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On the non-uniform null controllability of a linear KdV equation

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Abstract. In this paper we consider a linear KdV equation with a transport term posed on a finite interval with the boundary conditions considered by Colin and Ghidaglia. The main results concern the behavior of the cost of null controllability with respect to the dispersion coefficient when the control acts on the left endpoint. In particular, for any final time we prove that this cost grows exponentially as the dispersion coefficient vanishes and the transport coefficient is negative.

Keywords: uniform controllability, dispersion limit, Carleman inequalities

1. Introduction

Let $T > 0$, $L > 0$ and $Q := (0, T) \times (0, L)$. We consider the following controlled Korteweg–de Vries (KdV) equation posed on a finite domain:

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } Q, \\ y_{|x=0} = v, \qquad y_{x\,|x=L} = 0, \qquad y_{xx\,|x=L} = 0 & \text{in } (0, T), \\ y_{|t=0} = y_0 & \text{in } (0, L), \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is the dispersion coefficient, $M \in \mathbb{R}$ is the transport coefficient, $y_0 \in L^2(0, L)$ is the initial condition and $v = v(t)$ stands for the control. Here and all along the paper we use the notation, for a given function $f = f(t, x)$,

$$f_{|x=x_0} := f(\cdot, x_0) \quad \text{and} \quad f_{|t=t_0} := f(t_0, \cdot).$$

The boundary conditions in (1.1) were proposed by Colin and Ghidaglia in [4] (see also [3]) as a model for propagation of surface water waves in the situation where a wave maker is putting energy on a finite-length channel from the left extremity and the right one is free.

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Most results for (1.1) are related to its well-posedness of its nonlinear version (see [13] and the references therein). However, let us observe that if $M > 0$, this system is not dissipative and therefore the existence of solutions needs to be justified. This will be done later in Section 2.

As for the controllability problem, in [2] the authors considered controls in all the boundary conditions and all the possible combinations. For controllability results of the KdV equation in a finite interval with another boundary conditions, see [1,15,16] and the references therein.

In this paper, we are mainly interested in the study of how the size of the control behaves as a function of the dispersion coefficient ε. To this end, we define the quantity

$$C_{\text{cost}}^{\varepsilon,0} := \sup_{\substack{y_0 \in L^2(0,L) \\ y_0 \neq 0}} \min_{\substack{v \in L^2(0,T) \\ y_{|t=T}=0}} \frac{\|v\|^2_{L^2(0,T)}}{\|y_0\|^2_{L^2(0,L)}}, \tag{1.2}$$

which stands for the *cost of null controllability* of (1.1). Notice that $C_{\text{cost}}^{\varepsilon,0}$ is the best constant such that, for all $y_0 \in L^2(0, L)$ and $v \in L^2(0, T)$ such that $y_{|t=T} = 0$, the estimate

$$\|v\|^2_{L^2(0,T)} \leqslant C_{\text{cost}}^{\varepsilon,0} \|y_0\|^2_{L^2(0,L)}$$

is satisfied.

Our first result states an improvement of the cost of the control with respect to the one in [12]. From this work, one can deduce that there exists $v \in L^2(0, T)$ such that

$$C_{\text{cost}}^{\varepsilon,0} \leqslant C \exp\bigl(C\varepsilon^{-1}\bigr). \tag{1.3}$$

The first main result of this paper is the following theorem.

Theorem 1.1. *Let $T, L, \varepsilon > 0$ and $M \in \mathbb{R}$. Then, for any $y_0 \in L^2(0, L)$, there exists a control $v \in L^2(0, T)$ such that the associated solution of (1.1) satisfies $y_{|t=T} = 0$. Furthermore, we have the estimate*

$$C_{\text{cost}}^{\varepsilon,0} \leqslant \bar{C} \exp\bigl(C\bigl(\varepsilon^{-1/2}T^{-1/2} + M^{1/2}\varepsilon^{-1/2} + MT\bigr)\bigr), \tag{1.4}$$

if $M > 0$, and

$$C_{\text{cost}}^{\varepsilon,0} \leqslant \bar{C} \exp\bigl(C\bigl(\varepsilon^{-1/2}T^{-1/2} + |M|^{1/2}\varepsilon^{-1/2}\bigr)\bigr), \tag{1.5}$$

if $M < 0$, where $C > 0$ is a constant independent of T, M and ε and $\bar{C} > 0$ depends at most polynomially on ε^{-1}, T^{-1}, T and $|M|^{-1}$.

