Insensitizing controls for the Boussinesq system with no control on the temperature equation

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Abstract

In this paper we prove the existence of controls insensitizing the L^2-norm of the solution of the Boussinesq system. The novelty here is that no control is used on the temperature equation. Furthermore, the control acting on the fluid equation can be chosen to have one vanishing component. It is well known that the insensitizing control problem is equivalent to a null controllability result for a cascade system, which is obtained thanks to a suitable Carleman estimate for the adjoint of the linearized system and an inverse mapping theorem. The particular form of the adjoint equation will allow us to obtain the null controllability of the linearized system.

Keywords: Navier-Stokes system, Boussinesq system, null controllability, Carleman inequalities, insensitizing controls

1 Introduction

Let Ω be a nonempty bounded connected open subset of $\mathbb{R}^N$ $(N = 2$ or $3)$ of class $\mathcal{C}^\infty$. Let $T > 0$ and let $\omega \subset \Omega$ be a (small) nonempty open subset which is the *control set* and $\mathcal{O}_1, \mathcal{O}_2 \subset \Omega$ which are called the *observatories* or *observation sets*. Throughout this paper, we will use the notation $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$.

Let us recall the definition of some usual spaces in the context of incompressible fluids:

$$\mathcal{V} = \{y \in \mathcal{C}_0^\infty(\Omega)^N \,:\, \nabla \cdot y = 0 \text{ in } \Omega\}.$$

We denote by H the closure of the space $\mathcal{V}$ in $L^2(\Omega)$ and by V its closure in $H_0^1(\Omega)$.

We consider the Boussinesq system

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = f + v\mathbb{1}_\omega + \theta\, e_N, \quad \nabla\cdot y = 0 & \text{in } Q,\\ \theta_t - \Delta\theta + y\cdot\nabla\theta = f_0 & \text{in } Q,\\ y = 0, \quad \theta = 0 & \text{on } \Sigma,\\ y(0) = y^0 + \tau\widehat{y}_0, \quad \theta(0) = \theta^0 + \tau\widehat{\theta}_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where

$$e_N = \begin{cases} (0,1) & \text{if } N = 2,\\ (0,0,1) & \text{if } N = 3, \end{cases}$$

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stands for the gravity vector field, $y = y(x,t)$ represents the velocity of the fluid at the point x and time t, $\theta = \theta(x,t)$ their temperature. Moreover, $v = v(x,t)$ stands for the control which acts only over the set ω, $(f, f_0) \in L^2(Q)^{N+1}$ are given externally applied forces and the initial state $(y(0), \theta(0))$ is partially unknown in the following sense:

- $y^0 \in H$ and $\theta^0 \in L^2(\Omega)$ are known,
- $\widehat{y}_0 \in H$ and $\widehat{\theta}_0 \in L^2(\Omega)$ are unknown with $\|\widehat{y}_0\|_{L^2(\Omega)^N} = \|\widehat{\theta}_0\|_{L^2(\Omega)} = 1$, and
- τ is a small unknown real number.

As it was introduced by J.-L. Lions in [24], we are interested in *insensitizing* the functional

$$J_\tau(y,\theta) := \frac{1}{2}\iint_{\mathcal{O}_1\times(0,T)} |y|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\iint_{\mathcal{O}_2\times(0,T)} |\theta|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{1.2}$$

In this context, this means to find a control $v \in L^2(\omega \times (0,T))$ such that

$$\left.\frac{\partial J_\tau(y,\theta)}{\partial \tau}\right|_{\tau=0} = 0 \quad \forall\,(\widehat{y}_0, \widehat{\theta}_0) \in L^2(\Omega)^{N+1} \text{ such that } \|\widehat{y}_0\|_{L^2(\Omega)^N} = \|\widehat{\theta}_0\|_{L^2(\Omega)} = 1, \tag{1.3}$$

that is, that the uncertainty of the initial condition is not perceived by the observation made by the functional.

The first results concerning the existence of insensitizing controls were obtained for the heat equation in [5, 28]. Later on, in the papers [6, 7, 8] the authors deal with different types of nonlinearities in the equation and/or the boundary conditions. In [16], the existence of controls insensitizing the gradient of the solution is established. As long as insensitizing controls for fluid equations are concerned, we can mention [17] for the Stokes system and [18, 12] for the Navier-Stokes equations. In particular, [12] obtains insensitizing vector controls with one component equal to zero. This result was later extended to the Boussinesq system (1.1), where the authors proved the existence of insensitizing controls for (1.1) with v having up to two vanishing components and a control in the temperature equation.

This paper is thought to be a complement to [11]. Here, we are interested in insensitizing controls acting only on the fluid equation. Furthermore, we will see that this control can be chosen to have one vanishing component.

It is well known that (1.3) is equivalent to the (partial) null controllability of the cascade system (see for instance [5]):

$$\begin{cases} w_t - \Delta w + (w\cdot\nabla)w + \nabla p_0 = f + v\mathbb{1}_\omega + r\,e_N, & \nabla\cdot w = 0 \quad \text{in } Q,\\ -z_t - \Delta z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + q\nabla r + \nabla p_1 = w\mathbb{1}_{\mathcal{O}_1}, & \nabla\cdot z = 0 \quad \text{in } Q,\\ r_t - \Delta r + w\cdot\nabla r = f_0 & \text{in } Q,\\ -q_t - \Delta q - w\cdot\nabla q = z_N + r\,\mathbb{1}_{\mathcal{O}_2} & \text{in } Q,\\ w = z = 0, \quad r = q = 0 & \text{on } \Sigma,\\ w(0) = y^0, \quad z(T) = 0, \quad r(0) = \theta^0, \quad q(T) = 0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

that is, we look for a control $v \in L^2(\omega\times(0,T))^N$, with $v_i \equiv 0$ for a given $i \in \{1,\dots,N-1\}$, such that $z(0) = 0$ and $q(0) = 0$ in Ω. Indeed, for every $\widehat{y}_0 \in L^2(\Omega)^N$ and every $\widehat{\theta}_0 \in L^2(\Omega)$ we have

$$\left.\frac{\partial J_\tau(y,\theta)}{\partial \tau}\right|_{\tau=0} = \iint_{\mathcal{O}\times(0,T)} \left(w\cdot y^\tau + r\,\theta^\tau\right) \mathrm{d}x\,\mathrm{d}t, \tag{1.5}$$

where $w := y_{|\tau=0}$, $r := \theta_{|\tau=0}$, $y^\tau := \dfrac{\partial y}{\partial \tau}_{|\tau=0}$ and $\theta^\tau := \dfrac{\partial \theta}{\partial \tau}_{|\tau=0}$. In fact, (y^τ, θ^τ) is the solution of

$$\begin{cases} y_t^\tau - \Delta y^\tau + (y^\tau \cdot \nabla) w + (w \cdot \nabla) y^\tau + \nabla p^\tau = \theta^\tau e_N, \quad \nabla \cdot y^\tau = 0 & \text{in } Q, \\ \theta_t^\tau - \Delta \theta^\tau + (y^\tau \cdot \nabla) r + (w \cdot \nabla) \theta^\tau = 0 & \text{in } Q, \\ y^\tau = 0, \quad \theta^\tau = 0 & \text{on } \Sigma, \\ y^\tau(0) = \widehat{y}_0, \quad \theta^\tau(0) = \widehat{\theta}_0 & \text{in } \Omega. \end{cases}$$

Using (1.4)-(1.5), we find that

$$\left. \frac{\partial J_\tau(y,\theta)}{\partial \tau} \right|_{\tau=0} = \int_\Omega \left(z(0) \cdot \widehat{y}_0 + q(0) \widehat{\theta}_0 \right) \mathrm{d}x$$

for all $(\widehat{y}^0, \widehat{\theta}_0) \in L^2(\Omega)^{N+1}$ such that $\|\widehat{y}_0\|_{L^2(\Omega)^N} = \|\widehat{\theta}_0\|_{L^2(\Omega)^N} = 1$, from where we can conclude.

The null controllability of system (1.4) is the main result of this paper.

Theorem 1.1. *Let $i \in \{1, \dots, N-1\}$ and $m \geq 10$ be a real number. Assume that $\omega \cap \mathcal{O}_1 \neq \emptyset$, $y^0 \equiv 0$ and $\theta^0 \equiv 0$. Then, there exist $\delta > 0$ and $C > 0$, depending on ω, Ω, $\mathcal{O}_1$, $\mathcal{O}_2$ and T, such that for any $f \in L^2(Q)^N$ and any $f_0 \in L^2(Q)$ satisfying $\|e^{C/t^m}(f, f_0)\|_{L^2(Q)^{N+1}} < \delta$, there exists a control $v \in L^2(Q)^N$ with $v_i \equiv 0$ such that the corresponding solution (w, z, r, q, v) to (1.4) satisfies $z(0) = 0$ and $q(0) = 0$ in Ω.*

Remark 1.2. *A related problem to the null controllability of* (1.4) *is the usual null controllability of* (1.1) *(with $\tau = 0$, $f \equiv 0$ and $f_0 \equiv 0$). Our method does not seem to allow to give a positive answer to this problem when no control is acting on the heat equation, even if we do not seek to remove one of the components of the control in the fluid equation (see [9, Remark 3]). Therefore, it is quite surprising that for system* (1.4) *a result of this kind can be obtained.*

In the case $N = 2$, the controllability result of Theorem 1.1 is optimal in the sense that we are able to consider only one scalar control. It is natural to ask if, in the case $N = 3$, a result of this type can be achieved with just one controlled equation. Unfortunately, the method used here does not provide an answer to this question. However, one could try to adapt the arguments used in [14], where the authors proved the local null controllability of the 3-dimensional Navier-Stokes system by means of a control having two vanishing components using Coron's return method [13]. Of course, this is a more difficult problem since the number of equations in (1.4) is much higher than the ones in [14].

