case of vanishing diffusion (μy_{xx} with small $\mu > 0$) [7] and vanishing dispersion [11,12] for large control times T. For (1.1), this question has a negative answer in the case $v_1 \equiv v_2 \equiv 0$. Indeed, in [1], we proved that there exist initial conditions such that the L^2-norm of any control v_0 driving the solution of (1.1) to zero is exponentially increasing as ε goes to zero. Furthermore, this holds for any $T > 0$.

This paper is a continuation of [1]. We are interested in investigating the uniform null controllability in two cases:

- Case 1: $v_1 \equiv 0$.
- Case 2: $v_0 \equiv 0$.

Indeed, we are able to obtain uniform null controllability of (1.1) as long as we control with v_2 the boundary condition $y_{xx|x=L}$ and the initial condition $y_0 \in L^2(0, L)$ satisfies, for some $L_0 \in (0, L)$,

$$y_0 = 0 \text{ in } (L_0, L). \tag{1.3}$$

The result corresponding to the first case is the following.

Theorem 1.1. *Let $L, \varepsilon, M > 0$, $L_0 \in (0, L)$ and set $v_1 \equiv 0$. There exists $K > 1$, independent of ε and M, such that for every $T \geq K/M$ and every $y_0 \in L^2(0, L)$ satisfying (1.3), there are controls v_0^ε and v_2^ε belonging to $L^2(0, T)$ such that $y_{|t=T} = 0$ in $(0, L)$ and*

$$\|v_0^\varepsilon\|^2_{L^2(0,T)} + \|v_2^\varepsilon\|^2_{L^2(0,T)} \leq \bar{C} \exp\left(-\frac{CM^{1/2}}{\varepsilon^{1/2}}\right)\|y_0\|^2_{L^2(0,L)} \tag{1.4}$$

where the constant $\bar{C}$ depends at most polynomially on ε^{-1}, ε, M^{-1} and M, and C only depends on L and L_0.

A direct consequence of Theorem 1.1 is that the *cost of null controllability* is uniformly bounded with respect to ε. Furthermore, it goes to zero as ε vanishes. Indeed, we define this cost as

$$C_0^\varepsilon := \sup_{\substack{y_0 \in L^2(0,L) \\ (1.3), y_0 \neq 0}} \min_{\substack{v_0^\varepsilon, v_2^\varepsilon \in L^2(0,T) \\ y_{|t=T}=0}} \frac{\|v_0^\varepsilon\|^2_{L^2(0,T)} + \|v_2^\varepsilon\|^2_{L^2(0,T)}}{\|y_0\|^2_{L^2(0,L)}}. \tag{1.5}$$

Notice that C_0^ε is the best constant such that (1.4) holds (see, for instance, [8]). With this notation, we deduce from Theorem 1.1 the following result.

Corollary 1.2. *Let $L, M > 0$, $L_0 \in (0, L)$ and $T > 0$ as in Theorem 1.1. Then,*

$$\lim_{\varepsilon \to 0} C_0^\varepsilon = 0.$$

Some remarks are in order. Notice that in [1], it is shown that, when only v_0 is acting, the cost explodes as $\varepsilon \to 0$, even if T is arbitrarily large. Although that result is proved for a larger class of initial conditions, Corollary 1.2 says that the controls can be uniformly bounded with respect to ε if a control on $y_{xx|x=L}$ is permitted.

Concerning the second case, we are able to prove the following

Theorem 1.3. *Let $L, \varepsilon, M > 0$, $L_0 \in (0, L)$ and set $v_0 \equiv 0$. There exist $C > 0$ and $K > 1$, both independent of ε and M such that for every $T \geq K/M$ and every $y_0 \in L^2(0, L)$ satisfying (1.3), there are controls v_1^ε and v_2^ε belonging to $L^2(0, T)$ such that $y_{|t=T} = 0$ in $(0, L)$ and*

$$\|v_1^\varepsilon\|^2_{L^2(0,T)} + \|v_2^\varepsilon\|^2_{L^2(0,T)} \leq CM \exp\left(-\frac{CM^{1/2}}{\varepsilon^{1/2}}\right)\|y_0\|^2_{L^2(0,L)}. \tag{1.6}$$

Similarly as before, the cost of null controllability in this case is defined as

$$C_1^\varepsilon := \sup_{\substack{y_0 \in L^2(0,L) \\ (1.3), y_0 \neq 0}} \min_{\substack{v_1^\varepsilon, v_2^\varepsilon \in L^2(0,T) \\ y_{|t=T}=0}} \frac{\|v_1^\varepsilon\|^2_{L^2(0,T)} + \|v_2^\varepsilon\|^2_{L^2(0,T)}}{\|y_0\|^2_{L^2(0,L)}} \tag{1.7}$$

and we deduce from (1.6) the following asymptotic behavior.

Corollary 1.4. *Let $L, M > 0$, $L_0 \in (0, L)$ and $T > 0$ as in Theorem 1.3. Then,*

$$\lim_{\varepsilon \to 0} C_1^\varepsilon = 0.$$

From [12], it is not hard to convince ourselves that controlling from the left extreme of the interval $(0, L)$ is not so different than controlling from the right when using (1.2). In fact, the uniform null controllability result proved in [12, Theorem 1.1] will hold if u_1 and u_2 are active in (1.2) and $u_0 = 0$ (see also [7], where the same happens for vanishing diffusion). For the boundary conditions in (1.1), we see from Corollary 1.4 that the behavior of the cost when we control from the right-end of $(0, L)$ is dramatically different than controlling from the left (see [1, Corollary 1.4]).

It comes natural to ask if the results of Theorems 1.1 and 1.3 would hold if $v_2 = 0$, but v_0 and v_1 are active. The approach followed in this article does not allow us to give a positive nor negative answer to this question. However, given that what enables to obtain these results is to use v_2, and in view of the negative result from [1], we conjecture that the cost in this case cannot be uniformly bounded. Furthermore, it should explode as the dispersion coefficient vanishes. This case remains, for the time being, an open problem.

To prove Theorems 1.1 and 1.3, we follow the ideas of [7,12]. It consists in two main steps. First, we prove an observability inequality for the solutions of the adjoint equation (see (3.1) below). This is obtained by means of Carleman estimates. Second, we prove an exponential dissipation estimate that holds for large times. From the results in [1], an estimate like the one proved in [12] cannot hold. Nevertheless, the fact that we allow v_2 to act on the equation helps to obtain a different kind of dissipation estimate that allows to counteract the constant coming from the Carleman estimate for large times (see Section 3 for details).

This work is organized as follows. In Section 2, we justify the existence and uniqueness of solutions. Then, in Section 3 we prove a new exponential dissipation estimate for the solutions of the adjoint equation of (1.1). The proofs of Theorems 1.1 and 1.3 can be found in Sections 4 and 5, respectively. Finally, in Appendix A, we prove a new Carleman estimate for KdV.

2. On the existence and uniqueness of solutions of (1.1)

The well-posedness of the KdV equation has been the object of several works. Most results are related for the non-linear version

$$y_t + y_{xxx} + yy_x = 0.$$

For well-posedness results concerning the boundary conditions in (1.1), we refer to [15] and the references therein.

A common strategy to prove existence is by means of continuous semi-groups which requires the linear operator of the equation to be dissipative. Notice that this is not the case for (1.1) when $M > 0$. It is not our intention here to give sharp regularity, but to justify the existence and uniqueness of solutions of (1.1) in this case.

Proposition 2.1. *Let $\varepsilon, T, M > 0$, $v_0, v_1, v_2 \in L^2(0,T)$ and $y_0 \in L^2(0,L)$. Then, there exists a unique solution $y \in L^2(0,T; H^{-2}(0,L)) \cap \mathcal{C}^0([0,T]; H^{-5}(0,L))$ of (1.1).*

Proof. We start by lifting the boundary conditions. Let ε and M be arbitrary positive constants. We consider the function

$$z(t,x) = \int_0^t y(s,x)\,\mathrm{d}s - \frac{(L-x)^3}{L^3}\int_0^t v_0(s)\,\mathrm{d}s - x\int_0^t v_1(s)\,\mathrm{d}s + \frac{x(2L-x)}{2}\int_0^t v_2(s)\,\mathrm{d}s$$

where y is supposed to satisfy (1.1) with $v_0, v_1, v_2 \in L^2(0,T)$. It is clear then that z satisfies the equation

$$\begin{cases} z_t + \varepsilon z_{xxx} - M z_x = f & \text{in } Q, \\ z_{|x=0} = 0, \quad z_{x|x=L} = 0, \quad z_{xx|x=L} = 0 & \text{in } (0,T), \\ z_{|t=0} = 0 & \text{in } (0,L), \end{cases} \tag{2.1}$$

with

$$\begin{aligned} f(t,x) := &-\frac{(L-x)^3}{L^3}v_0(t) - xv_1(t) + \frac{x(2L-x)}{2}v_2(t) + \left(\frac{6\varepsilon}{L^3} - \frac{3M(L-x)^2}{L^3}\right)\int_0^t v_0(s)\,\mathrm{d}t \\ &+ M\int_0^t v_1(s)\,\mathrm{d}s - M(L-x)\int_0^t v_2(s)\,\mathrm{d}s + y_0. \end{aligned} \tag{2.2}$$

Then, if we prove the existence (and uniqueness) of solution z of (2.1), we would have proved the existence (and uniqueness) of a solution of (1.1) by simply defining

$$y(t,x) := z_t(t,x) + \frac{(L-x)^3}{L^3}v_0(t) + xv_1(t) - \frac{x(2L-x)}{2}v_2(t). \tag{2.3}$$

The following lemma was proved in [1].