We remark that (1.4)–(1.5) say that the cost of the control is at most of order $\exp(C\varepsilon^{-1/2})$, whereas in (1.3) is of order $\exp(C\varepsilon^{-1})$. This difference becomes of great importance when studying its behavior in the limit $\varepsilon \to 0$.

Before stating the second and third main results of this paper, let us consider the transport equation

$$y_t - My_x = 0 \quad \text{in } Q. \tag{1.6}$$

Since (1.6) is controllable if and only if $T \geqslant L/|M|$ (see, for instance, [5, Theorem 2.6, p. 29]), with a control $y_{|x=0} = v_1$ if $M < 0$ and a control $y_{|x=L} = v_2$ if $M > 0$. Furthermore, the cost of null controllability is equal to zero. Indeed, the solution of (1.6) can be brought to zero at time T just by taking $v_1 \equiv 0$ when $M < 0$, and by taking $v_2 \equiv 0$ when $M > 0$. Thus, we should expect that the cost would decrease to zero in this case as $\varepsilon \to 0$, or at least if the final time T is large enough. On the other hand, if $T < L/|M|$ it is expected that the cost of the control would explode as ε tends to zero.

In [10], the authors consider this problem for the classical boundary conditions

$$y_{|x=0} = v(t), \qquad y_{|x=L} = 0, \qquad y_{x\,|x=L} = 0 \quad \text{in } (0, T),$$

and in [9] with controls in all the boundary terms. We refer also to [6] and [11] for the case of vanishing viscosity in one and arbitrary space dimension, respectively.

In these works, the strategy relies on the combination of a suitable Carleman inequality, which gives an observability constant that explodes with ε, with an exponential dissipation estimate for the adjoint equation such that for T large enough counteracts the previous constant. It has been pointed out in [9] and [10] that such a result can only be expected for (1.1) when $M > 0$ due to the asymmetric effect of the dispersion term.

We will prove that the cost of null controllability explodes as $\varepsilon \to 0$, even if T is large. Actually, we will prove it for the quantity

$$C_{\text{cost}}^{\varepsilon,-1} := \sup_{\substack{y_0 \in H_n^3(0,L) \\ y_0 \neq 0}} \min_{\substack{v \in H^{-1}(0,T) \\ y_{|t=T}=0}} \frac{\|v\|^2_{H^{-1}(0,T)}}{\|y_0\|^2_{H_n^3(0,L)}}, \tag{1.7}$$

where the space $H_n^3(0, L)$ is defined as follows: for any $a < b$, let

$$H_n^3(a, b) := \{h \in H^3(a, b) \colon h'(b) = h''(b) = 0\},$$

endowed with the norm

$$\|h\|^2_{H_n^3(a,b)} := \|h\|^2_{L^2(a,b)} + \|h'''\|^2_{L^2(a,b)}.$$

The following theorem states a lower bound for this new cost of null controllability.

Theorem 1.2. *Let $T, L, \varepsilon, M > 0$ and $\gamma, \delta \in (0, 1)$. Then,*

$$C_{\text{cost}}^{\varepsilon,-1} > \frac{(1-\gamma)\bar{K}(T, L, \varepsilon, M, \delta)}{C_{-1}(T, L, \varepsilon, M, \delta)C_1(\delta L, L)} - 1,$$

where

$$\bar{K}(T, L, \varepsilon, M, \delta) := \frac{\sinh^2((1-\delta)LM^{1/2}\varepsilon^{-1/2})}{\varepsilon T M(1-\delta)L}$$

and, $C_{-1}(T, L, \varepsilon, M, \delta)$ and $C_1(\delta L, L)$ are given by (4.2) and (4.25), respectively.

Remark 1.3. From the expression of $C_{-1}(T, L, \varepsilon, M, \delta)$ in (4.2), one can see that it depends polynomially on ε and ε^{-1}.

From Theorem 1.2, we can deduce the following result that shows that the cost defined in (1.7) blows up as ε vanishes.

Corollary 1.4. *Let $T, L, M > 0$ and $\kappa \in (0,1)$. Then, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$C_{\text{cost}}^{\varepsilon,-1} \geqslant \exp\left(\frac{(2-\kappa)LM^{1/2}}{\varepsilon^{1/2}}\right).$$

Notice that this implies that the cost defined in (1.2) goes to infinity as ε goes to zero since $C_{\text{cost}}^{\varepsilon,-1} \leqslant C_{\text{cost}}^{\varepsilon,0}$. What is interesting about this result is that it differs from the one obtained in [10] (and [6] for that matter). Notice that for $M > 0$ and $\varepsilon \to 0$ in (1.1), it seems very difficult (if not impossible) to prove convergence in some sense to a solution of the transport equation (1.6) since we do not know the value at the right-end of the interval $(0, L)$. Actually, this convergence question has not been addressed even for the classical boundary conditions (see [10]). Nevertheless, one could have expected to obtain an appropriate dissipation estimate as in [9], but Corollary 1.4 shows that this is not the case.