As a corollary of Theorem 1.1, we obtain the existence of insensitizing controls for the Boussinesq system (1.1):

Corollary 1.3. *Under the hypothesis of Theorem 1.1, there exist insensitizing controls for the functional* (1.2) *with no control in the temperature equation in* (1.1)*. Furthermore, this control can be chosen to have the i-th component equal to zero, as long as $i \neq N$.*

Our method of proof does not allow to drop the assumption $\omega \cap \mathcal{O}_1 \neq \emptyset$. However, it has been proved in [21] that this is not a necessary condition for ε-insensitivity to hold for some parabolic equations (see also [26]). In this subject, it has been discovered in [4] that new phenomena arises in parabolic systems where the control and coupling sets do not meet. In particular, there exists a minimal time of controllability depending on the location of these sets. Therefore, condition $\omega \cap \mathcal{O}_1 \neq \emptyset$ seems natural for controllability in arbitrary time $T > 0$. We refer to [2, 3] for controllability results concerning this geometric condition in the context of hyperbolic systems.

The hypothesis on the initial conditions is related to the fact that we deal with a system that mixes forward and backward equations. Even for the simpler case of the heat equation, it is a hard task to characterize the initial conditions that can be insensitized. In [28] and [29], the authors analyze the initial data that can or cannot be insensitized under different configurations of the observatory and control sets.

Thus, we concentrate in proving Theorem 1.1. The strategy is classical: first, we study the null controllability of the linearized system around zero:

$$\begin{cases} w_t - \Delta w + \nabla p_0 = f^w + v\mathbb{1}_\omega + r\, e_N, & \nabla\cdot w = 0 \quad \text{in } Q,\\ -z_t - \Delta z + \nabla p_1 = f^z + w\mathbb{1}_{\mathcal{O}_1}, & \nabla\cdot z = 0 \quad \text{in } Q,\\ r_t - \Delta r = f^r & \text{in } Q,\\ -q_t - \Delta q = f^q + z_N + r\,\mathbb{1}_{\mathcal{O}_2} & \text{in } Q,\\ w = z = 0, \quad r = q = 0 & \text{on } \Sigma,\\ w(0) = y^0, \quad z(T) = 0, \quad r(0) = \theta^0, \quad q(T) = 0 & \text{in } \Omega, \end{cases} \tag{1.6}$$

where f^w, f^z, f^r and f^q will be taken to decrease exponentially to zero at $t = 0$, and then we go back to the nonlinear problem by means of an inverse mapping theorem. The null controllability of (1.6) is to be understood in the sense of Theorem 1.1. The main tool to achieve the null controllability of (1.6), and the second main result of this paper (Proposition 3.2), is an *observability inequality* like

$$\begin{aligned} \iint_Q e^{-C_0/t^m}(|\varphi|^2 + |\psi|^2 + |\phi|^2 + |\sigma|^2)\,\mathrm{d}x\,\mathrm{d}t &\le C\big\|e^{-C_1/2t^m}(g^\varphi, g^\psi, g^\phi, g^\sigma)\big\|_X^2\\ &+ C\iint_{\omega\times(0,T)} e^{-C_1/t^m}\big((N-2)|\varphi_j|^2 + |\varphi_N|^2\big)\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{1.7}$$

with $j \in \{1,\dots,N-1\}\setminus\{i\}$, for the solutions of the adjoint equation with source terms:

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_1 = g^\varphi + \psi\,\mathbb{1}_{\mathcal{O}_1}, & \nabla\cdot\varphi = 0 \quad \text{in } Q,\\ \psi_t - \Delta\psi + \nabla\pi_2 = g^\psi + \sigma e_N, & \nabla\cdot\psi = 0 \quad \text{in } Q,\\ -\phi_t - \Delta\phi = g^\phi + \varphi_N + \sigma\,\mathbb{1}_{\mathcal{O}_2} & \text{in } Q,\\ \sigma_t - \Delta\sigma = g^\sigma & \text{in } Q,\\ \varphi = \psi = 0, \quad \phi = \sigma = 0 & \text{on } \Sigma,\\ \varphi(T) = 0, \quad \psi(0) = \psi^0, \quad \phi(T) = 0, \quad \sigma(0) = \sigma^0 & \text{in } \Omega. \end{cases} \tag{1.8}$$

Here $\psi^0 \in H$, $\sigma^0 \in L^2(\Omega)$, X is an appropriate Banach space and $2C_1 > C_0$.

There are some remarks to be made about inequality (1.7). Notice that are not local terms of ϕ present in the right-hand side, which is the main difference with the result obtained in [11] (see inequality (1.7) in that reference). Actually, if there were such local terms, it would not be possible to estimate them since ϕ does not interact as a right-hand side with the other equations. Therefore, we need a new approach to manipulate the equations in (1.8). In particular, we take advantage of $\phi(T) \equiv 0$ and energy estimates to have a weighted global integral of ϕ in the right-hand side of (1.7) without adding terms in ω.

In consequence, we are not able to remove the local term of φ_N using the equation satisfied by ϕ and, therefore, the N-th component of v in (1.6) is kept. In fact, the role of v_N is to control the effects coming from the equation of r, which is not directly controlled, and acts precisely on the N-th equation of w through the coupling re_N. To know that if v_N can be disregarded from (1.6) and still have null controllability remains an open question.

The paper is organized like this: in Section 2, we introduce some notation and previous results that are used later on. Section 3 is dedicated to prove (1.7). In Sections 4 and 5 we prove the null controllability results for systems (1.6) and (1.1), respectively.

2 Technical results and notations

2.1 Some notations

We denote by $X_0 := L^2(Q)$ and $Y_0 := L^2(0,T;H)$. For n a positive integer we define the spaces X_n and Y_n as follows:

$$X_n := L^2(0,T;H^{2n}(\Omega)\cap H_0^1(\Omega))\cap H^n(0,T;L^2(\Omega)),$$
$$Y_n := L^2(0,T;H^{2n}(\Omega)^N\cap V)\cap H^n(0,T;L^2(\Omega)^N),$$

endowed with the norms

$$\|u\|_{X_n}^2 := \|u\|_{L^2(0,T;H^{2n}(\Omega))}^2 + \|u\|_{H^n(0,T;L^2(\Omega))}^2$$

and

$$\|u\|_{Y_n}^2 := \|u\|_{L^2(0,T;H^{2n}(\Omega)^N)}^2 + \|u\|_{H^n(0,T;L^2(\Omega)^N)}^2,$$

respectively.

The following subspaces will be useful (see Section 4). First, for every positive integer n, we set

$$X_{n,0} := \{u\in X_n : [\mathcal{L}^k u]_{|\Sigma} = 0, \quad [\mathcal{L}^k u](0) = 0, \quad k = 0,\dots,n-1\},$$

endowed with the equivalent norm (see Lemma 2.3 below),

$$\|u\|_{X_{n,0}} := \|\mathcal{L}^n u\|_{L^2(Q)}$$

where we have denoted

$$\mathcal{L} := \partial_t - \Delta.$$

Next, let

$$Y_{1,0} := \{u\in Y_1 :\ u(0) = 0\}$$

and

$$Y_{2,0} := \{u\in Y_{1,0}\cap L^2(0,T;H^4(\Omega)^N)\cap H^2(0,T;L^2(\Omega)^N) : (\mathcal{L}_H u)_{|\Sigma} = 0, \quad (\mathcal{L}_H u)(0) = 0\}$$

endowed with the equivalent norm (see Lemma 2.4 below)

$$\|u\|_{Y_{n,0}} := \|\mathcal{L}_H^n u\|_{L^2(Q)^N}, \quad n = 1,2.$$

Here, $\mathcal{L}_H := \partial_t - \mathcal{P}_L(\Delta)$, where $\mathcal{P}_L$ denotes the Leray projector over the space H, i.e $\mathcal{P}_L : L^2(Q)^N \to L^2(Q)^N$, $\mathcal{P}_L u = u - \nabla p$, where $\Delta p = \nabla\cdot u$ in Ω and $\nabla p\cdot\vec{n} = u\cdot\vec{n}$ on $\partial\Omega$ (see [27, pages 16-18]).

2.2 Carleman estimates

Let ω_0 be a nonempty open subset of $\mathbb{R}^N$ such that $\omega_0 \Subset \omega\cap\mathcal{O}_1$ and $\eta\in\mathcal{C}^8(\overline{\Omega})$ such that

$$|\nabla\eta| > 0 \text{ in } \overline{\Omega\setminus\omega_0},\ \eta > 0 \text{ in } \Omega \text{ and } \eta\equiv 0 \text{ on } \partial\Omega.$$

The existence of such a function η is given in [15].

We consider the following weight functions as in [11]:

$$\begin{aligned}
&\alpha(x,t) := \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\ell(t)^m}, \quad \xi(x,t) := \frac{e^{\lambda\eta(x)}}{\ell(t)^m},\\
&\alpha^*(t) := \max_{x\in\overline{\Omega}}\alpha(x,t), \quad \xi^*(t) := \min_{x\in\overline{\Omega}}\xi(x,t),\\
&\widehat{\alpha}(t) := \min_{x\in\overline{\Omega}}\alpha(x,t), \quad \widehat{\xi}(t) := \max_{x\in\overline{\Omega}}\xi(x,t),
\end{aligned} \tag{2.1}$$

where $\lambda \geq 1$, $m \geq 10$ and $\ell \in \mathcal{C}^\infty([0,T])$ is a positive function in $(0,T)$ satisfying

$$\begin{aligned} &\ell(t) = t \quad \forall\, t \in [0, T/4], \quad \ell(t) = T - t \quad \forall\, t \in [3T/4, T], \\ &\ell(t) \leq \ell(T/2),\ \forall\, t \in [0,T]. \end{aligned}$$

Notice that from (2.1), we obtain the following properties:

$$|\partial_t^n \alpha|,\ |\partial_t^n \xi| \leq C\xi^{(1+n/m)}, \quad |\partial_x^l \alpha|,\ |\partial_x^l \xi| \leq C\xi \tag{2.2}$$

where n is any nonnegative integer, $l \in \mathbb{N}^N$ and

$$e^{-as\alpha^*}(\xi^*)^c \leq C, \tag{2.3}$$

for any $a > 0$ and $c \in \mathbb{R}$. In (2.2) and (2.3), $C > 0$ is a constant only depending on Ω, λ, η, ℓ, l, a and c. These properties are also valid if we replace (α, ξ) by (α^*, ξ^*).

The first result is a Carleman inequality proved in [11]. More precisely it corresponds to estimate (3.10) in Paragraph 3.1.3 of that reference.