Lemma 2.2. *Let $\varepsilon, T, M > 0$ and $f \in L^2(Q)$. Then, there exists a unique solution z of (2.1) which belongs to $L^2(0,T; H^1(0,L)) \cap L^\infty(0,T; L^2(0,L))$.*

Remark 2.3. The notion of solution in Lemma 2.2 is the one from [17], that is, z verifies equation (2.1) in $\mathcal{D}'(0,T; H^{-2})$. The same notion of (mild or weak) solution is considered for equation (1.1).

Since f defined in (2.2) clearly belongs to $L^2(Q)$, the existence and uniqueness of z solution of (2.1) is ensured from Lemma 2.2. Now, using the equation satisfied by z and (2.3), we deduce that $y \in L^2(0,T;H^{-2}) \cap H^1(0,T;H^{-5})$ satisfies (1.1) (in the sense of Remark 2.3). This concludes the proof of Proposition 2.1. □

3. An exponential dissipation estimate

This section is devoted to the proof of an exponential dissipation estimate for the solutions of the adjoint equation of (1.1)

$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xxx} + M\varphi_x = 0 & \text{in } Q, \\ \varphi_{|x=0} = 0, \quad \varphi_{x|x=0} = 0, \quad (\varepsilon\varphi_{xx} - M\varphi)_{|x=L} = 0 & \text{in } (0,T), \\ \varphi_{|t=T} = \varphi_T & \text{in } (0,L), \end{cases} \tag{3.1}$$

with $\varphi_T \in L^2(0,L)$. In [12], it was proven that for the boundary conditions

$$\varphi_{x=0} = \varphi_{x|x=0} = \varphi_{|x=L} = 0 \quad \text{in } (0,T)$$

an estimate of the kind ([12, Proposition 3.2])

$$\int_0^L |\varphi_{|t=t_1}|^2 \,\mathrm{d}x \le \exp\left(-\frac{C(M,L)(t_2-t_1)}{\varepsilon^{1/2}}\right)\int_0^L |\varphi_{|t=t_2}|^2 \,\mathrm{d}x$$

holds as soon as $t_2 - t_1 > L/M$. Normally, one would expect that the solutions of (3.1) verify the same type of inequality, but this would imply an uniform null controllability result (for large times) with $v_1 = v_2 = 0$ in (1.1). This would contradict [1, Theorem 1.2].

Nevertheless, we are able to prove a new kind of dissipation estimate which is stated in the following proposition.

Proposition 3.1. *Let $\varepsilon, M > 0$ and $L_0 \in (0,L)$. For every pair $(t_1,t_2) \in (0,T)^2$ such that $t_2 - t_1 > L/M$ and for every $\varphi_T \in L^2(0,L)$, the solution φ of (3.1) satisfies*

$$\int_0^{L_0} |\varphi_{|t=t_1}|^2 \,\mathrm{d}x \le \exp\left(-\frac{2(L-L_0)^{1/2}(M(t_2-t_1)-L_0)}{3\sqrt{7}\varepsilon^{1/2}(t_2-t_1)^{1/2}}\right)\int_0^L |\varphi_{|t=t_2}|^2 \,\mathrm{d}x$$
$$+ M\exp\left(-\frac{2(L-L_0)^{3/2}}{3\sqrt{7}\varepsilon^{1/2}(t_2-t_1)^{1/2}}\right)\int_{t_1}^{t_2} |\varphi_{|x=L}|^2 \,\mathrm{d}t. \tag{3.2}$$

Proof. We follow the steps used in [12] (see also [1]). Let $\rho(t,x) := M(T-t) - x$. We multiply the equation by $\exp(r\rho)\varphi$, $r > 0$, and integrate in $(0,L)$. After integration by parts, and taking into account the boundary condition in (3.1), we obtain

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L e^{r\rho}|\varphi|^2 \,\mathrm{d}x - \frac{\varepsilon r^3}{2}\int_0^L e^{r\rho}|\varphi|^2 \,\mathrm{d}x + \frac{3\varepsilon r}{2}\int_0^L e^{r\rho}|\varphi_x|^2 \,\mathrm{d}x + \frac{\varepsilon}{2}e^{r\rho_{|x=L}}|\varphi_{x|x=L}|^2$$
$$= \frac{M}{2}e^{r\rho_{|x=L}}|\varphi_{|x=L}|^2 + \frac{\varepsilon r^2}{2}e^{r\rho_{|x=L}}|\varphi_{|x=L}|^2 + \varepsilon r e^{r\rho_{|x=L}}\varphi_{x|x=L}\varphi_{|x=L}. \tag{3.3}$$

Let us estimate the last two terms in the right-hand side of (3.3). On the one hand, using Cauchy–Schwarz's inequality, we have that

$$\varepsilon r e^{r\rho_{|x=L}}\varphi_{x|x=L}\varphi_{|x=L} \le \frac{\varepsilon}{2}e^{r\rho_{|x=L}}|\varphi_{x|x=L}|^2 + \frac{\varepsilon r^2}{2}e^{r\rho_{|x=L}}|\varphi_{|x=L}|^2.$$

On the other hand, since $\varphi_{|x=0} = 0$,

$$\varepsilon r^2 e^{r\rho_{|x=L}}|\varphi_{|x=L}|^2 = \varepsilon r^2 e^{r\rho_{|x=L}}\int_0^L (|\varphi|^2)_x \,\mathrm{d}x = 2\varepsilon r^2 e^{r\rho_{|x=L}}\int_0^L \varphi_x\varphi \,\mathrm{d}x.$$

Notice that $\exp(r\rho)$ reaches its minimum at $x = L$. Therefore, by Cauchy–Schwarz's inequality we get

$$\varepsilon r^2 e^{r\rho_{|x=L}}|\varphi_{|x=L}|^2 \le \frac{3\varepsilon r}{2}\int_0^L e^{r\rho}|\varphi_x|^2 \,\mathrm{d}x + \frac{2\varepsilon r^3}{3}\int_0^L e^{r\rho}|\varphi|^2 \,\mathrm{d}x.$$

Going back to (3.3), we obtain

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L e^{r\rho}|\varphi|^2 \,\mathrm{d}x - \frac{7\varepsilon r^3}{6}\int_0^L e^{r\rho}|\varphi|^2 \,\mathrm{d}x \le \frac{M}{2}e^{r\rho_{|x=L}}|\varphi_{|x=L}|^2.$$

From this inequality, we deduce

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-7/3\varepsilon r^3(T-t)}\int_0^L e^{r\rho}|\varphi|^2 \,\mathrm{d}x\right) \le M e^{-7/3\varepsilon r^3(T-t)}e^{r\rho_{|x=L}}|\varphi_{|x=L}|^2$$

and integrating between t_1 and t_2 yields

$$\begin{aligned} e^{-7/3\varepsilon r^3(T-t_1)}\int_0^L e^{r\rho_{|t=t_1}}|\varphi_{|t=t_1}|^2 \,\mathrm{d}x &\le e^{-7/3\varepsilon r^3(T-t_2)}\int_0^L e^{r\rho_{|t=t_2}}|\varphi_{|t=t_2}|^2 \,\mathrm{d}x \\ &\quad + M\int_{t_1}^{t_2} e^{-7/3\varepsilon r^3(T-t)}e^{r\rho_{|x=L}}|\varphi_{|x=L}|^2 \,\mathrm{d}t. \end{aligned} \tag{3.4}$$

Since

$$\exp\big(r\rho_{|t=t_2}\big) \le \exp\big(r\rho(t_2, 0)\big) \text{ in } (0, L)$$

and

$$\exp\big(-7/3\varepsilon r^3(T-t)\big)\exp\big(r\rho_{|x=L}\big) \le \exp\big(-7/3\varepsilon r^3(T-t_2)\big)\exp\big(r\rho(t_1, L)\big) \text{ in } (t_1, t_2)\times(0, L),$$

we get in (3.4) the estimate

$$\int_0^L e^{-rx}|\varphi_{|t=t_1}|^2 \,\mathrm{d}x \le e^{7/3\varepsilon r^3(t_2-t_1)}e^{-rM(t_2-t_1)}\int_0^L |\varphi_{|t=t_2}|^2 \,\mathrm{d}x + M e^{7/3\varepsilon r^3(t_2-t_1)}e^{-rL}\int_{t_1}^{t_2}|\varphi_{|x=L}|^2 \,\mathrm{d}t. \tag{3.5}$$