Finally, when $M < 0$ we are able to obtain an explosion result when T is smaller than $L/|M|$.

Theorem 1.5. *Let $M < 0$ and $T < L/|M|$. Then, there exist a constant $C > 0$ (independent of ε), $\varepsilon_0 > 0$ and initial conditions $y_0 \in L^2(0, L)$ such that, if $v \in L^2(0, T)$ is a control such that the solution y of (1.1) satisfies $y_{|t=T} = 0$, then, for every $\varepsilon \in (0, \varepsilon_0)$,*

$$\|v\|^2_{L^2(0,T)} \geqslant \exp\left(\frac{C}{\varepsilon^{1/2}}\right)\|y_0\|^2_{L^2(0,L)}. \tag{1.8}$$

In particular,

$$C_{\text{cost}}^{\varepsilon,0} \geqslant \exp\left(\frac{C}{\varepsilon^{1/2}}\right).$$

The rest of the paper is organized as follows. In Section 2 we prove the existence of solutions of equation (1.1). In Section 3, we prove Theorem 1.1. We prove an observability inequality for the adjoint system (3.2) and then applying the Hilbert Uniqueness Method (HUM). The observability inequality is proved by means of a suitable Carleman estimate. In Section 4, we prove Theorem 1.2. In Section 5, we prove Theorem 1.5. Finally, we give the proof of the Carleman estimate in Appendix A.

2. Existence of solutions when $M > 0$

In this section we will prove the existence (and uniqueness) of a solution of (1.1) when $M > 0$, $v \in L^2(0, T)$ and $y_0 \in L^2(0, L)$. Let us first remark that if y solves (1.1), then

$$z(t,x) := \int_0^t y(s,x)\,\mathrm{d}s - \frac{(L-x)^3}{L^3}\int_0^t v(s)\,\mathrm{d}s$$

would solve

$$\begin{cases} z_t + \varepsilon z_{xxx} - Mz_x = f & \text{in } Q, \\ z_{|x=0} = 0, \qquad z_{x|x=L} = 0, \qquad z_{xx|x=L} = 0 & \text{in } (0,T), \\ z_{|t=0} = 0 & \text{in } (0,L), \end{cases} \tag{2.1}$$

with

$$f(t,x) := -\frac{(L-x)^3}{L^3}v(t) + \left(\frac{6\varepsilon}{L^3} - \frac{3M(L-x)^2}{L^3}\right)\int_0^t v(s)\,\mathrm{d}t + y_0. \tag{2.2}$$

Then, if we prove the existence (and uniqueness) of solution z of (2.1), we would have proved the existence (and uniqueness) of a solution of (1.1) by simply defining

$$y(t,x) := z_t(t,x) + \frac{(L-x)^3}{L^3}v(t). \tag{2.3}$$

Lemma 2.1. *Let $\varepsilon, T, M > 0$ and $f \in L^2(Q)$. Then, there exists a unique solution z of (2.1) which belongs to $\mathcal{X}_T := L^\infty(0,T;L^2(0,L)) \cap L^2(0,T;H^1(0,L))$.*

Proof. We will use the contracting map fixed-point theorem. We consider, for any $g \in L^2(0,T; H^1(0,L))$, the following problem

$$\begin{cases} z_t + \varepsilon z_{xxx} = Mg_x + f & \text{in } Q, \\ z_{|x=0} = 0, \qquad z_{x|x=L} = 0, \qquad z_{xx|x=L} = 0 & \text{in } (0,T), \\ z_{|t=0} = 0 & \text{in } (0,L). \end{cases} \tag{2.4}$$

Notice that the operator defined by

$$A := -\varepsilon z_{xxx}$$

is dissipative with

$$\mathcal{D}(A) = \{h \in H^3(0,L) \colon h(0) = h'(L) = h''(L) = 0\}.$$

It is easy to check that the adjoint of A is also dissipative, thus the existence of a unique z solution of (2.4) is ensured.