Lemma 2.1. *Let $\omega, \mathcal{O} \subset \Omega$ and assume $\omega \cap \mathcal{O} \neq \emptyset$. Then, there exists a constant $\widehat{\lambda}_0 > 0$ such that for any $\lambda \geq \widehat{\lambda}_0$, there exists $C > 0$ depending only on λ, Ω, ω, η and ℓ such that for any $j \in \{1, \dots, N-1\}$, any $g^\varphi \in Y_0$, any $g^\psi \in Y_2$, any $g \in X_2$ and any $\psi^0 \in H$, the solution of*

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_1 = g^\varphi + \psi\, \mathbb{1}_{\mathcal{O}}, & \nabla\cdot\varphi = 0 \quad \text{in } Q, \\ \psi_t - \Delta\psi + \nabla\pi_2 = g^\psi + g\, e_N, & \nabla\cdot\psi = 0 \quad \text{in } Q, \\ \varphi = \psi = 0 & \text{on } \Sigma, \\ \varphi(T) = 0, \quad \psi(0) = \psi^0 & \text{in } \Omega, \end{cases} \tag{2.4}$$

satisfies

$$\begin{aligned} s^4 \iint_Q e^{-11s\alpha^*}(\xi^*)^4|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + s^5 \iint_Q e^{-9s\alpha^*}(\xi^*)^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\ \leq C\big\|s^{9/2}e^{-9/2s\alpha}\xi^{9/2}\,g^\varphi\big\|_{Y_0}^2 + C\big\|e^{-7/2s\alpha^*}\,g^\psi\big\|_{Y_2}^2 + C\big\|s^{1/2}e^{-4s\alpha^*}(\xi^*)^{1/2-2/m}\,g\big\|_{X_2}^2 \\ + Cs^{13}\iint_{\omega_0\times(0,T)} e^{-9s\alpha}\xi^{13}\big((N-2)|\varphi_j|^2 + |\varphi_N|^2\big)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.5}$$

for every $s \geq C$.

To prove (2.5) it suffices to combine a Carleman estimate for φ with local terms of φ_j and φ_N (see for instance [10, Proposition 2.1] and [11, Lemma 2.1]), and a Carleman estimate for ψ with local terms of $\Delta\psi_j$ and $\Delta\psi_N$ (see for instance [12, Proposition 3.2] and [11, Lemma 2.3]). Then, we can use the coupling in the first equation of (2.4) to eliminate the local terms of ψ. Namely, we use that

$$\Delta\psi_k = -(\Delta\varphi_k)_t - \Delta^2\varphi_k + \partial_k\nabla\cdot g^\varphi - \Delta g_k^\varphi \quad \text{in } (0,T)\times\mathcal{O},\ k = j, N.$$

More details can be found in [11, section 3.1.3].

The next result is a special Carleman estimate for the solutions of the heat equation with

$$\mathcal{D}(\cdot) := [\partial_1^2 + (N-2)\partial_2^2](\cdot) \tag{2.6}$$

as local term.

Lemma 2.2. *There exists a constant* $\widehat{\lambda}_1 > 0$ *such that for any* $\lambda \geq \widehat{\lambda}_1$*, there exists* $C > 0$ *depending only on* λ*,* Ω*,* ω*,* η *and* ℓ *such that for any* $g \in X_3$ *and any* $u^0 \in L^2(\Omega)$*, the solution* u *of*

$$\begin{cases} u_t - \Delta u = g & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = u^0 & \text{in } \Omega, \end{cases} \tag{2.7}$$

satisfies

$$I(u) \leq C \big\| e^{-7/2s\alpha^*} g \big\|_{X_3}^2 + Cs^5 \iint_{\omega_0 \times (0,T)} e^{-8s\alpha} \xi^5 |\mathcal{D}u|^2 \, \mathrm{d}x \, \mathrm{d}t, \tag{2.8}$$

for every $s \geq C$*, where*

$$\begin{aligned} I(u) := &\iint_Q e^{-8s\alpha} (s^{-1}\xi^{-1} |\nabla^3 \mathcal{D}u|^2 + s\xi |\nabla^2 \mathcal{D}u|^2 + s^3\xi^3 |\nabla \mathcal{D}u|^2 + s^5\xi^5 |\mathcal{D}u|^2) \, \mathrm{d}x \, \mathrm{d}t \\ &+ s^5 \iint_Q e^{-8s\alpha^*} (\xi^*)^5 |u|^2 \, \mathrm{d}x \, \mathrm{d}t + \sum_{k=1}^{4} \big\| s^{5/2-k} e^{-4s\alpha^*} (\xi^*)^{5/2-k-k/m} u \big\|_{X_k}^2 . \end{aligned} \tag{2.9}$$

Estimate (2.8) is entirely proved in [11, section 3.3.2] (inequality (3.22)). Let us give the guidelines of the proof. We start by applying the operator $\nabla\nabla\mathcal{D}$ to equation (2.7) and then use a Carleman estimate with non homogeneous boundary conditions, namely, the one proved in [20] (see also [10, Lemma 2.2]). Then, we use regularity results for the heat equation to estimate the boundary terms. Actually, it would suffice to assume $g \in X_2$, but for later purposes we need further regularity. Finally, we can recover the L^2-norm using

$$\int_\Omega \big(|\partial_1 u|^2 + (N-2) |\partial_2 u|^2 \big) \, \mathrm{d}x = - \int_\Omega u \, \mathcal{D}u \, \mathrm{d}x,$$

together with Poincaré's and Young's inequalities.

2.3 Regularity results

Here, we state some regularity results concerning the heat and Stokes equations, respectively. The first one is (see for instance [23, Chapter 4])

Lemma 2.3. *For every* $T > 0$ *and every* $g \in X_n$ *(*n *any nonnegative integer), there exists a unique solution* $u \in X_{n+1}$ *to the heat equation* (2.7) *with* $u^0 \equiv 0$ *and there exists a constant* $C > 0$ *depending only on* Ω *such that*

$$\|u\|_{X_{n+1}} \leq C \|g\|_{X_n} . \tag{2.10}$$

The second one can be found in [22, Theorem 6, pages 100-101] (see also [27])

Lemma 2.4. *For every* $T > 0$*, every* $u^0 \in V$ *and every* $g \in L^2(Q)^N$*, there exists a unique solution*

$$u \in L^2(0,T; H^2(\Omega)^N) \cap H^1(0,T; H) \cap L^\infty(0,T; V)$$

to the Stokes system

$$\begin{cases} u_t - \Delta u + \nabla p = g, \quad \nabla \cdot u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = u^0 & \text{in } \Omega, \end{cases}$$

for some $p \in L^2(0,T;H^1(\Omega))$*, and there exists a constant* $C>0$ *depending only on* Ω *such that*

$$\|u\|^2_{L^2(0,T;H^2(\Omega)^N)} + \|u\|^2_{H^1(0,T;L^2(\Omega)^N)} + \|u\|^2_{L^\infty(0,T;V)} \le C\left(\|g\|^2_{L^2(Q)^N} + \|u^0\|^2_V\right). \quad (2.11)$$

Moreover, if $g \in L^2(0,T;H^2(\Omega)^N) \cap H^1(0,T;L^2(\Omega)^N)$ *and* $u^0 \in H^3(\Omega)^N \cap V$ *satisfy the compatibility condition:*

$$\nabla \bar{p} = \Delta u^0 + g(0) \text{ on } \partial\Omega,$$

where $\bar{p}$ *is any solution of the Neumann boundary-value problem*

$$\begin{cases} \Delta \bar{p} = \nabla \cdot g(0) & \text{in } Q,\\ \dfrac{\partial \bar{p}}{\partial n} = \Delta u^0 \cdot n + g(0)\cdot n & \text{on } \Sigma, \end{cases}$$

then $u \in Y_2$ *and there exists a constant* $C>0$ *depending only on* Ω *such that*

$$\|u\|^2_{Y_2} \le C\left(\|g\|^2_{Y_1} + \|u^0\|^2_{H^3(\Omega)}\right). \quad (2.12)$$

3 Carleman estimate

In this section we prove a Carleman inequality for system (1.8). To do this, we proceed in two steps. First, we prove a Carleman estimate for the equations in (1.8) not involving ϕ, with a local term as in (1.7). Next, we incorporate the weighted norm of ϕ to the previous inequality, provided that we change the weight functions.

3.1 Carleman estimate without ϕ

In this section we will first prove a Carleman estimate concerning φ, ψ and σ.

Proposition 3.1. *Assume* $\omega \cap \mathcal{O}_1 \neq \emptyset$*. Then, there exists a constant* $\widehat{\lambda}_3 > 0$ *such that for any* $\lambda \ge \widehat{\lambda}_3$*, there exists* $C>0$ *depending only on* λ*,* Ω*,* ω*,* η *and* ℓ *such that for any* $j \in \{1,\dots,N-1\}$*, any* $g^\varphi \in Y_0$*, any* $g^\psi \in Y_2$*, any* $g^\sigma \in X_3$*, any* $\psi^0 \in H$ *and any* $\sigma^0 \in L^2(\Omega)$*, the solution of* (1.8) *satisfies*

$$\begin{aligned} s^4 \iint_Q e^{-11s\alpha^*}(\xi^*)^4|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + s^5\iint_Q e^{-9s\alpha^*}(\xi^*)^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + s^5\iint_Q e^{-8s\alpha^*}(\xi^*)^5|\sigma|^2\,\mathrm{d}x\,\mathrm{d}t \\ \le Cs^{15}\iint_Q e^{-16s\alpha+8s\alpha^*}\xi^{18}|g^\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + C\left\|e^{-7/2s\alpha^*}\,g^\psi\right\|^2_{Y_2} + C\left\|e^{-7/2s\alpha^*}\,g^\sigma\right\|^2_{X_3} \\ + Cs^{19}\iint_{\omega\times(0,T)} e^{-16s\alpha+8s\alpha^*}\xi^{22}((N-2)|\varphi_j|^2 + |\varphi_N|^2)\,\mathrm{d}x\,\mathrm{d}t, \quad (3.1) \end{aligned}$$

for every $s \ge C$*.*

Proof. We start by applying the Lemma 2.1 with $\mathcal{O} = \mathcal{O}_1$ and $g = \sigma$:

$$\begin{aligned} s^4 \iint_Q e^{-11s\alpha^*}(\xi^*)^4|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + s^5\iint_Q e^{-9s\alpha^*}(\xi^*)^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\ \le C\left\|s^{9/2}e^{-9/2s\alpha}\xi^{9/2}\,g^\varphi\right\|^2_{Y_0} + C\left\|e^{-7/2s\alpha^*}\,g^\psi\right\|^2_{Y_2} \\ + C\left\|s^{1/2}e^{-4s\alpha^*}(\xi^*)^{1/2-2/m}\,\sigma\right\|^2_{X_2} + Cs^{13}\iint_{\omega_0\times(0,T)} e^{-9s\alpha}\xi^{13}((N-2)|\varphi_j|^2+|\varphi_N|^2)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