We observe in (3.5) that, to have any hope to choose $r > 0$ such that the argument in the exponential is negative, we need to truncate the integral in the left-hand side so that the term $\exp(-rL)$ does not completely disappear. Thus, for any $L_0 \in (0, L)$, we get

$$\int_0^{L_0} |\varphi_{|t=t_1}|^2 \,\mathrm{d}x \le e^{7/3\varepsilon r^3(t_2-t_1)} e^{-r(M(t_2-t_1)-L_0)} \int_0^L |\varphi_{|t=t_2}|^2 \,\mathrm{d}x + M e^{7/3\varepsilon r^3(t_2-t_1)} e^{-r(L-L_0)} \int_{t_1}^{t_2} |\varphi_{|x=L}|^2 \,\mathrm{d}t. \tag{3.6}$$

The objective now is to choose $r > 0$ appropriately. We first remark that

$$\frac{7}{3}\varepsilon r^3(t_2-t_1) - r(M(t_2-t_1)-L_0) < 0 \quad \text{if } 0 < r < \frac{\sqrt{3}(M(t_2-t_1)-L_0)^{1/2}}{\sqrt{7}\varepsilon^{1/2}(t_2-t_1)^{1/2}}$$

and

$$\frac{7}{3}\varepsilon r^3(t_2-t_1) - r(L-L_0) < 0 \quad \text{if } 0 < r < \frac{\sqrt{3}(L-L_0)^{1/2}}{\sqrt{7}\varepsilon^{1/2}(t_2-t_1)^{1/2}}.$$

Thus, $r > 0$ needs to be chosen to satisfy

$$r < \min\left\{ \frac{\sqrt{3}(M(t_2-t_1)-L_0)^{1/2}}{\sqrt{7}\varepsilon^{1/2}(t_2-t_1)^{1/2}}, \frac{\sqrt{3}(L-L_0)^{1/2}}{\sqrt{7}\varepsilon^{1/2}(t_2-t_1)^{1/2}} \right\} = \frac{\sqrt{3}(L-L_0)^{1/2}}{\sqrt{7}\varepsilon^{1/2}(t_2-t_1)^{1/2}}.$$

The last equality holds since $L < M(t_2 - t_1)$. Let

$$r := \frac{(L-L_0)^{1/2}}{\sqrt{7}\varepsilon^{1/2}(t_2-t_1)^{1/2}}.$$

With this choice of r in (3.6) we obtain

$$\begin{aligned} \int_0^{L_0} |\varphi_{|t=t_1}|^2 \,\mathrm{d}x \le \exp\left(\frac{(L-L_0)^{3/2} - 3(L-L_0)^{1/2}(M(t_2-t_1)-L_0)}{3\sqrt{7}\varepsilon^{1/2}(t_2-t_1)^{1/2}} \right) \int_0^L |\varphi_{|t=t_2}|^2 \,\mathrm{d}x \\ + M \exp\left(-\frac{2(L-L_0)^{3/2}}{3\sqrt{7}\varepsilon^{1/2}(t_2-t_1)^{1/2}} \right) \int_{t_1}^{t_2} |\varphi_{|x=L}|^2 \,\mathrm{d}t. \end{aligned}$$

Using again that $L < M(t_2 - t_1)$ to estimate the first exponential, we finally deduce (3.2). □

4. Proof of Theorem 1.1

It is well known that the null controllability result stated in Theorem 1.1 is equivalent to the observability inequality (see [16])

$$\|\varphi_{|t=0}\|^2_{L^2(0,L_0)} \le C_{obs}\left(\|\varphi_{xx|x=0}\|^2_{L^2(0,T)} + \|\varphi_{|x=L}\|^2_{L^2(0,T)}\right) \tag{4.1}$$

where φ is the solution of (3.1) and $C_{obs} > 0$ is a constant independent of φ. We prove (4.1) by means of a Carleman estimate (proved in [1]) and the exponential dissipation estimate proved in Proposition 3.1.

Carleman estimates have been largely used since the celebrated work [10]. Indeed, they are a powerful tool to prove observability for parabolic ([9,7]) and dispersive ([17,11]) equations.

Let us introduce the weight function

$$\alpha(t,x) = \frac{p(x)}{t^{1/2}(T-t)^{1/2}} \quad (t,x) \in (0,T) \times (0,L),$$

where $p(x)$ is a polynomial of degree 2, strictly positive, increasing and concave. We have the following result.

Lemma 4.1. *([1, Proposition 3.2]) Let $T, \varepsilon > 0$ and $M \in \mathbb{R} \setminus \{0\}$. There exists a positive constant C independent of T, ε and M such that, for any solution φ of (3.1), we have*

$$\iint\limits_Q e^{-2s\alpha_{|x=L}} \left(s^5 \alpha^5_{|x=0} |\varphi|^2 + s^3 \alpha^3_{|x=0} |\varphi_x|^2 + s\alpha_{|x=0} |\varphi_{xx}|^2 \right) \mathrm{d}x\,\mathrm{d}t$$
$$\leq \bar{C} \exp\left(C|M|^{1/2}\varepsilon^{-1/2} \right) s^5 \int_0^T e^{-2s\alpha_{|x=0}} \alpha^5_{|x=0} |\varphi_{xx|x=0}|^2\,\mathrm{d}t \tag{4.2}$$

for all $s \geq C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$ and $\bar{C}$ depends at most polynomially on ε^{-1}, ε, $|M|^{-1}$ and $|M|$.

Now we are in position to prove (4.1). From Lemma 4.1, fixing $s = C(T + \varepsilon^{-1/2}T^{1/2} + M^{1/2}\varepsilon^{-1/2}T)$ (recall that $M > 0$), we obtain

$$\int_{T/4}^{3T/4} \int_0^L |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \bar{C} \exp\left(\frac{C}{\varepsilon^{1/2}} \left(\frac{1}{T^{1/2}} + M^{1/2} \right) \right) \int_0^T |\varphi_{xx|x=0}|^2\,\mathrm{d}t. \tag{4.3}$$

Now, we take $t_1 = 0$ in Proposition 3.1 and integrate with respect to $t_2 = t$ between $T/4$ and $3T/4$ in (3.2), provided that $T > 4L/M$. This way, we have

$$\int_0^{L_0} |\varphi_{|t=0}|^2\,\mathrm{d}x \leq \frac{2}{T} \int_{T/4}^{3T/4} \exp\left(-\frac{2(L-L_0)^{1/2}(Mt - L_0)}{3\sqrt{7}\varepsilon^{1/2}t^{1/2}} \right) \int_0^L |\varphi|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \frac{2M}{T} \int_{T/4}^{3T/4} \exp\left(-\frac{2(L-L_0)^{3/2}}{3\sqrt{7}\varepsilon^{1/2}t^{1/2}} \right) \int_0^t |\varphi_{|x=L}|^2\,\mathrm{d}s\,\mathrm{d}t. \tag{4.4}$$

Since in $(T/4, 3T/4)$ it is verified that

$$\exp\left(-\frac{2(L-L_0)^{1/2}(Mt-L_0)}{3\sqrt{7}\varepsilon^{1/2}t^{1/2}} \right) \leq \exp\left(-\frac{(L-L_0)^{1/2}(MT-4L_0)}{3\sqrt{7}\varepsilon^{1/2}T^{1/2}} \right)$$

and

$$\exp\left(-\frac{2(L-L_0)^{3/2}}{3\sqrt{7}\varepsilon^{1/2}t^{1/2}} \right) \leq \exp\left(-\frac{4(L-L_0)^{3/2}}{3\sqrt{21}\varepsilon^{1/2}T^{1/2}} \right),$$

we obtain from (4.4)

$$
\int_0^{L_0} |\varphi_{|t=0}|^2 \,\mathrm{d}x \le M \exp\Big(-\frac{4(L-L_0)^{3/2}}{3\sqrt{21}\varepsilon^{1/2}T^{1/2}}\Big) \int_0^T |\varphi_{|x=L}|^2 \,\mathrm{d}t
+ \frac{2}{T}\exp\Big(-\frac{(L-L_0)^{1/2}(MT-4L_0)}{3\sqrt{7}\varepsilon^{1/2}T^{1/2}}\Big) \int_{T/4}^{3T/4}\int_0^L |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \tag{4.5}
$$

for every $T > 4L/M$. Going back to (4.3) we find that

$$
\int_0^{L_0} |\varphi_{|t=0}|^2 \,\mathrm{d}x \le M \exp\Big(-\frac{4(L-L_0)^{3/2}}{3\sqrt{21}\varepsilon^{1/2}T^{1/2}}\Big) \int_0^T |\varphi_{|x=L}|^2 \,\mathrm{d}t
+ \frac{\bar{C}}{T}\exp\Big(\frac{C(1+M^{1/2}T^{1/2}) - (3\sqrt{7})^{-1}(L-L_0)^{1/2}(MT-4L_0)}{\varepsilon^{1/2}T^{1/2}}\Big) \int_0^T |\varphi_{xx|x=0}|^2 \,\mathrm{d}t. \tag{4.6}
$$

Notice that at this point, observability inequality (4.1) is proved and therefore the existence of two controls $v_0^\varepsilon, v_2^\varepsilon \in L^2(0,T)$ driving the solution y of (1.1) to zero at time T.

Let us now deduce (1.4). From (4.6), it suffices to consider $T = K/M$ for $K > 0$ sufficiently large to obtain (1.4). This concludes the proof of Theorem 1.1.

5. Proof of Theorem 1.3

The proof of Theorem 1.3 follows the same strategy of Theorem 1.1, that is, we prove the observability inequality

$$
\|\varphi_{|t=0}\|^2_{L^2(0,L_0)} \le C_{obs}\big(\|\varphi_{x|x=L}\|^2_{L^2(0,T)} + \|\varphi_{|x=L}\|^2_{L^2(0,T)}\big). \tag{5.1}
$$

In this case, a new Carleman inequality is needed, one with observation terms as in (5.1). To accomplish this, we need to consider the weight function

$$
\beta(t,x) = \frac{q(x)}{t^{1/2}(T-t)^{1/2}} \tag{5.2}
$$

where $q(x)$ is a polynomial of degree 2, strictly positive, decreasing and concave. We further assume that q satisfies

$$
2q(L) > q(0). \tag{5.3}
$$

(Take, for instance, $q(x) = -\frac{x^2}{L^2} - \frac{x}{L} + 6$.) Let us state the new Carleman inequality.

Proposition 5.1. *Let $T, \varepsilon > 0$ and $M \in \mathbb{R}\setminus\{0\}$. There exists a positive constant C independent of T, ε and M such that, for any solution φ of (3.1), we have*

$$
\iint_Q e^{-2s\beta}\Big(s^5\beta^5|\varphi|^2 + s^3\beta^3|\varphi_x|^2 + s\beta|\varphi_{xx}|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \le Cs^5\int_0^T e^{-2s\beta_{|x=L}}\beta^5_{|x=L}|\varphi_{|x=L}|^2
+ Cs^5(1+MT^5)\int_0^T e^{-2s(2\beta_{|x=L}-\beta_{|x=0})}\beta^5_{|x=L}|\varphi_{x|x=L}|^2\,\mathrm{d}t \tag{5.4}
$$

for all $s \ge C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$.

Notice that, unlike the Carleman estimate proved in [14], (5.4) holds for the optimal power of $t(T-t)$ for the KdV equation, which is crucial to obtain (1.6). Furthermore, let us recall that in [1] a change of unknowns is considered to deal with the boundary condition $\varepsilon\varphi_{xx|x=L} - M\varphi_{|x=L} = 0$ and prove (4.2). This is indeed the obstruction found in [14]. Here, we are able to deal with this boundary condition by slightly changing the weights of the observation terms.

The details of the proof of Proposition 5.1 is in Appendix A.

The proof of the observability inequality (5.1), and in consequence of Theorem 1.3, is actually analogous to (5.1). However, we sketch it to clarify the dependence on the parameters in (1.6).

From Proposition 5.1, fixing $s = C(T + \varepsilon^{-1/2}T^{1/2} + M^{1/2}\varepsilon^{-1/2}T)$, we obtain

$$\int_{T/4}^{3T/4}\int_0^L |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \le C\exp\left(\frac{C}{\varepsilon^{1/2}}\left(\frac{1}{T^{1/2}} + M^{1/2}\right)\right)\int_0^T |\varphi_{|x=L}|^2\,\mathrm{d}t + C(1+MT)\exp\left(\frac{C}{\varepsilon^{1/2}}\left(\frac{1}{T^{1/2}} + M^{1/2}\right)\right)\int_0^T |\varphi_{x|x=L}|^2\,\mathrm{d}t. \tag{5.5}$$

We combine (5.5) with (4.5). This gives, for every $T > 4L/M$,

$$\begin{aligned}\int_0^{L_0} |\varphi_{|t=0}|^2\,\mathrm{d}x \le{}& M\exp\left(-\frac{4(L-L_0)^{3/2}}{3\sqrt{21}\varepsilon^{1/2}T^{1/2}}\right)\int_0^T |\varphi_{|x=L}|^2\,\mathrm{d}t \\ &+ \frac{C}{T}\exp\left(\frac{C(1+M^{1/2}T^{1/2}) - (3\sqrt{7})^{-1}(L-L_0)^{1/2}(MT-4L_0)}{\varepsilon^{1/2}T^{1/2}}\right)\int_0^T |\varphi_{|x=L}|^2\,\mathrm{d}t \\ &+ C\Big(\frac{1}{T} + M\Big)\exp\left(\frac{C(1+M^{1/2}T^{1/2}) - (3\sqrt{7})^{-1}(L-L_0)^{1/2}(MT-4L_0)}{\varepsilon^{1/2}T^{1/2}}\right)\int_0^T |\varphi_{x|x=L}|^2\,\mathrm{d}t.\end{aligned}$$

It suffices now to take $T = K/M$, with $K > 0$ large enough to obtain (1.6).

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Appendix A. Proof of Proposition 5.1

Let us start by noticing that from (5.2) we have

$$C_0 \le T\beta, \quad C_0\beta \le -\beta_x \le C_1\beta, \quad C_0\beta \le -\beta_{xx} \le C_1\beta, \tag{A.1}$$

and

$$|\beta_t| + |\beta_{xt}| + |\beta_{xxt}| \le C_1 T\beta^3, \quad |\beta_{tt}| \le C_1 T^2\beta^5, \tag{A.2}$$

for every $(t,x) \in Q$ and for some positive constants C_0 and C_1 that do not depend on T. As in [11] and [12] we define $\psi := e^{-s\beta}\varphi$. From (3.1) we obtain

$$L_1\psi + L_2\psi = L_3\psi,$$

where

$$L_1\psi := \varepsilon\psi_{xxx} + \psi_t + 3\varepsilon s^2\beta_x^2\psi_x - M\psi_x,$$
$$L_2\psi := (\varepsilon s^3\beta_x^3 + s\beta_t - Ms\beta_x)\psi + 3\varepsilon s\beta_{xx}\psi_x + 3\varepsilon s\beta_x\psi_{xx}$$

and

$$L_3\psi := -3\varepsilon s^2\beta_x\beta_{xx}\psi.$$

Notice that

$$\psi_{|x=0} = 0 \tag{A.3}$$
$$\psi_{x|x=0} = 0. \tag{A.4}$$

Taking the L^2-norm we have

$$\|L_1\psi\|^2_{L^2(Q)} + \|L_2\psi\|^2_{L^2(Q)} + 2(L_1\psi,\, L_2\psi)_{L^2(Q)} = \|L_3\psi\|^2_{L^2(Q)}. \tag{A.5}$$

In what follows we compute the L^2-product. Then, we come back to the original variable φ. This procedure is actually quite standard. The boundary condition at $x = L$ in (3.1) leaves a boundary term that cannot be trivially estimated. The main novelty here is the estimation of the remaining boundary term (see (A.16) below) without the necessity of changing the power of $t(T-t)$ in (5.2) as in [14, Proposition 3]. Let us denote by $(L_i\psi)_j$ the j-th term of $L_i\psi$. We have divided the rest of the proof in four main steps.

Step 1. Computation of the L^2-product.

Step 1.1. Computing $((L_1\psi)_1,\, L_2\psi)_{L^2(Q)}$.

• Let us integrate by parts twice in space the first term:

$$\begin{aligned}
((L_1\psi)_1,\, (L_2\psi)_1)_{L^2(Q)} = &-\frac{1}{2}\varepsilon \iint\limits_Q (\varepsilon s^3\beta_x^3 + s\beta_t - Ms\beta_x)\partial_x|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t\\
&-\varepsilon \iint\limits_Q (3\varepsilon s^3\beta_x^2\beta_{xx} + s\beta_{xt} - Ms\beta_{xx})\psi\psi_{xx}\,\mathrm{d}x\,\mathrm{d}t\\
&+\varepsilon \int_0^T (\varepsilon s^3\beta_x^3 + s\beta_t - Ms\beta_x)_{|x=L}\psi_{|x=L}\psi_{xx|x=L}\,\mathrm{d}t\\
= &\,\frac{3}{2}\varepsilon \iint\limits_Q (3\varepsilon s^3\beta_x^2\beta_{xx} + s\beta_{xt} - Ms\beta_{xx})|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t\\
&-\frac{1}{2}\varepsilon \int_0^T (\varepsilon s^3\beta_x^3 + s\beta_t - Ms\beta_x)_{|x=L}|\psi_{x|x=L}|^2\,\mathrm{d}t\\
&-3\varepsilon^2 s^3 \iint\limits_Q \beta_{xx}^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\varepsilon \int_0^T (6\varepsilon s^3\beta_x\beta_{xx}^2 + s\beta_{xxt})_{|x=L}|\psi_{|x=L}|^2\,\mathrm{d}t
\end{aligned}$$

$$-\varepsilon\int_0^T (3\varepsilon s^3\beta_x^2\beta_{xx} + s\beta_{xt} - Ms\beta_{xx})_{|x=L}\psi_{|x=L}\psi_{x|x=L}\,\mathrm{d}t$$

$$+\varepsilon\int_0^T (\varepsilon s^3\beta_x^3 + s\beta_t - Ms\beta_x)_{|x=L}\psi_{|x=L}\psi_{xx|x=L}\,\mathrm{d}t.$$