(3.2)

The idea now is to combine this inequality with (2.8) applied to the equation satisfied by σ. Indeed, we set $u=\sigma$ and $g=g^\sigma$ in Lemma 2.2. We have

$$I(\sigma) \leq C\left\|e^{-7/2s\alpha^*} g^\sigma\right\|^2_{X_3} + C_0 s^5 \iint_{\omega_0\times(0,T)} e^{-8s\alpha}\xi^5|\mathcal{D}\sigma|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{3.3}$$

for every $s\geq C$. Recall that $I(\sigma)$ is given by (2.9) and $\mathcal{D}\sigma := \left[\partial_1^2+(N-2)\partial_2^2\right]\sigma$. On the other hand, notice that from the equations in (1.8) we have

$$\Delta\pi_1 = \nabla\cdot g^\varphi \quad \text{in } \mathcal{O}_1\times(0,T)$$

and

$$\Delta\pi_2 = \nabla\cdot g^\psi + \partial_N\sigma \quad \text{in } Q.$$

Then, we can obtain the following relation between σ and φ_N in the set $(\omega_0\cap\mathcal{O}_1)\times(0,T)$:

$$\mathcal{D}\sigma = [-\Delta\partial_t^2+\Delta^3]\varphi_N - (\Delta g_N^\varphi)_t + \Delta^2 g_N^\varphi + (\partial_N\nabla\cdot g^\varphi)_t - \Delta(\partial_N\nabla\cdot g^\varphi) - \Delta g_N^\psi + \partial_N\nabla\cdot g^\psi. \tag{3.4}$$

Next, we take $\omega_0\Subset\omega'\Subset(\mathcal{O}_1\cap\omega)$ and $\zeta(x)\in\mathcal{C}_0^8(\omega')$ such that $0\leq\zeta\leq1$ and $\zeta_{|\omega_0}\equiv1$, and using (3.4) we have

$$\begin{aligned}
s^5\iint_{\omega_0\times(0,T)} e^{-8s\alpha}\xi^5|\mathcal{D}\sigma|^2\,\mathrm{d}x\,\mathrm{d}t &\leq s^5\iint_{\omega'\times(0,T)}\zeta(x)e^{-8s\alpha}\xi^5\mathcal{D}\sigma\big(\big[-\Delta\partial_t^2+\Delta^3\big]\varphi_N\big)\,\mathrm{d}x\,\mathrm{d}t\\
&+ s^5\iint_{\omega'\times(0,T)}\zeta(x)e^{-8s\alpha}\xi^5\mathcal{D}\sigma\big(-(\Delta g_N^\varphi)_t+\Delta^2g_N^\varphi+(\partial_N\nabla\cdot g^\varphi)_t-\Delta(\partial_N\nabla\cdot g^\varphi)\big)\,\mathrm{d}x\,\mathrm{d}t\\
&+ s^5\iint_{\omega'\times(0,T)}\zeta(x)e^{-8s\alpha}\xi^5\mathcal{D}\sigma\big(-\Delta g_N^\psi+\partial_N\nabla\cdot g^\psi\big)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Let us call by A_1, A_2 and A_3 the integrals in the right-hand side, respectively. Integrating by parts and using $m\geq10$ and Young's inequality, we obtain

$$\begin{aligned}
|A_1| &\leq \frac{1}{6C_0}I(\sigma) + Cs^{19}\iint_{\omega'\times(0,T)} e^{-16s\alpha+8s\alpha^*}\xi^{22}|\varphi_N|^2\,\mathrm{d}x\,\mathrm{d}t,\\
|A_2| &\leq \frac{1}{6C_0}I(\sigma) + Cs^{15}\iint_Q e^{-16s\alpha+8s\alpha^*}\xi^{18}|g^\varphi|^2\,\mathrm{d}x\,\mathrm{d}t,\\
|A_3| &\leq \frac{1}{6C_0}I(\sigma) + Cs^{11}\iint_Q e^{-16s\alpha+8s\alpha^*}\xi^{14}|g^\psi|^2\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where C_0 is the constant appearing in (3.3). Notice that the assumption $g^\sigma\in X_3$ comes into play when estimating $|A_1|$. Putting these estimates together, we get

$$\begin{aligned}
s^5\iint_{\omega_0\times(0,T)} e^{-8s\alpha}\xi^5|\mathcal{D}\sigma|^2\,\mathrm{d}x\,\mathrm{d}t &\leq \frac{1}{2C_0}I(\sigma) + Cs^{19}\iint_{\omega'\times(0,T)} e^{-16s\alpha+8s\alpha^*}\xi^{22}|\varphi_N|^2\,\mathrm{d}x\,\mathrm{d}t\\
&+ Cs^{15}\iint_Q e^{-16s\alpha+8s\alpha^*}\xi^{18}|g^\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{11}\iint_Q e^{-16s\alpha+8s\alpha^*}\xi^{14}|g^\psi|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.5}$$

Combining (3.3) and (3.5), and then with (3.2) we obtain

$$\begin{aligned}
&s^4\iint_Q e^{-11s\alpha^*}(\xi^*)^4|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + s^5\iint_Q e^{-9s\alpha^*}(\xi^*)^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + s^5\iint_Q e^{-8s\alpha^*}(\xi^*)^5|\sigma|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\leq Cs^{15}\iint_Q e^{-16s\alpha+8s\alpha^*}\xi^{18}|g^\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + C\left\|e^{-7/2s\alpha^*} g^\psi\right\|^2_{Y_2} + C\left\|e^{-7/2s\alpha^*} g^\sigma\right\|^2_{X_3}\\
&+(N-2)Cs^{13}\iint_{\omega_0\times(0,T)} e^{-9s\alpha^*}\xi^{13}|\varphi_j|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{19}\iint_{\omega'\times(0,T)} e^{-16s\alpha+8s\alpha^*}\xi^{22}|\varphi_N|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

From this we get in particular (3.1). ☐

3.2 Carleman estimate involving ϕ

To prove (1.7) we would usually apply a Carleman inequality for ϕ, but this would make appear a local term of ϕ. Since ϕ is not coupled with the other equations, it can not be easily, at least at first sight, estimated by the other variables. However, we will take advantage of the homogeneous initial condition to finish the proof of (1.7).

First, we will deduce a Carleman inequality with weights similar as those in (2.1), but such that they are not decreasing in time and do not vanish in $t = T$. To this end, we define

$$\begin{aligned}
&\beta(x,t) := \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\widetilde{\ell}(t)^m}, \quad \gamma(x,t) := \frac{e^{\lambda\eta(x)}}{\widetilde{\ell}(t)^m},\\
&\beta^*(t) := \max_{x\in\overline{\Omega}} \alpha(x,t), \quad \gamma^*(t) = \min_{x\in\overline{\Omega}} \gamma(x,t),\\
&\widehat{\beta}(t) := \min_{x\in\overline{\Omega}} \beta(x,t), \quad \widehat{\gamma}(t) := \max_{x\in\overline{\Omega}} \gamma(x,t),
\end{aligned}$$

where

$$\widetilde{\ell}(t) = \begin{cases} \ell(t) & 0 \le t \le T/2,\\ \|\ell\|_\infty & T/2 < t \le T. \end{cases}$$

Notice that properties (2.2) and (2.3) are still valid for these weights.

Using energy estimates and $\varphi(T) = 0$, we can show from (3.1) that

$$\begin{aligned}
&\iint_Q e^{-11s\beta^*}(\gamma^*)^4|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + \iint_Q e^{-9s\beta^*}(\gamma^*)^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + \iint_Q e^{-8s\beta^*}(\gamma^*)^5|\sigma|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le C\big\|e^{-7/2s\beta^*} g^\varphi\big\|^2_{Y_0} + C\big\|e^{-7/2s\beta^*} g^\psi\big\|^2_{Y_2} + C\big\|e^{-7/2s\beta^*} g^\sigma\big\|^2_{X_3}\\
&\qquad + C\iint_{\omega\times(0,T)} e^{-16s\widehat{\beta}+8s\beta^*}\widehat{\gamma}^{22}\big((N-2)|\varphi_j|^2 + |\varphi_N|^2\big)\,\mathrm{d}x\,\mathrm{d}t. \qquad (3.6)
\end{aligned}$$

We refer to [10, section 3] or [9, section 3] for a detailed proof of (3.6).

Now we are ready to state the following

Proposition 3.2. *Assume $\omega \cap \mathcal{O}_1 \neq \emptyset$. Then, there exists a constant $\widehat{\lambda}_4 > 0$ such that for any $\lambda \ge \widehat{\lambda}_4$, there exists $C > 0$ depending only on λ, Ω, ω, η and ℓ such that for any $j \in \{1, \dots, N-1\}$, any $g^\varphi \in Y_0$, any $g^\psi \in Y_2$, any, $g^\phi \in X_0$, any $g^\sigma \in X_3$, any $\psi^0 \in H$ and any $\sigma^0 \in L^2(\Omega)$, the solution of (1.8) satisfies*

$$\begin{aligned}
&\iint_Q e^{-11s\beta^*}(\gamma^*)^4|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + \iint_Q e^{-9s\beta^*}(\gamma^*)^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + \iint_Q e^{-11s\beta^*}|\phi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&+ \iint_Q e^{-8s\beta^*}(\gamma^*)^5|\sigma|^2\,\mathrm{d}x\,\mathrm{d}t \le C\iint_{\omega\times(0,T)} e^{-16s\widehat{\beta}+8s\beta^*}\widehat{\gamma}^{22}\big((N-2)|\varphi_j|^2 + |\varphi_N|^2\big)\,\mathrm{d}x\,\mathrm{d}t\\
&+ C\big\|e^{-7/2s\beta^*} g^\varphi\big\|^2_{Y_0} + C\big\|e^{-7/2s\beta^*} g^\psi\big\|^2_{Y_2} + C\big\|e^{-11/2s\beta^*} g^\phi\big\|^2_{X_0} + C\big\|e^{-7/2s\beta^*} g^\sigma\big\|^2_{X_3}, \qquad (3.7)
\end{aligned}$$

for every $s \ge C$.

Proof. Let $\rho(t) := e^{-11/2s\beta^*}$. We can easily check that $\rho\phi$ satisfies the equation:

$$\begin{cases} -(\rho\phi)_t - \Delta(\rho\phi) = \rho\, g^\phi + \rho\,\varphi_N + \rho\,\sigma 1\!\!1_{\mathcal{O}_2} - \rho'\phi & \text{in } Q,\\ \rho\phi = 0 & \text{on } \Sigma,\\ (\rho\phi)(T) = 0 & \text{in } \Omega. \end{cases}$$

We multiply this equation by $\rho\,\phi$ and integrate over Ω. We get

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^2|\phi|^2\,\mathrm{d}x+\int_\Omega \rho^2|\nabla\phi|^2\,\mathrm{d}x \\ =\int_\Omega \rho^2 g^\phi\,\phi\,\mathrm{d}x+\int_\Omega \rho^2\varphi_N\,\phi\,\mathrm{d}x+\int_\Omega \rho^2\sigma\,\phi\,\mathbb{1}_{\mathcal{O}_2}\,\mathrm{d}x-\int_\Omega \rho'\rho|\phi|^2\,\mathrm{d}x.$$

Since ρ is a positive and non-decreasing function we have $\rho'\rho\geq 0$ and thus

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^2|\phi|^2\,\mathrm{d}x+\int_\Omega \rho^2|\nabla\phi|^2\,\mathrm{d}x\leq\int_\Omega \rho^2 g^\phi\,\phi\,\mathrm{d}x+\int_\Omega \rho^2\varphi_N\,\phi\,\mathrm{d}x+\int_\Omega \rho^2\sigma\,\phi\,\mathbb{1}_{\mathcal{O}_2}\,\mathrm{d}x,$$

and after using Poincaré's and Young's inequalities we obtain

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^2|\phi|^2\,\mathrm{d}x+\frac{1}{2}\int_\Omega \rho^2|\nabla\phi|^2\,\mathrm{d}x\leq C\int_\Omega \rho^2|g^\phi|^2\,\mathrm{d}x+C\int_\Omega \rho^2|\varphi_N|^2\,\mathrm{d}x+C\int_\Omega \rho^2|\sigma|^2\,\mathrm{d}x.$$

Integrating in time, we find that

$$\iint_Q \rho^2|\phi|^2\,\mathrm{d}x\,\mathrm{d}t\leq C\iint_Q \rho^2|g^\phi|^2\,\mathrm{d}x+C\iint_Q \rho^2|\varphi_N|^2\,\mathrm{d}x\,\mathrm{d}t+C\iint_Q \rho^2|\sigma|^2\,\mathrm{d}x\,\mathrm{d}t,$$

where we have also used that $\phi(T)=0$, which is essential to obtain the previous estimate. This, together with (3.6), leads to (3.7). □

4 Null controllability of the linear system

In this section we deal with the null controllability of the linear system

$$\begin{cases}\mathcal{L}w+\nabla p_0=f^w+v\mathbb{1}_\omega+r\,e_N, & \nabla\cdot w=0 \quad \text{in } Q,\\ \mathcal{L}^*z+\nabla p_1=f^z+w\mathbb{1}_{\mathcal{O}_1}, & \nabla\cdot z=0 \quad \text{in } Q,\\ \mathcal{L}r=f^r & \text{in } Q,\\ \mathcal{L}^*q=f^q+z_N+r\,\mathbb{1}_{\mathcal{O}_2} & \text{in } Q,\\ w=z=0,\quad r=q=0 & \text{on } \Sigma,\\ w(0)=0,\quad z(T)=0,\quad r(0)=0,\quad q(T)=0 & \text{in } \Omega,\end{cases}\tag{4.1}$$

that is, for a given $i\in\{1,\dots,N-1\}$, we will prove the existence of a solution (w,p_0,z,p_1,r,q,v) of (4.1), with $v_i\equiv 0$, such that $z(0)=0$ and $q(0)=0$ in Ω. Here, we have use the notation $\mathcal{L}^*:=-\partial_t-\Delta$, the formal adjoint of the parabolic operator $\mathcal{L}$ defined in Section 2.1.

Let us first state an observability inequality from (3.7) which will make things easier and clearer in what is to come.

Lemma 4.1. *Let $j\in\{1,\dots,N-1\}$ and let g^φ, g^ϕ, s and λ be like in Proposition 3.2. Furthermore, assume that $g^\psi\in Y_{2,0}$ and $g^\sigma\in X_{3,0}$. Then, there exists a constant $C>0$ (depending on s and λ) such that every solution $(\varphi,\pi_1,\psi,\pi_2,\phi,\sigma)$ of (1.8) satisfies*

$$\iint_Q e^{-12s\beta^*}\left(|\varphi|^2+|\psi|^2+|\phi|^2+|\sigma|^2\right)\mathrm{d}x\,\mathrm{d}t\leq C\iint_{\omega\times(0,T)} e^{-7s\beta^*}\left((N-2)|\varphi_j|^2+|\varphi_N|^2\right)\mathrm{d}x\,\mathrm{d}t \\ +C\|e^{-7/2s\beta^*}g^\varphi\|^2_{Y_0}+C\|e^{-7/2s\beta^*}g^\psi\|^2_{Y_{2,0}}+C\|e^{-7/2s\beta^*}g^\phi\|^2_{X_0}+C\|e^{-7/2s\beta^*}g^\sigma\|^2_{X_{3,0}}.\tag{4.2}$$

Inequality (4.2) is obtained directly from the properties of the weight functions and the equivalence between norms mentioned in Section 2.1.

The rest of the section goes as in [11]. The idea is to look for a solution in an appropriate weighted functional space. To this end, we introduce, for $i \in \{1, \dots, N-1\}$, the spaces

$$\begin{aligned} E_i = \{(w, p_0, z, p_1, r, q, v) :\ & e^{7/2s\beta^*} v 1\!\!1_\omega \in L^2(Q)^N, \quad v_i \equiv 0, \\ & e^{7/2s\beta^*}(\gamma^*)^{-1-1/m} w \in Y_1, \quad e^{7/2s\beta^*}(\gamma^*)^{-6-6/m} z \in Y_1, \quad z(T) = 0, \\ & e^{7/2s\beta^*}(\gamma^*)^{-1-1/m} r \in X_1, \quad e^{7/2s\beta^*}(\gamma^*)^{-11-11/m} q \in X_1, \quad q(T) = 0, \\ & e^{6s\beta^*}(\mathcal{L}w + \nabla p_0 - v 1\!\!1_\omega - r\, e_N, \quad \mathcal{L}^* z + \nabla p_1 - w 1\!\!1_{\mathcal{O}_1}) \in L^2(Q)^{2N}, \\ & e^{6s\beta^*}(\mathcal{L}r, \quad \mathcal{L}^* q - z_N - r 1\!\!1_{\mathcal{O}_2}) \in L^2(Q)^2\}. \end{aligned}$$

It is clear that E_i is a Banach space endowed with its natural norm.

Proposition 4.2. *Assume the hypothesis of Lemma 4.1, $i \in \{1, \dots, N-1\}$ and*

$$e^{6s\beta^*}(f^w, f^z, f^r, f^q) \in L^2(Q)^{2N+2}. \tag{4.3}$$

Then, we can find a control $v \in L^2(Q)^N$ such that the associated solution (w, p_0, z, p_1, r, q) to (4.1) *satisfies $(w, p_0, z, p_1, r, q, v) \in E_i$. In particular, $v_i \equiv 0$ and $(z(0), q(0)) = (0, 0)$ in Ω.*

Proof. The proof follows the same strategy used in [11], although the arguments were introduced in [15] and [19]. However, we include the proof for the sake of completeness. See [14] for a proof in a more general framework. Let P_0 be the space of functions $(\varphi, \pi_1, \psi, \pi_2, \phi, \sigma) \in \mathcal{C}^\infty(\overline{Q})^{2N+4}$ such that

- $\nabla \cdot \varphi = \nabla \cdot \psi = 0$,
- $\varphi_{|\Sigma} = \psi_{|\Sigma} = 0, \quad \phi_{|\Sigma} = \sigma_{|\Sigma} = 0$,
- $\varphi(T) = \psi(0) = 0, \quad \phi(T) = \sigma(0) = 0$,
- $\displaystyle\int_\Omega \pi_1\, dx = \int_\Omega \pi_2\, dx = 0$,
- $\nabla \cdot (\mathcal{L}\psi + \nabla \pi_2 - \sigma\, e_N) = 0$,
- $\left(\mathcal{L}_H^k[e^{-7/2s\beta^*}(\mathcal{L}\psi + \nabla \pi_2 - \sigma\, e_N)]\right)_{|\Sigma} = 0, \quad k = 0, 1$,
- $\left(\mathcal{L}_H^k[e^{-7/2s\beta^*}(\mathcal{L}\psi + \nabla \pi_2 - \sigma\, e_N)]\right)(0) = 0, \quad k = 0, 1$,
- $\mathcal{L}^k[e^{-7/2s\beta^*}\mathcal{L}\sigma]_{|\Sigma} = 0, \quad k = 0, 1, 2$,
- $\mathcal{L}^k[e^{-7/2s\beta^*}\mathcal{L}\sigma](0) = 0, \quad k = 0, 1, 2$.