From (A.1)–(A.2) and Young's inequality, we obtain

$$((L_1\psi)_1,\,(L_2\psi)_1)_{L^2(Q)} \geq \frac{9}{2}\varepsilon^2 s^3 \iint_Q \beta_x^2\beta_{xx}|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t - C\varepsilon s(T+|M|T^2)\iint_Q \beta^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$-C\varepsilon^2 s^3 T^2 \iint_Q \beta^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t - C\Big(\varepsilon^2 s^3 + \varepsilon s(T+|M|T^2)\Big)\int_0^T \beta^3_{|x=L}|\psi_{x|x=L}|^2\,\mathrm{d}t$$

$$-C\Big(\varepsilon^2 s^5 + \varepsilon^2 s^3 T^2 + \varepsilon^{1/2}s^2(T^{3/2}+|M|^{3/2}T^3) + \varepsilon s T^2(T+|M|T^2)\Big)\int_0^T \beta^5_{|x=L}|\psi_{|x=L}|^2\,\mathrm{d}t$$

$$-C\Big(\varepsilon^2 s + \varepsilon^{3/2}(T^{1/2}+|M|^{1/2}T)\Big)\int_0^T \beta_{|x=L}|\psi_{xx|x=L}|^2\,\mathrm{d}t.$$

- In the second term, we integrate by parts again in space:

$$((L_1\psi)_1,\,(L_2\psi)_2)_{L^2(Q)} = -3\varepsilon^2 s\iint_Q \beta_{xx}|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t + 3\varepsilon^2 s\int_0^T \beta_{xx|x=L}\psi_{x|x=L}\psi_{xx|x=L}\,\mathrm{d}t$$

$$\geq -3\varepsilon^2 s\iint_Q \beta_{xx}|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t - C\varepsilon^2 s\int_0^T \beta_{|x=L}|\psi_{xx|x=L}|^2\,\mathrm{d}t - C\varepsilon^2 sT^2\int_0^T \beta^3_{|x=L}|\psi_{x|x=L}|^2\,\mathrm{d}t.$$

To obtain the above inequality, we have used the properties (A.1) and Young's inequality.

- Analogously for the third term:

$$((L_1\psi)_1,\,(L_2\psi)_3)_{L^2(Q)} = -\frac{3}{2}\varepsilon^2 s\iint_Q \beta_{xx}|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{3}{2}\varepsilon^2 s\int_0^T \beta_{x|x=L}|\psi_{xx|x=L}|^2\,\mathrm{d}t$$

$$-\frac{3}{2}\varepsilon^2 s\int_0^T \beta_{x|x=0}|\psi_{xx|x=0}|^2\,\mathrm{d}t \geq -\frac{3}{2}\varepsilon^2 s\iint_Q \beta_{xx}|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$-\frac{3}{2}\varepsilon^2 s\int_0^T \beta_{x|x=0}|\psi_{xx|x=0}|^2\,\mathrm{d}t - C\varepsilon^2 s\int_0^T \beta_{|x=L}|\psi_{xx|x=L}|^2\,\mathrm{d}t.$$

- Putting together these computations, we obtain

$$
\begin{aligned}
((L_1\psi)_1,\, L_2\psi)_{L^2(Q)} &\geq \frac{9}{2}\varepsilon^2 s^3 \iint_Q \beta_x^2 \beta_{xx} |\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t - \frac{9}{2}\varepsilon^2 s \iint_Q \beta_{xx} |\psi_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&- C\varepsilon s(T+|M|T^2) \iint_Q \beta^3 |\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t - C\varepsilon^2 s^3 T^2 \iint_Q \beta^5 |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&- \frac{3}{2}\varepsilon^2 s \int_0^T \beta_{x|x=0} |\psi_{xx|x=0}|^2 \,\mathrm{d}t - C\Big(\varepsilon^2 s^3 + \varepsilon^2 s T^2 + \varepsilon s(T+|M|T^2)\Big) \int_0^T \beta^3{}_{|x=L} |\psi_{x|x=L}|^2 \,\mathrm{d}t \\
&- C\Big(\varepsilon^2 s^5 + \varepsilon^2 s^3 T^2 + \varepsilon^{1/2} s^2 (T^{3/2} + |M|^{3/2} T^3) + \varepsilon s T^2 (T+|M|T^2)\Big) \int_0^T \beta^5_{|x=L} |\psi_{|x=L}|^2 \,\mathrm{d}t \\
&- C\Big(\varepsilon^2 s + \varepsilon^{3/2}(T^{1/2} + |M|^{1/2} T)\Big) \int_0^T \beta_{|x=L} |\psi_{xx|x=L}|^2 \,\mathrm{d}t.
\end{aligned}
\tag{A.6}
$$

Step 1.2. Computing $((L_1\psi)_2,\, L_2\psi)_{L^2(Q)}$.

- We integrate by parts in time the first term and use (A.1)–(A.2):

$$
\begin{aligned}
((L_1\psi)_2,\, (L_2\psi)_1)_{L^2(Q)} &= -\frac{1}{2} \iint_Q (3\varepsilon s^3 \beta_x^2 \beta_{xt} + s\beta_{tt} - Ms\beta_{xt}) |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\geq -C\Big(\varepsilon s^3 T + s(T^2 + |M|T^3)\Big) \iint_Q \beta^5 |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

- The second term:

$$
((L_1\psi)_2,\, (L_2\psi)_2)_{L^2(Q)} = 3\varepsilon s \iint_Q \beta_{xx} \psi_x \psi_t \,\mathrm{d}x\,\mathrm{d}t.
$$

- In the third term, we integrate by parts first in space and then in time. We obtain, together with (A.1)–(A.2),

$$
\begin{aligned}
((L_1\psi)_2,\, (L_2\psi)_3)_{L^2(Q)} &= -\frac{3}{2}\varepsilon s \iint_Q \beta_x \partial_t |\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t - 3\varepsilon s \iint_Q \beta_{xx} \psi_x \psi_t \,\mathrm{d}x\,\mathrm{d}t \\
&+ 3\varepsilon s \int_0^T \beta_{x|x=L} \psi_{x|x=L} \psi_{t|x=L} \,\mathrm{d}t = \frac{3}{2}\varepsilon s \iint_Q \beta_{xt} |\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&- 3\varepsilon s \iint_Q \beta_{xx} \psi_x \psi_t \,\mathrm{d}x\,\mathrm{d}t + 3\varepsilon s \int_0^T \beta_{x|x=L} \psi_{x|x=L} \psi_{t|x=L} \,\mathrm{d}t \\
&\geq -3\varepsilon s \iint_Q \beta_{xx} \psi_x \psi_t \,\mathrm{d}x\,\mathrm{d}t + 3\varepsilon s \int_0^T \beta_{x|x=L} \psi_{x|x=L} \psi_{t|x=L} \,\mathrm{d}t - C\varepsilon s T \iint_Q \beta^3 |\psi_x|^2.
\end{aligned}
$$

- Putting together these inequalities, we get

$$((L_1\psi)_2,\, L_2\psi)_{L^2(Q)} \geq -C\Big(\varepsilon s^3 T + s(T^2+|M|T^3)\Big)\iint\limits_Q \beta^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$- C\varepsilon s T\iint\limits_Q \beta^3|\psi_x|^2 + 3\varepsilon s\int_0^T \beta_{x|x=L}\psi_{x|x=L}\psi_{t|x=L}\,\mathrm{d}t. \tag{A.7}$$

Step 1.3. Computing $((L_1\psi)_3,\, L_2\psi)_{L^2(Q)}$.

- We integrate by parts in space the first term and from (A.1)–(A.2) we obtain

$$\begin{aligned}
&((L_1\psi)_3,\, (L_2\psi)_1)_{L^2(Q)}\\
&= -\frac{1}{2}\varepsilon\iint\limits_Q (15\varepsilon s^5\beta_x^4\beta_{xx} + 6s^3\beta_x\beta_{xx}\beta_t + 3s^3\beta_x^2\beta_{xt} - 9Ms^3\beta_x^2\beta_{xx})|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&+ \frac{3}{2}\varepsilon\int_0^T(\varepsilon s^5\beta_x^5 + s^3\beta_x^2\beta_t - Ms^3\beta_x^3)_{|x=L}|\psi_{|x=L}|^2\,\mathrm{d}t\\
&\geq \frac{15}{2}C_0^5\varepsilon^2 s^5\iint\limits_Q\beta^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t - C\varepsilon s^3(T+|M|T^2)\iint\limits_Q\beta^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&- C\varepsilon(\varepsilon s^5 + s^3(T+|M|T^2))\int_0^T\beta^5_{|x=L}|\psi_{|x=L}|^2\,\mathrm{d}t.
\end{aligned}$$

- The second term is

$$((L_1\psi)_3,\, (L_2\psi)_2)_{L^2(Q)} = 9\varepsilon^2 s^3\iint\limits_Q\beta_x^2\beta_{xx}|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t.$$

- In the third term, integration by parts in space and (A.1) yield

$$\begin{aligned}
((L_1\psi)_3,\, (L_2\psi)_3)_{L^2(Q)} &= -\frac{27}{2}\varepsilon^2 s^3\iint\limits_Q\beta_x^2\beta_{xx}|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{9}{2}\varepsilon^2 s^3\int_0^T\beta^3_{x|x=L}|\psi_{x|x=L}|^2\,\mathrm{d}t\\
&\geq -\frac{27}{2}\varepsilon^2 s^3\iint\limits_Q\beta_x^2\beta_{xx}|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t - C\varepsilon^2 s^3\int_0^T\beta^3_{|x=L}|\psi_{x|x=L}|^2\,\mathrm{d}t.
\end{aligned}$$

- Putting together these estimates, we get

$$((L_1\psi)_3,\, L_2\psi)_{L^2(Q)} \geq -\frac{9}{2}\varepsilon^2 s^3\iint\limits_Q\beta_x^2\beta_{xx}|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{15}{2}C_0^5\varepsilon^2 s^5\iint\limits_Q\beta^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$- C\varepsilon s^3(T+|M|T^2)\iint\limits_Q\beta^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t - C\varepsilon^2 s^3\int_0^T\beta^3_{|x=L}|\psi_{x|x=L}|^2\,\mathrm{d}t \tag{A.8}$$
$$- C\varepsilon\Big(\varepsilon s^5 + s^3(T+|M|T^2)\Big)\int_0^T\beta^5_{|x=L}|\psi_{|x=L}|^2\,\mathrm{d}t.$$

Step 1.4. Computing $((L_1\psi)_4,\, L_2\psi)_{L^2(Q)}$.

- As before, we integrate by parts in space the first term. From (A.1)–(A.2), this gives

$$\begin{aligned}((L_1\psi)_4,\, (L_2\psi)_1)_{L^2(Q)} =& \frac{M}{2}\iint\limits_Q (3\varepsilon s^3\beta_x^2\beta_{xx} + s\beta_{xt} - Ms\beta_{xx})|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\ &- \frac{M}{2}\int\limits_0^T (\varepsilon s^3\beta_x^3 + s\beta_t - Ms\beta_x)_{|x=L}|\psi_{|x=L}|^2\,\mathrm{d}t\\ &\geq -C(\varepsilon s^3 + s(T+|M|T^2))|M|T^2\iint\limits_Q \beta^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\ &- C(\varepsilon s^3 + s(T+|M|T^2))|M|T^2\int\limits_0^T \beta^5_{|x=L}|\psi_{|x=L}|^2\,\mathrm{d}t.\end{aligned}$$

- From (A.1), the second term gives directly

$$((L_1\psi)_4,\, (L_2\psi)_2)_{L^2(Q)} \geq -C|M|\varepsilon sT^2\iint\limits_Q \beta^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t.$$

- The third and final term gives, after integration by parts and taking (A.1) into account,

$$\begin{aligned}((L_1\psi)_4,\, (L_2\psi)_3)_{L^2(Q)} &= \frac{3}{2}M\varepsilon s\iint\limits_Q \beta_{xx}|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t - \frac{3}{2}M\varepsilon s\int\limits_0^T \beta_{x|x=L}|\psi_{x|x=L}|^2\,\mathrm{d}t\\ &\geq -C|M|\varepsilon sT^2\iint\limits_Q \beta^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t - C|M|\varepsilon sT^2\int\limits_0^T \beta^3_{|x=L}|\psi_{x|x=L}|^2\,\mathrm{d}t.\end{aligned}$$

- Putting together these expressions, we obtain:

$$\begin{aligned}((L_1\psi)_4,\, L_2\psi)_{L^2(Q)} \geq& -C\Big(\varepsilon s^3 + s\big(T+|M|T^2\big)\Big)|M|T^2\iint\limits_Q \beta^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\ &- C|M|\varepsilon sT^2\iint\limits_Q \beta^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t - C|M|\varepsilon sT^2\int\limits_0^T \beta^3_{|x=L}|\psi_{x|x=L}|^2\,\mathrm{d}t\\ &- C(\varepsilon s^3 + s(T+|M|T^2))|M|T^2\int\limits_0^T \beta^5_{|x=L}|\psi_{|x=L}|^2\,\mathrm{d}t.\end{aligned}\tag{A.9}$$

Step 2. The entire product $(L_1\psi,\, L_2\psi)_{L^2(Q)}$.

We gather estimates (A.6)–(A.9). We find the following positive terms:

$$A_1 := \frac{15}{2}C_0^5\varepsilon^2 s^5\iint\limits_Q \beta^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t, \quad A_2 := \frac{9}{2}C_0\varepsilon^2 s\iint\limits_Q \beta|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t$$

and

$$A_3 := \frac{3}{2} C_0 \varepsilon^2 s \int_0^T \beta_{|x=0} |\psi_{xx|x=0}|^2 \, dt.$$

Now, let us estimate the nonpositive integrals coming from the addition of (A.6)–(A.9) in terms of A_i.

We begin with the terms concerning $|\psi|^2$ in Q. They are all bounded by

$$C\Big(s^3\big(\varepsilon T + \varepsilon^2 T^2 + |M|\varepsilon T^2\big) + s\big(T^2 + |M|T^3 + |M|^2 T^4\big)\Big) \iint_Q \beta^5 |\psi|^2 \, dx \, dt$$

which are absorbed by A_1 by taking $s \geq C(T + T^{1/2}\varepsilon^{-1/2} + |M|^{1/2}\varepsilon^{-1/2} T)$.

With $s \geq C(T + T^{1/2}\varepsilon^{-1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$, the integrals concerning $|\psi_{|x=L}|^2$, $|\psi_{x|x=L}|^2$ and $|\psi_{xx|x=L}|^2$, can be bounded by

$$C\varepsilon^2 \int_0^T \Big(s\beta_{|x=L} |\psi_{xx|x=L}|^2 + s^3 \beta^3_{|x=L} |\psi_{x|x=L}|^2 + s^5 \beta^5_{|x=L} |\psi_{|x=L}|^2 \Big) \, dt. \tag{A.10}$$

Now, we deal with the terms containing $|\psi_x|^2$ in Q. They can all be estimated by

$$C\varepsilon s(T + |M|T^2) \iint_Q \beta^3 |\psi_x|^2 \, dx \, dt.$$

Integration by parts in space shows that

$$\begin{aligned}
C\varepsilon s(T + |M|T^2) \iint_Q \beta^3 |\psi_x|^2 \, dx \, dt &= \frac{3}{2} C\varepsilon s(T + |M|T^2) \iint_Q (2\beta\beta_x^2 + \beta^2 \beta_{xx}) |\psi|^2 \, dx \, dt \\
&- \frac{3}{2} C\varepsilon s(T + |M|T^2) \int_0^T \beta^2_{|x=L} \beta_{x|x=L} |\psi_{|x=L}|^2 \, dt - C\varepsilon s(T + |M|T^2) \iint_Q \beta^3 \psi \psi_{xx} \, dx \, dt \\
&+ C\varepsilon s(T + |M|T^2) \int_0^T \beta^3_{|x=L} \psi_{|x=L} \psi_{x|x=L} \, dt \\
&\geq -C\big(\varepsilon s T^2 (T + |M|T^2) + \varepsilon^{1/2} s^2 (T^{3/2} + |M|^{3/2} T^3)\big) \iint_Q \beta^5 |\psi|^2 \, dx \, dt \\
&- C\varepsilon^{3/2} (T^{1/2} + |M|^{1/2} T) \iint_Q \beta |\psi_{xx}|^2 \, dx \, dt - C\varepsilon\big(sT^2(T + |M|T^2)\big) \int_0^T \beta^5_{|x=L} |\psi_{|x=L}|^2 \, dt \\
&- C\varepsilon s(T + |M|T^2) \int_0^T \beta^3_{|x=L} |\psi_{x|x=L}|^2 \, dt.
\end{aligned} \tag{A.11}$$

Notice that here we have used (A.4) and Young's inequality. Taking $s \geq C(T + \varepsilon^{-1/2} T^{1/2} + |M|^{1/2} \varepsilon^{-1/2} T)$, the first two integrals can be absorbed by A_1 and A_2, respectively, and the two remaining can be estimated by (A.10).

Finally, all these estimates give

$$\begin{aligned}(L_1\psi,\, L_2\psi)_{L^2(Q)} \geq{}& C_0\varepsilon^2 s^5 \iint\limits_Q \beta^5|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + C_0\varepsilon^2 s \iint\limits_Q \beta|\psi_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &+ C_0\varepsilon^2 s \int_0^T \beta_{|x=0}|\psi_{xx|x=0}|^2 \,\mathrm{d}t + 3\varepsilon s \int_0^T \beta_{x|x=L}\psi_{x|x=L}\psi_{t|x=L} \,\mathrm{d}t \\ &- C\varepsilon^2 \int_0^T \left(s\beta_{|x=L}|\psi_{xx|x=L}|^2 + s^3\beta^3_{|x=L}|\psi_{x|x=L}|^2 + s^5\beta^5_{|x=L}|\psi_{|x=L}|^2 \right) \mathrm{d}t\end{aligned}$$

for every $s \geq C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$. Coming back to (A.5), and together with

$$\|L_3\psi\|^2_{L^2(Q)} \leq C\varepsilon^2 s^4 T \iint\limits_Q \beta^5|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t,$$

we obtain the following inequality for ψ:

$$\begin{aligned}\varepsilon^2 s^5 \iint\limits_Q \beta^5|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t &+ \varepsilon^2 s \iint\limits_Q \beta|\psi_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s \int_0^T \beta_{|x=0}|\psi_{xx|x=0}|^2 \,\mathrm{d}t \\ &\leq C\varepsilon^2 \int_0^T \left(s\beta_{|x=L}|\psi_{xx|x=L}|^2 + s^3\beta^3_{|x=L}|\psi_{x|x=L}|^2 + s^5\beta^5_{|x=L}|\psi_{|x=L}|^2 \right) \mathrm{d}t \\ &\qquad - 3\varepsilon s \int_0^T \beta_{x|x=L}\psi_{x|x=L}\psi_{t|x=L} \,\mathrm{d}t \end{aligned} \tag{A.12}$$

for every $s \geq C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$.