We define the bilinear form

$$\begin{aligned} & a\big((\widetilde{\varphi}, \widetilde{\pi}_1, \widetilde{\psi}, \widetilde{\pi}_2, \widetilde{\phi}, \widetilde{\sigma}), (\varphi, \pi_1, \psi, \pi_2, \phi, \sigma)\big) \\ & := \iint_Q e^{-7s\beta^*}(\mathcal{L}^*\widetilde{\varphi} + \nabla \widetilde{\pi}_1 - \widetilde{\psi} 1\!\!1_{\mathcal{O}_1}) \cdot (\mathcal{L}^*\varphi + \nabla \pi_1 - \psi 1\!\!1_{\mathcal{O}_1})\, \mathrm{d}x\, \mathrm{d}t \\ & + \iint_Q \mathcal{L}_H^2[e^{-7/2s\beta^*}(\mathcal{L}\widetilde{\psi} + \nabla \widetilde{\pi}_2 - \widetilde{\sigma}\, e_N)] \cdot \mathcal{L}_H^2[e^{-7/2s\beta^*}(\mathcal{L}\psi + \nabla \pi_2 - \sigma\, e_N)]\, \mathrm{d}x\, \mathrm{d}t \\ & + \iint_Q e^{-7s\beta^*}(\mathcal{L}^*\widetilde{\phi} - \widetilde{\varphi}_N - \widetilde{\sigma}\, 1\!\!1_{\mathcal{O}_2})(\mathcal{L}^*\phi - \varphi_N - \sigma\, 1\!\!1_{\mathcal{O}_2})\, \mathrm{d}x\, \mathrm{d}t \end{aligned}$$

$$+\iint_Q \mathcal{L}^3[e^{-7/2s\beta^*}\mathcal{L}\widetilde{\sigma}]\mathcal{L}^3[e^{-7/2s\beta^*}\mathcal{L}\sigma]\,\mathrm{d}x\,\mathrm{d}t$$
$$+\iint_{\omega\times(0,T)} e^{-7s\beta^*}\big((N-2)\widetilde{\varphi}_j\,\varphi_j+\widetilde{\varphi}_N\,\varphi_N\big)\,\mathrm{d}x\,\mathrm{d}t,$$

where $j\in\{1,\dots,N-1\}\setminus\{i\}$ and a linear form

$$\langle G,(\varphi,\pi_1,\psi,\pi_2,\phi,\sigma)\rangle := \iint_Q f^w\cdot\varphi\ \mathrm{d}x\,\mathrm{d}t+\iint_Q f^z\cdot\psi\,\mathrm{d}x\,\mathrm{d}t+\iint_Q f^r\,\phi\ \mathrm{d}x\,\mathrm{d}t+\iint_Q f^q\,\sigma\,\mathrm{d}x\,\mathrm{d}t.$$

Thanks to (4.2), we have that $a(\cdot,\cdot):P_0\times P_0\mapsto\mathbb{R}$ is a symmetric, definite positive bilinear form on P_0. We denote by P the completion of P_0 for the norm induced by $a(\cdot,\cdot)$. Then, $a(\cdot,\cdot)$ is well-defined, continuous and definite positive on P. Furthermore, in view of the Carleman estimate (4.2) and the assumptions (4.3), the linear form $(\varphi,\pi_1,\psi,\pi_2,\phi,\sigma)\mapsto\langle G,(\varphi,\pi_1,\psi,\pi_2,\phi,\sigma)\rangle$ is well-defined and continuous on P. Hence, from Lax-Milgram's lemma, we deduce that the variational problem:

$$\begin{cases}\text{Find } (\widetilde{\varphi},\widetilde{\pi}_1,\widetilde{\psi},\widetilde{\pi}_2,\widetilde{\phi},\widetilde{\sigma})\in P \text{ such that, } \forall\,(\varphi,\pi_1,\psi,\pi_2,\phi,\sigma)\in P,\\ a\big((\widetilde{\varphi},\widetilde{\pi}_1,\widetilde{\psi},\widetilde{\pi}_2,\widetilde{\phi},\widetilde{\sigma}),(\varphi,\pi_1,\psi,\pi_2,\phi,\sigma)\big)=\langle G,(\varphi,\pi_1,\psi,\pi_2,\phi,\sigma)\rangle,\end{cases}\tag{4.4}$$

possesses a unique solution $(\widehat{\varphi},\widehat{\pi}_1,\widehat{\psi},\widehat{\pi}_2,\widehat{\phi},\widehat{\sigma})$.

We define $\widehat{v}$ by

$$\big\{\ \widehat{v}_k := -e^{-7s\beta^*}\widehat{\varphi}_k\mathbb{1}_\omega,\ k\neq i,\quad \widehat{v}_i\equiv 0\quad\text{in } Q.\tag{4.5}$$

It is readily checked from (4.4) and (4.5) that

$$\begin{aligned}&\iint_Q\big(|\widetilde{w}|^2+|\widetilde{z}|^2+|\widetilde{r}|^2+|\widetilde{q}|^2\big)\ \mathrm{d}x\,\mathrm{d}t\\ &+\iint_{\omega\times(0,T)} e^{7s\beta^*}\ \big((N-2)|\widehat{v}_j|^2+|\widehat{v}_N|^2\big)\ \mathrm{d}x\,\mathrm{d}t<+\infty,\end{aligned}\tag{4.6}$$

where we have denoted $\widetilde{w}$, $\widetilde{z}$, $\widetilde{r}$ and $\widetilde{q}$ by

$$\begin{cases}\widetilde{w}:=e^{-7/2s\beta^*}(\mathcal{L}^*\widehat{\varphi}+\nabla\widehat{\pi}_1-\widehat{\psi}\mathbb{1}_{\mathcal{O}_1}),\\ \widetilde{z}:=\mathcal{L}_H^2[e^{-7/2s\beta^*}(\mathcal{L}\widehat{\psi}+\nabla\widehat{\pi}_2-\widehat{\sigma}\,e_N)],\\ \widetilde{r}:=e^{-7/2s\beta^*}(\mathcal{L}^*\widehat{\phi}-\widehat{\varphi}_N-\widehat{\sigma}\,\mathbb{1}_{\mathcal{O}_2}),\\ \widetilde{q}:=\mathcal{L}^3[e^{-7/2s\beta^*}\mathcal{L}\widehat{\sigma}].\end{cases}\tag{4.7}$$

Now, we take $(\widehat{w},\widehat{z},\widehat{r},\widehat{q})$, together with some pressures $(\widehat{p}_0,\widehat{p}_1)$, to be the (weak) solution of (4.1) with $v=\widehat{v}$, i.e., they verify

$$\begin{cases}\mathcal{L}\widehat{w}+\nabla\widehat{p}_0=f^w+\widehat{v}\mathbb{1}_\omega+\widehat{r}\,e_N, & \nabla\cdot\widehat{w}=0 & \text{in } Q,\\ \mathcal{L}^*\widehat{z}+\nabla\widehat{p}_1=f^z+\widehat{w}\mathbb{1}_{\mathcal{O}_1}, & \nabla\cdot\widehat{z}=0 & \text{in } Q,\\ \mathcal{L}\widehat{r}=f^r & & \text{in } Q,\\ \mathcal{L}^*\widehat{q}=f^q+\widehat{z}_N+\widehat{r}\,\mathbb{1}_{\mathcal{O}_2} & & \text{in } Q,\\ \widehat{w}=\widehat{z}=0,\quad \widehat{r}=\widehat{q}=0 & & \text{on } \Sigma,\\ \widehat{w}(0)=0,\quad \widehat{z}(T)=0,\quad \widehat{r}(0)=0, & \widehat{q}(T)=0 & \text{in } \Omega.\end{cases}\tag{4.8}$$

Notice that $(\widehat{w},\widehat{p}_0,\widehat{z},\widehat{p}_1,\widehat{r},\widehat{q})$ is well defined by Lemmas 2.3 and 2.4 since $\widehat{v}\in L^2(Q)^N$ (by (4.5)-(4.6)) and (4.3).

In the following, we will prove the following exponential decay properties

$$
\begin{aligned}
&e^{7/2s\beta^*}(\gamma^*)^{-1-1/m}\widehat{w} \in Y_1, \quad e^{7/2s\beta^*}(\gamma^*)^{-6-6/m}\widehat{z} \in Y_1 \\
&e^{7/2s\beta^*}(\gamma^*)^{-1-1/m}\widehat{r} \in X_1, \quad e^{7/2s\beta^*}(\gamma^*)^{-11-11/m}\widehat{q} \in X_1,
\end{aligned} \tag{4.9}
$$

which will complete the proof of Proposition 4.2.

First, let us prove that $(\widetilde{w}, \widetilde{z}, \widetilde{r}, \widetilde{q})$ defined in (4.7) is actually the solution (in the sense of transposition) of

$$
\begin{cases}
e^{-7/2s\beta^*}\widetilde{w} = \widehat{w} & \text{in } Q, \\
e^{-7/2s\beta^*}(\mathcal{L}_H^*)^2\widetilde{z} = \widehat{z}, \quad \nabla\cdot\widetilde{z} = 0 & \text{in } Q, \\
e^{-7/2s\beta^*}\widetilde{r} = \widehat{r} & \text{in } Q, \\
e^{-7/2s\beta^*}(\mathcal{L}^*)^3\widetilde{q} = \widehat{q} & \text{in } Q,
\end{cases} \tag{4.10}
$$

such that

$$
\begin{cases}
(\mathcal{L}_H^*)^\ell\widetilde{z} = 0 & \text{on } \Sigma, \quad \ell = 0, 1, \\
(\mathcal{L}_H^*)^\ell\widetilde{z}(T) = 0 & \text{in } \Omega, \quad \ell = 0, 1, \\
(\mathcal{L}^*)^k\widetilde{q} = 0 & \text{on } \Sigma, \quad k = 0, \dots, 2, \\
(\mathcal{L}^*)^k\widetilde{q}(T) = 0 & \text{in } \Omega, \quad k = 0, \dots, 2.
\end{cases} \tag{4.11}
$$

Indeed, from (4.4) for every $(\varphi, \pi_1, \psi, \pi_2, \phi, \sigma) \in P_0$ we obtain the following:

$$
\begin{aligned}
&\iint_Q \widetilde{w}\cdot e^{-7/2s\beta^*}(\mathcal{L}^*\varphi + \nabla\pi_1 - \psi\mathbb{1}_{\mathcal{O}_1})\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widetilde{z}\cdot\mathcal{L}_H^2[e^{-7/2s\beta^*}(\mathcal{L}\psi + \nabla\pi_2 - \sigma\,e_N)]\,\mathrm{d}x\,\mathrm{d}t \\
&+ \iint_Q \widetilde{r}\,e^{-7/2s\beta^*}(\mathcal{L}^*\phi - \varphi_N - \sigma\,\mathbb{1}_{\mathcal{O}_2})\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widetilde{q}\,\mathcal{L}^3[e^{-7/2s\beta^*}\mathcal{L}\sigma]\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \iint_Q \varphi\cdot(\mathcal{L}\widehat{w} + \nabla\widehat{p}_0 - \widehat{r}\,e_N)\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \psi\cdot(\mathcal{L}^*\widehat{z} + \nabla\widehat{p}_1 - \widehat{w}\mathbb{1}_{\mathcal{O}_1})\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \iint_Q \phi\,\mathcal{L}\widehat{r}\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \sigma\,(\mathcal{L}^*\widehat{q} - \widehat{z}_N - \widehat{r}\mathbb{1}_{\mathcal{O}_2})\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \iint_Q \widehat{w}\cdot(\mathcal{L}^*\varphi + \nabla\pi_1 - \psi\mathbb{1}_{\mathcal{O}_1})\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widehat{z}\cdot(\mathcal{L}\psi + \nabla\pi_2 - \sigma\,e_N)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \iint_Q \widehat{r}\,(\mathcal{L}^*\phi - \varphi_N - \sigma\,\mathbb{1}_{\mathcal{O}_2})\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widehat{q}\,\mathcal{L}\sigma\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Notice that for the first equality, we have only used the definitions (4.5) and (4.7) in (4.4), together with the equation (4.8). For the second one, we have used integration by parts in time and space.