At this point, there are two tasks left to do: first, to go back to the original variable φ; and second, to estimate the last boundary term in (A.12). This is the purpose of the following pages.

Step 3. Coming back to the original variable.

Let us now retrieve the original variable φ. First, we point out that the same computations made in (A.11) yield

$$\begin{aligned}\varepsilon^2 s^3 \iint\limits_Q \beta^3|\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t \leq{}& C\varepsilon^2 s^5 \iint\limits_Q \beta^5|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + C\varepsilon^2 s \iint\limits_Q \beta|\psi_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &+ C\varepsilon^2 \int_0^T \left(s^3\beta^3_{|x=L}|\psi_{x|x=L}|^2 + s^5\beta^5_{|x=L}|\psi_{|x=L}|^2 \right) \mathrm{d}t\end{aligned}$$

as long as $s \geq CT$. This means that we can add this term to the left-hand side of (A.12), and together with $\psi = e^{-s\beta}\varphi$, we obtain directly that

$$\varepsilon^2 s^5 \iint\limits_Q e^{-2s\beta}\beta^5|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s^3 \iint\limits_Q \beta^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s \iint\limits_Q \beta|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \varepsilon^2 s \int_0^T \beta_{|x=0}|\psi_{xx|x=0}|^2\,\mathrm{d}t \le -3\varepsilon s \int_0^T \beta_{x|x=L}\psi_{x|x=L}\psi_{t|x=L}\,\mathrm{d}t$$
$$+ C\varepsilon^2 \int_0^T \left(s\beta_{|x=L}|\psi_{xx|x=L}|^2 + s^3\beta^3_{|x=L}|\psi_{x|x=L}|^2 + s^5 e^{-2s\beta_{|x=L}}\beta^5_{|x=L}|\varphi_{|x=L}|^2\right)\mathrm{d}t \tag{A.13}$$

for every $s \ge C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$.

Taking the derivative with respect to x in $\psi = e^{-s\beta}\varphi$, we find that

$$s^{3/2}e^{-s\beta}\beta^{3/2}\varphi_x = s^{3/2}\beta^{3/2}\psi_x + s^{5/2}e^{-s\beta}\beta^{3/2}\beta_x\varphi, \tag{A.14}$$

and taking the $L^2(Q)$-norm, we see that we can add

$$\varepsilon^2 s^3 \iint\limits_Q e^{-2s\beta}\beta^3|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t$$

to the left-hand side of (A.13) (recall (A.1)). Similarly, from (A.1) and

$$s^{1/2}e^{-s\beta}\beta^{1/2}\varphi_{xx} = s^{1/2}\beta^{1/2}\psi_{xx} + 2s^{3/2}\beta^{1/2}\beta_x\psi_x + e^{-s\beta}\beta^{1/2}(s^{3/2}\beta^{1/2}\beta_{xx} + s^{5/2}\beta_x^2)\varphi, \tag{A.15}$$

we can add

$$\varepsilon^2 s \iint\limits_Q e^{-2s\beta}\beta|\varphi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t$$

to the left-hand side of (A.13) if $s \ge CT$. Taking the value at $x = L$ in (A.14) and (A.15), we obtain from (A.13)

$$\varepsilon^2 s^5 \iint\limits_Q e^{-2s\beta}\beta^5|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s^3 \iint\limits_Q e^{-2s\beta}\beta^3|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s \iint\limits_Q e^{-2s\beta}\beta|\varphi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \varepsilon^2 s \int_0^T e^{-2s\beta_{|x=0}}\beta_{|x=0}|\varphi_{xx|x=0}|^2\,\mathrm{d}t \le -3\varepsilon s \int_0^T \beta_{x|x=L}\psi_{x|x=L}\psi_{t|x=L}\,\mathrm{d}t$$
$$+ C\varepsilon^2 \int_0^T e^{-2s\beta_{|x=L}}\left(s\beta_{|x=L}|\varphi_{xx|x=L}|^2 + s^3\beta^3_{|x=L}|\varphi_{x|x=L}|^2 + s^5\beta^5_{|x=L}|\varphi_{|x=L}|^2\right)\mathrm{d}t$$

for every $s \ge C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$. Since $\varphi_{xx|x=L} = M\varepsilon^{-1}\varphi_{|x=L}$, we get

$$\varepsilon^2 s^5 \iint\limits_Q e^{-2s\beta}\beta^5|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s^3 \iint\limits_Q e^{-2s\beta}\beta^3|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s \iint\limits_Q e^{-2s\beta}\beta|\varphi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \varepsilon^2 s \int_0^T e^{-2s\beta_{|x=0}}\beta_{|x=0}|\varphi_{xx\,|x=0}|^2\,\mathrm{d}t \le -3\varepsilon s \int_0^T \beta_{x\,|x=L}\psi_{x\,|x=L}\psi_{t\,|x=L}\,\mathrm{d}t$$
$$+ C\varepsilon^2 \int_0^T e^{-2s\beta_{|x=L}}\left(s^3\beta^3_{|x=L}|\varphi_{x\,|x=L}|^2 + s^5\beta^5_{|x=L}|\varphi_{|x=L}|^2\right)\mathrm{d}t \tag{A.16}$$

for every $s \ge C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$.

Step 4. Estimation of the remaining boundary term and conclusion.

Finally, we are in position to estimate the boundary term

$$B := 3\varepsilon s \int_0^T \beta_{x\,|x=L}\psi_{x\,|x=L}\psi_{t\,|x=L}\,\mathrm{d}t.$$

The idea is to write B in terms of φ and use the equation in (3.1). The same idea was used in [14], but here we do not need to change the power of $t(T-t)$ in (5.2).

As before, it is easy to see from $\psi = e^{-s\beta}\varphi$ that

$$B = 3\varepsilon s^3 \int_0^T \left(e^{-2s\beta}\beta_t\beta_x^2\right)_{|x=L}|\varphi_{|x=L}|^2\,\mathrm{d}t - 3\varepsilon s^2 \int_0^T \left(e^{-2s\beta}\beta_t\beta_x\right)_{|x=L}\varphi_{|x=L}\varphi_{x\,|x=L}\,\mathrm{d}t$$
$$+ \frac{3}{2}\varepsilon s^2 \int_0^T \left(e^{-2s\beta}{\beta_x}^2\right)_{t\,|x=L}|\varphi_{|x=L}|^2\,\mathrm{d}t + 3\varepsilon s \int_0^T \left(e^{-2s\beta}\beta_x\right)_{|x=L}\varphi_{x\,|x=L}\varphi_{t\,|x=L}\,\mathrm{d}t.$$

From (A.1)–(A.2) and Young's inequality, we see that

$$B \le C\varepsilon^2 \int_0^T e^{-2s\beta_{|x=L}}\left(s^3\beta^3_{|x=L}|\varphi_{x\,|x=L}|^2 + s^5\beta^5_{|x=L}|\varphi_{|x=L}|^2\right)\mathrm{d}t + 3\varepsilon s \int_0^T e^{-2s\beta_{|x=L}}\beta_{x\,|x=L}\varphi_{x\,|x=L}\varphi_{t\,|x=L}\,\mathrm{d}t \tag{A.17}$$

for every $s \ge C(T + \varepsilon^{-1/2}T^{1/2})$. Now, the last term in this inequality can be estimated as follows:

$$\widetilde{B} := 3\varepsilon s \int_0^T e^{-2s\beta_{|x=L}}\beta_{x\,|x=L}\varphi_{x\,|x=L}\varphi_{t\,|x=L}\,\mathrm{d}t$$
$$\le C\varepsilon s^3 \int_0^T e^{-2s(2\beta_{|x=L}-\beta_{|x=0})}\beta^2_{x\,|x=L}\beta^2_{|x=0}|\varphi_{x\,|x=L}|^2\,\mathrm{d}t + \varepsilon s^{-1}\int_0^T \beta^{-2}_{|x=0}e^{-2s\beta_{|x=0}}|\varphi_{t\,|x=L}|^2\,\mathrm{d}t$$
$$\le C\varepsilon^2 s^5 \int_0^T e^{-2s(2\beta_{|x=L}-\beta_{|x=0})}\beta^5_{|x=L}|\varphi_{x\,|x=L}|^2\,\mathrm{d}t + \varepsilon s^{-1}\int_0^T \beta^{-2}_{|x=0}e^{-2s\beta_{|x=0}}|\varphi_{t\,|x=L}|^2\,\mathrm{d}t \tag{A.18}$$