From this last equality, we obtain for all $(h^w, h^z, h^r, h^q) \in L^2(Q)^{2N+2}$

$$
\begin{aligned}
&\iint_Q \widetilde{w}\cdot h^w\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widetilde{z}\cdot h^z\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widetilde{r}\,h^r\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widetilde{q}\,h^q\,\mathrm{d}x\,\mathrm{d}t \\
&= \iint_Q \widehat{w}\cdot\Phi^w\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widehat{z}\cdot\Phi^z\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widehat{r}\,\Phi^r\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widehat{q}\,\Phi^q\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{4.12}
$$

where $(\Phi^w, \Phi^z, \Phi^r, \Phi^q)$ is the unique solution of

$$
\begin{cases}
e^{-7/2s\beta^*}\Phi^w = h^w, & \text{in } Q, \\
\mathcal{L}_H^2[e^{-7/2s\beta^*}\Phi^z] = h^z, \quad \nabla\cdot\Phi^z = 0, & \text{in } Q, \\
e^{-7/2s\beta^*}\Phi^r = h^r, & \text{in } Q, \\
\mathcal{L}^3[e^{-7/2s\beta^*}\Phi^q] = h^q, & \text{in } Q,
\end{cases} \tag{4.13}
$$

such that

$$\begin{cases} \mathcal{L}_H^\ell(e^{-7/2s\beta^*}\Phi^z) = 0 & \text{on } \Sigma, \quad \ell = 0,1, \\ \mathcal{L}_H^\ell(e^{-7/2s\beta^*}\Phi^z)(0) = 0 & \text{in } \Omega, \quad \ell = 0,1, \\ \mathcal{L}^k(e^{-7/2s\beta^*}\Phi^q) = 0 & \text{on } \Sigma, \quad k = 0,\dots,2, \\ \mathcal{L}^k(e^{-7/2s\beta^*}\Phi^q)(0) = 0 & \text{in } \Omega, \quad k = 0,\dots,2. \end{cases} \tag{4.14}$$

It is classical to show that (4.12)-(4.14) is equivalent to (4.10)-(4.11).

Next, since this paper is also meant to be a complement of [11], we concentrate in proving the part of (4.9) concerning $\widehat{q}$. The part concerning $\widehat{z}$ was proved in detail in [11]. Namely, it was proved that

$$e^{7/2s\beta^*}(\gamma^*)^{-5-5/m}\widehat{z} \in L^2(Q)^N. \tag{4.15}$$

Nevertheless, following the arguments that we will show now, (4.15) can be readily deduced. We begin by considering the functions

$$q_{*,0} := e^{7/2s\beta^*}(\gamma^*)^{-5-5/m}\widehat{q}, \quad f^q_{*,0} := e^{7/2s\beta^*}(\gamma^*)^{-5-5/m}(f^q + \widehat{z}_N + \widehat{r}1_{\mathcal{O}_2}).$$

Notice that, from (4.3), (4.6), (4.10) and (4.15), we have $f^q_{*,0} \in L^2(Q)$. Then, by (4.8) $q_{*,0}$ verifies

$$\begin{cases} \mathcal{L}^* q_{*,0} = f^q_{*,0} - (e^{7/2s\beta^*}(\gamma^*)^{-5-5/m})_t\widehat{q} & \text{in } Q, \\ q_{*,0} = 0 & \text{on } \Sigma, \\ q_{*,0}(T) = 0 & \text{in } \Omega. \end{cases}$$

From (4.10), we have the identity:

$$(e^{7/2s\beta^*}(\gamma^*)^{-5-5/m})_t\widehat{q} = c_3(t)(\mathcal{L}^*)^3\widetilde{q},$$

where we have denoted by $c_k(t)$ a function such that (see (2.2))

$$\left|\frac{\mathrm{d}^\ell c_k}{\mathrm{d}t^\ell}\right| \le C < \infty, \quad \forall\, \ell = 0,\dots,k. \tag{4.16}$$

On the other hand, for any $h \in X_{2,0}$ we have

$$\iint_Q q_{*,0}\, h \,\mathrm{d}x\,\mathrm{d}t = \iint_Q f^q_{*,0}\,\Phi\,\mathrm{d}x\,\mathrm{d}t - \iint_Q c_3(t)(\mathcal{L}^*)^3\widetilde{q}\,\Phi\,\mathrm{d}x\,\mathrm{d}t,$$

where $\Phi \in X_{3,0}$ is the solution of

$$\begin{cases} \mathcal{L}\Phi = h & \text{in } Q, \\ \Phi = 0 & \text{on } \Sigma, \\ \Phi(0) = 0 & \text{in } \Omega. \end{cases}$$

Using (4.11), we can integrate by parts to obtain

$$\begin{aligned} \iint_Q q_{*,0}\, h \,\mathrm{d}x\,\mathrm{d}t &= \iint_Q f^q_{*,0}\,\Phi\,\mathrm{d}x\,\mathrm{d}t - \iint_Q (\mathcal{L}^*)^2\widetilde{q}\,(\mathcal{L}[c_3(t)\Phi])\,\mathrm{d}x\,\mathrm{d}t \\ &= \iint_Q f^q_{*,0}\,\Phi\,\mathrm{d}x\,\mathrm{d}t - \iint_Q \widetilde{q}\,\mathcal{L}^3[c_3(t)\Phi]\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Notice that

$$\mathcal{L}^3[c_3(t)\Phi] = c_3'''(t)\Phi + 3c_3''(t)\mathcal{L}\Phi + 3c_3'(t)\mathcal{L}^2\Phi + c_3(t)\mathcal{L}^3\Phi$$

and since

$$\|\Phi\|_{X_3} \le C\|h\|_{X_{2,0}},$$

(from regularity result (2.10) and the equivalence between norms), we obtain from the last equality, together with (4.16),

$$\iint_Q q_{*,0}\, h \,\mathrm{d}x\,\mathrm{d}t \le C\left[\left\|f^q_{*,0}\right\|_{L^2(Q)} + \|\widehat{q}\|_{L^2(Q)}\right]\|h\|_{X_{2,0}}, \quad \forall h \in X_{2,0}. \tag{4.17}$$

Now, let us set

$$q_{*,1} := e^{7/2s\beta^*}(\gamma^*)^{-8-8/m}\widehat{q}, \quad f^q_{*,1} := e^{7/2s\beta^*}(\gamma^*)^{-8-8/m}(f^q + \widehat{z}_N + \widehat{r}\mathbb{1}_{\mathcal{O}_2}).$$

Same as before, $q_{*,1}$ solves

$$\begin{cases} \mathcal{L}^* q_{*,1} = f^q_{*,1} - (e^{7/2s\beta^*}(\gamma^*)^{-8-8/m})_t\widehat{q} & \text{in } Q,\\ q_{*,1} = 0 & \text{on } \Sigma,\\ q_{*,1}(T) = 0 & \text{in } \Omega,\end{cases}$$

and for any $h \in X_{1,0}$ we have

$$\iint_Q q_{*,1}\, h \,\mathrm{d}x\,\mathrm{d}t = \iint_Q f^q_{*,1}\,\Phi\,\mathrm{d}x\,\mathrm{d}t - \iint_Q (e^{7/2s\beta^*}(\gamma^*)^{-8-8/m})_t\widehat{q}\,\Phi\,\mathrm{d}x\,\mathrm{d}t.$$

Moreover, since

$$\iint_Q (e^{7/2s\beta^*}(\gamma^*)^{-8-8/m})_t\widehat{q}\,\Phi\,\mathrm{d}x\,\mathrm{d}t = \iint_Q q_{*,0}\, c_2(t)\Phi\,\mathrm{d}x\,\mathrm{d}t,$$

and $c_2(t)\Phi \in X_{2,0}$, we obtain from (4.17), with $c_2(t)\Phi$ instead of h:

$$\iint_Q c_2(t)\Phi\, q_{*,0}\,\mathrm{d}x\,\mathrm{d}t \le C\left[\left\|f^q_{*,0}\right\|_{L^2(Q)} + \|\widehat{q}\|_{L^2(Q)}\right]\|c_2(t)\Phi\|_{X_{2,0}}.$$

Going back to $q_{*,1}$, we get using (4.16) that

$$\iint_Q q_{*,1}\, h \,\mathrm{d}x\,\mathrm{d}t \le C\left[\left\|f^q_{*,0}\right\|_{L^2(Q)} + \|\widehat{q}\|_{L^2(Q)}\right]\|\Phi\|_{X_{2,0}}.$$

Notice that we have also used $(\gamma^*)^{-8-8/m} \le C(\gamma^*)^{-5-5/m}$. From (2.10) we have

$$\|\Phi\|_{X_2} \le C\|h\|_{X_{1,0}},$$

and therefore

$$\iint_Q q_{*,1}\, h \,\mathrm{d}x\,\mathrm{d}t \le C\left[\left\|f^q_{*,0}\right\|_{L^2(Q)} + \|\widehat{q}\|_{L^2(Q)}\right]\|h\|_{X_{1,0}}, \quad \forall h \in X_{1,0}. \tag{4.18}$$

Finally, we set

$$q_{*,2} := e^{7/2s\beta^*}(\gamma^*)^{-10-10/m}\widehat{q}, \quad f^q_{*,2} := e^{7/2s\beta^*}(\gamma^*)^{-10-10/m}(f^q + \widehat{z}_N + \widehat{r}\mathbb{1}_{\mathcal{O}}).$$

As before, we find that for any $h \in X_0$ we have

$$\iint_Q q_{*,2}\, h \,\mathrm{d}x\,\mathrm{d}t = \iint_Q f^q_{*,2}\,\Phi\,\mathrm{d}x\,\mathrm{d}t - \iint_Q (e^{7/2s\beta^*}(\gamma^*)^{-10-10/m})_t\widehat{q}\,\Phi\,\mathrm{d}x\,\mathrm{d}t$$

and

$$\iint_Q (e^{7/2s\beta^*}(\gamma^*)^{-10-10/m})_t\widehat{q}\,\Phi\,\mathrm{d}x\,\mathrm{d}t = \iint_Q c_1(t)\Phi\, q_{*,1}\,\mathrm{d}x\,\mathrm{d}t,$$

so we can use (4.18) with $c_1(t)\Phi \in X_{1,0}$ instead of h and we obtain, using $\|\Phi\|_{X_1} \le C\|h\|_{X_0}$,

$$\iint_Q q_{*,2}\, h \,\mathrm{d}x\,\mathrm{d}t \le C\left[\left\|f^q_{*,0}\right\|_{L^2(Q)} + \|\widehat{q}\|_{L^2(Q)}\right]\|h\|_{X_0}, \quad \forall h \in X_0. \tag{4.19}$$

Thus, we deduce that $q_{*,2} \in L^2(Q)$.

Finally, to complete the proof of (4.9), let

$$q_* := e^{7/2s\beta^*}(\gamma^*)^{-11-11/m}\widehat{q}, \quad f^q_* := e^{7/2s\beta^*}(\gamma^*)^{-11-11/m}(f^q + \widehat{z}_N + \widehat{r}\mathbb{1}_{\mathcal{O}_2}).$$

Then, q_* satisfies

$$\begin{cases} \mathcal{L}^* q_* = f^q_* + (e^{7/2s\beta^*}(\gamma^*)^{-11-11/m})_t\widehat{q} & \text{in } Q, \\ q_* = 0 & \text{on } \Sigma, \\ q_*(T) = 0 & \text{in } \Omega. \end{cases}$$

The right-hand side of this equation belongs to $L^2(Q)^N$ in view of (4.3), (4.6), (4.10), (4.15), (2.2) and $q_{*,2} \in L^2(Q)$. Thus, applying the regularity result for the heat equation (2.10), we deduce that $q_* \in X_1$. We complete the decay properties in (4.9) in the same manner by (2.12) for $\widehat{w}$ and $\widehat{z}$. This finishes the proof of Proposition 4.2. □

5 Null controllability of the nonlinear system

In this section we prove Theorem 1.1. We will prove that there exists a control v with $v_i \equiv 0$, for $i \in \{1, \dots, N-1\}$, such that the solution of the system

$$\begin{cases} \mathcal{L}w + (w\cdot\nabla)w + \nabla p_0 = f + v\mathbb{1}_\omega + r\,e_N, & \nabla\cdot w = 0 \quad \text{in } Q, \\ \mathcal{L}^* z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + q\nabla r + \nabla p_1 = w\mathbb{1}_{\mathcal{O}_1}, & \nabla\cdot z = 0 \quad \text{in } Q, \\ \mathcal{L}r + w\cdot\nabla r = f_0 & \text{in } Q, \\ \mathcal{L}^* q - w\cdot\nabla q = z_N + r\,\mathbb{1}_{\mathcal{O}_2} & \text{in } Q, \\ w = z = 0, \quad r = s = 0 & \text{on } \Sigma, \\ w(0) = 0, \quad z(T) = 0, \quad r(0) = 0, \quad q(T) = 0 & \text{in } \Omega, \end{cases} \tag{5.1}$$

satisfies $(z(0), q(0)) = (0, 0)$ in Ω.

We follow the arguments used in [19] (see also [10], [11] and [12]). Thanks to Proposition 4.2, we will be able to obtain the result for this nonlinear control problem by means of the following inverse mapping theorem (see [1]):

Theorem 5.1. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two Banach spaces and let $\mathcal{F} : \mathcal{G}_1 \to \mathcal{G}_2$ satisfy $\mathcal{F} \in \mathcal{C}^1(\mathcal{G}_1; \mathcal{G}_2)$. Assume that $g_1 \in \mathcal{G}_1$, $\mathcal{F}(g_1) = g_2$ and that $\mathcal{F}'(g_1) : \mathcal{G}_1 \to \mathcal{G}_2$ is surjective. Then, there exists $\delta > 0$ such that, for every $\bar{g} \in \mathcal{G}_2$ satisfying $\|\bar{g} - g_2\|_{\mathcal{G}_2} < \delta$, there exists a solution of the equation*

$$\mathcal{F}(g) = \bar{g}, \quad g \in \mathcal{G}_1.$$

Proof of Theorem 1.1. We start by defining the space:

$$L^2(e^{6s\beta^*}(0,T); L^2(\Omega)) := \{u \in L^2(Q) : e^{6s\beta^*}u \in L^2(Q)\}.$$

We will use Theorem 5.1 with the spaces

$$\mathcal{G}_1 := E_i,$$

$$\mathcal{G}_2 := L^2(e^{6s\beta^*}(0,T); L^2(\Omega)^{2N+2})$$

and the operator

$$\begin{aligned}\mathcal{F}(w,p_0,z,p_1,r,q,v) :=& (\mathcal{L}w + (w\cdot\nabla)w + \nabla p_0 - v\mathbb{1}_\omega - r\, e_N,\\ &\mathcal{L}^* z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + q\nabla r + \nabla p_1 - w\mathbb{1}_{\mathcal{O}_1},\\ &\mathcal{L}r + w\cdot\nabla r, \quad \mathcal{L}^* q - w\cdot\nabla q - z_N - r\,\mathbb{1}_{\mathcal{O}_2}),\end{aligned}$$

for $(w,p_0,z,p_1,r,q,v)\in\mathcal{G}_1$.

With this setting, we have the following lemma:

Lemma 5.2. *The operator $\mathcal{F}$ is of class $\mathcal{C}^1(\mathcal{G}_1;\mathcal{G}_2)$.*

Proof. Since all the terms in $\mathcal{F}$ are linear, except for $(w\cdot\nabla)w$, $(z\cdot\nabla^t)w-(w\cdot\nabla)z$, $q\nabla r$, $w\cdot\nabla r$ and $w\cdot\nabla q$, it is sufficient to prove that the bilinear operator

$$((w^1,p_0^1,z^1,p_1^1,r^1,q^1,v^1), \quad (w^2,p_0^2,z^2,p_1^2,r^2,q^2,v^2)) \to (w^1\cdot\nabla)w^2$$

is continuous from $\mathcal{G}_1\times\mathcal{G}_1$ to $L^2(e^{6s\beta^*}(0,T);L^2(\Omega)^N)$. Since $Y_1\subset L^\infty(0,T;V)$, we have that

$$e^{7/2s\beta^*}(\gamma^*)^{-1-1/m}w \in L^2(0,T;H^2(\Omega)^N)\cap L^\infty(0,T;V)$$

for any $(w,p_0,z,p_1,r,q,v)\in\mathcal{G}_1$. Therefore

$$e^{7/2s\beta^*}(\gamma^*)^{-1-1/m}w \in L^2(0,T;L^\infty(\Omega)^N)$$

and

$$\nabla(e^{7/2s\beta^*}(\gamma^*)^{-1-1/m}w) \in L^\infty(0,T;L^2(\Omega)^{N\times N}).$$

Thus, taking into account (2.3), we obtain

$$\begin{aligned}&\big\|e^{6s\beta^*}(w^1\cdot\nabla)w^2\big\|_{L^2(Q)^N}\\ &= \big\|e^{-s\beta^*}(\gamma^*)^{2+2/m}\left(e^{7/2s\beta^*}(\gamma^*)^{-1-1/m}w^1\cdot\nabla\right)e^{7/2s\beta^*}(\gamma^*)^{-1-1/m}w^2\big\|_{L^2(Q)^N}\\ &\le C\big\|e^{7/2s\beta^*}(\gamma^*)^{-1-1/m}w^1\big\|_{L^2(0,T;L^\infty(\Omega)^N)}\big\|e^{7/2s\beta^*}(\gamma^*)^{-1-1/m}w^2\big\|_{L^\infty(0,T;V)}.\end{aligned}$$

Of course, we can perform the same computations for the terms $(z\cdot\nabla^t)w,(w\cdot\nabla)z$. The terms concerning r and q are treated analogously since

$$e^{7/2s\beta^*}\left((\gamma^*)^{-1-1/m}\,r, \quad (\gamma^*)^{-11-11/m}\,q\right)\in L^\infty(0,T;H_0^1(\Omega)^2),$$

for any $(w,p_0,z,p_1,r,q,v)\in\mathcal{G}_1$. ☐

Now, since $\mathcal{F}'(0):\mathcal{G}_1\to\mathcal{G}_2$ is given by

$$\begin{aligned}\mathcal{F}'(0)(w,p_0,z,p_1,r,q,v) =& (\mathcal{L}w + \nabla p_0 - v\mathbb{1}_\omega - r\,e_N, \quad \mathcal{L}^*z + \nabla p_1 - w\mathbb{1}_{\mathcal{O}_1},\\ &\mathcal{L}r, \quad \mathcal{L}^*q - z_N - r\,\mathbb{1}_{\mathcal{O}_2}),\end{aligned}$$

for all $(w,p_0,z,p_1,r,s,v)\in\mathcal{G}_1$, from Proposition 4.2 it is deduced that $\mathcal{F}'(0)$ is a surjective functional. Together with Lemma 5.2, we can apply Theorem 5.1 with $g_1=0$ and $g_2=0$. Thus, there exists $\delta>0$ such that, if $\|e^{C/t^m}(f,f_0)\|_{L^2(Q)^{N+1}}\le\delta$, for some $C>0$, then there exists $(w,p_0,z,p_1,r,q,v)\in\mathcal{G}_1$ solution of (5.1). In particular, $v_i\equiv 0$ and $(z(0),q(0))=(0,0)$ and the proof of Theorem 1.1 is complete. ☐

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