for every $s \ge C\varepsilon^{-1/2}T^{1/2}$. To estimate the last term in this inequality, we multiply the equation on (3.1) by $4\varepsilon s^{-1}\beta^{-2}_{x=0}\exp(-2s\beta_{|x=0})\varphi_t$ and integrate in Q. Integration by parts in both space and time yield

$$2\varepsilon s^{-1}\int_0^T e^{-2s\beta_{|x=0}}\beta_{|x=0}^{-2}|\varphi_{t|x=L}|^2\,\mathrm{d}t = -4\varepsilon^2 s^{-1}\int_0^T e^{-2s\beta_{|x=0}}\beta_{|x=0}^{-2}\varphi_{xx|x=L}\varphi_{xt|x=L}\,\mathrm{d}t$$
$$-2\varepsilon^2 s^{-1}\iint_Q \left(e^{-2s\beta_{|x=0}}\beta_{|x=0}^{-2}\right)_t|\varphi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t - 2\varepsilon s^{-1}M\iint_Q \left(e^{-2s\beta_{|x=0}}\beta_{|x=0}^{-2}\right)_t|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{A.19}$$

Since $\varphi_{xx|x=L} = M\varepsilon^{-1}\varphi_{|x=L}$, the first term in the right-hand side of (A.19) satisfies

$$\begin{aligned}
&-4\varepsilon^2 s^{-1}\int_0^T e^{-2s\beta_{|x=0}}\beta_{|x=0}^{-2}\varphi_{xx|x=L}\varphi_{xt|x=L}\,\mathrm{d}t \\
&= -4\varepsilon s^{-1}M\int_0^T e^{-2s\beta_{|x=0}}\beta_{|x=0}^{-2}\varphi_{|x=L}\varphi_{xt|x=L}\,\mathrm{d}t \\
&= 4\varepsilon s^{-1}M\int_0^T \left(e^{-2s\beta_{|x=0}}\beta_{|x=0}^{-2}\right)_t\varphi_{|x=L}\varphi_{x|x=L}\,\mathrm{d}t + 4\varepsilon s^{-1}M\int_0^T e^{-2s\beta_{|x=0}}\beta_{|x=0}^{-2}\varphi_{t|x=L}\varphi_{x|x=L}\,\mathrm{d}t.
\end{aligned}$$

Using (A.1)–(A.2) and Young's inequality, we can obtain

$$\begin{aligned}
&-4\varepsilon^2 s^{-1}\int_0^T e^{-2s\beta_{|x=0}}\beta_{|x=0}^{-2}\varphi_{xx|x=L}\varphi_{xt|x=L}\,\mathrm{d}t \le \varepsilon s^{-1}\int_0^T e^{-2s\beta_{|x=0}}\beta_{|x=0}^{-2}|\varphi_{t|x=L}|^2\,\mathrm{d}t \\
&+ C\varepsilon s^{-1}M^2\int_0^T e^{-2s\beta_{|x=L}}\beta_{|x=L}^{-2}|\varphi_{x|x=L}|^2\,\mathrm{d}t + C\varepsilon^2 s^3\int_0^T e^{-2s\beta_{|x=L}}\beta_{|x=L}^{3}|\varphi_{x|x=L}|^2\,\mathrm{d}t \\
&+ C\varepsilon^2 s^5\int_0^T e^{-2s\beta_{|x=L}}\beta_{|x=L}^{5}|\varphi_{|x=L}|^2\,\mathrm{d}t,
\end{aligned} \tag{A.20}$$

for every $s \ge C(T + |M|^{1/2}\varepsilon^{-1/2}T)$. Notice that we also have used that $\exp(-2s\beta(t,\cdot))$ reaches its minimum at $x = 0$. Using again (A.1)–(A.2) for the two last terms of (A.19), they are estimated by

$$C\varepsilon^2(s^{-1}T^2 + T)\iint_Q e^{-2s\beta}\beta|\varphi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t + C\varepsilon\left(s^{-1}|M|T^4 + |M|T^3\right)\iint_Q e^{-2s\beta}\beta^3|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Therefore, taking into account (A.20) and coming back to (A.19), we obtain

$$\begin{aligned}
&\varepsilon s^{-1}\int_0^T e^{-2s\beta_{|x=L}}\beta_{|x=L}^{-2}|\varphi_{t|x=L}|^2\,\mathrm{d}t \le C\varepsilon s^{-1}M^2\int_0^T e^{-2s\beta_{|x=L}}\beta_{|x=L}^{-2}|\varphi_{x|x=L}|^2\,\mathrm{d}t \\
&+ C\varepsilon^2 s^3\int_0^T e^{-2s\beta_{|x=L}}\beta_{|x=L}^{3}|\varphi_{x|x=L}|^2\,\mathrm{d}t + C\varepsilon^2 s^5\int_0^T e^{-2s\beta_{|x=L}}\beta_{|x=L}^{5}|\varphi_{|x=L}|^2\,\mathrm{d}t \\
&+ C\varepsilon^2(s^{-1}T^2 + T)\iint_Q e^{-2s\beta}\beta|\varphi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t + C\varepsilon\left(s^{-1}|M|T^4 + |M|T^3\right)\iint_Q e^{-2s\beta}\beta^3|\varphi_x|^2\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

for every $s \geq C(T+|M|^{1/2}\varepsilon^{-1/2}T)$. Since this is the last term in (A.18), we plug the estimate of $\widetilde{B}$ in (A.17) and get

$$B \leq C\varepsilon^2 s^3 \int_0^T e^{-2s\beta_{|x=L}} \beta^3_{|x=L} |\varphi_{x|x=L}|^2 \,\mathrm{d}t + C\varepsilon^2 s^5 \int_0^T e^{-2s\beta_{|x=L}} \beta^5_{|x=L} |\varphi_{|x=L}|^2 \,\mathrm{d}t$$
$$+ C\varepsilon^2 s^5 \int_0^T e^{-2s(2\beta_{|x=L}-\beta_{|x=0})} \beta^5_{|x=L} |\varphi_{x|x=L}|^2 \,\mathrm{d}t + C\varepsilon s^{-1} M^2 \int_0^T e^{-2s\beta_{|x=L}} \beta^{-2}_{|x=L} |\varphi_{x|x=L}|^2 \,\mathrm{d}t$$
$$+ C\varepsilon^2 (s^{-1}T^2 + T) \iint_Q e^{-2s\beta} \beta |\varphi_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}t + C\varepsilon\big(s^{-1}|M|T^4 + |M|T^3\big) \iint_Q e^{-2s\beta} \beta^3 |\varphi_x|^2 \,\mathrm{d}x\,\mathrm{d}t$$

for every $s \geq C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$. Notice that since $2\beta_{|x=L} - \beta_{|x=0} < \beta_{|x=L}$ (recall that β is decreasing), we have the more compact form

$$B \leq C\varepsilon^2 (s^{-1}T^2 + T) \iint_Q e^{-2s\beta} \beta |\varphi_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$+ C\varepsilon\big(s^{-1}MT^4 + MT^3\big) \iint_Q e^{-2s\beta} \beta^3 |\varphi_x|^2 \,\mathrm{d}x\,\mathrm{d}t + C\varepsilon^2 s^5 \int_0^T e^{-2s\beta_{|x=L}} \beta^5_{|x=L} |\varphi_{|x=L}|^2 \,\mathrm{d}t$$
$$+ C\varepsilon^2 s^5 (1+|M|T) \int_0^T e^{-2s(2\beta_{|x=L}-\beta_{|x=0})} \beta^5_{|x=L} |\varphi_{x|x=L}|^2 \,\mathrm{d}t$$

for every $s \geq C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$.

Finally, we can use this estimate of B in (A.16). The global terms of φ_x and φ_{xx} are absorbed by the left-hand side of (A.16) by taking $s \geq C(T+|M|^{1/2}\varepsilon^{-1/2}T)$ and we deduce

$$\varepsilon^2 s^5 \iint_Q e^{-2s\beta} \beta^5 |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s^3 \iint_Q e^{-2s\beta} \beta^3 |\varphi_x|^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s \iint_Q e^{-2s\beta} \beta |\varphi_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \varepsilon^2 s \int_0^T e^{-2s\beta_{|x=0}} \beta_{|x=0} |\varphi_{xx|x=0}|^2 \,\mathrm{d}t \leq C\varepsilon^2 s^5 \int_0^T e^{-2s\beta_{|x=L}} \beta^5_{|x=L} |\varphi_{|x=L}|^2 \,\mathrm{d}t$$
$$+ C\varepsilon^2 s^5 (1+|M|T) \int_0^T e^{-2s(2\beta_{|x=L}-\beta_{|x=0})} \beta^5_{|x=L} |\varphi_{x|x=L}|^2 \,\mathrm{d}t$$

for every $s \geq C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$. The proof of Proposition 5.1 is complete.

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