Now, we multiply the equation in (2.4) by $(x+L)z$ and integrate by parts in space. We obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (x+L)|z|^2\,\mathrm{d}x + \frac{3\varepsilon}{2}\int_0^L |z_x|^2\,\mathrm{d}x \leqslant M\int_0^L (x+L)g_x z\,\mathrm{d}x + \int_0^L (x+L)fz\,\mathrm{d}x.$$

Using Young's and Poincaré's inequalities, we get

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (x+L)|z|^2\,\mathrm{d}x + \varepsilon\int_0^L |z_x|^2\,\mathrm{d}x \\ &\quad \leqslant 2M\left(\int_0^L (x+L)|z|^2\,\mathrm{d}x\right)^{1/2}\left(\int_0^L (x+L)|g_x|^2\,\mathrm{d}x\right)^{1/2} + C\varepsilon^{-1}\int_0^L |f|^2\,\mathrm{d}x, \end{aligned}$$

where $C > 0$ depends only on L. We integrate between 0 and t, and take the supremum:

$$\sup_{t\in(0,T)} \left\|(x+L)^{1/2} z\right\|^2_{L^2(0,L)} + \varepsilon\|z_x\|^2_{L^2(Q)}$$
$$\leqslant 2M \sup_{t\in(0,T)} \left\|(x+L)^{1/2} z\right\|_{L^2(0,L)} \int_0^T \left\|(x+L)^{1/2} g_x\right\|_{L^2(0,L)} \mathrm{d}t + C\varepsilon^{-1}\|f\|^2_{L^2(Q)}.$$

Using Young's inequality once more and (2.2) we obtain

$$\|z\|^2_{L^\infty(0,T;L^2(0,L))} + \varepsilon\|z\|^2_{L^2(0,T;H^1(0,L))}$$
$$\leqslant CM^2\|g\|^2_{L^1(0,T;H^1(0,L))} + C\varepsilon^{-1}\big(\big(1+\varepsilon^2T+M^2T\big)\|v\|^2_{L^2(0,T)} + T\|y_0\|^2_{L^2(0,L)}\big), \tag{2.5}$$

where C depends only on L.

We consider the norm

$$\|z\|^2_{\mathcal{X}_T} := \|z\|^2_{L^\infty(0,T;L^2(0,L))} + \varepsilon\|z\|^2_{L^2(0,T;H^1(0,L))}.$$

Let us now prove the existence of $\beta \in (0,T]$ such that the map

$$\mathcal{A} : \mathcal{X}_\beta \to \mathcal{X}_\beta$$
$$g \mapsto z$$

is a contraction. In view of (2.5), we have

$$\left\|\mathcal{A}(g)\right\|^2_{\mathcal{X}_\beta} \leqslant CM^2\varepsilon^{-1}\beta\|g\|^2_{\mathcal{X}_\beta} + C\varepsilon^{-1}\big(\big(1+\varepsilon^2T+M^2T\big)\|v\|^2_{L^2(0,T)} + T\|y_0\|^2_{L^2(0,L)}\big).$$

Let now, for $r > 0$:

$$B_r := \big\{h \in \mathcal{X}_\beta : \|h\|_{\mathcal{X}_\beta} \leqslant r\big\},$$

with

$$r^2 = 2C\varepsilon^{-1}\big(\big(1+\varepsilon^2T+M^2T\big)\|v\|^2_{L^2(0,T)} + T\|y_0\|^2_{L^2(0,L)}\big).$$

Choosing β such that

$$CM^2\varepsilon^{-1}\beta \leqslant \frac{1}{2}$$

we have for any $g \in B_r$:

$$\left\|\mathcal{A}(g)\right\|_{\mathcal{X}_\beta} \leqslant r.$$

Then, for any $g_1, g_2 \in B_r$:

$$\left\|\mathcal{A}(g_1) - \mathcal{A}(g_2)\right\|_{\mathcal{X}_\beta} \leqslant \frac{1}{\sqrt{2}}\|g_1 - g_2\|_{\mathcal{X}_\beta}.$$

Therefore, $\mathcal{A}$ is a contraction mapping on B_r and admits a unique fixed point $z = \mathcal{A}(z)$ in $\mathcal{X}_\beta$ which is easy to check that is the solution of (2.1) in $(0, \beta) \times (0, L)$. Since β does not depend on v nor y_0, we can repeat this argument in the time intervals $(n\beta, (n+1)\beta)$, for any $n \in \mathbb{N}$, with initial condition $z^{n\beta}_{|t=n\beta} \in L^2(0, L)$, where $z^{n\beta} \in \mathcal{X}_{n\beta}$ is the solution of (2.1) in $(0, n\beta) \times (0, L)$. Thus, $z \in \mathcal{X}_T$ and is a solution of (2.1). □

From (2.1)–(2.3), one sees that the solution of (1.1) belongs to $L^2(0, T; H^{-2}(0, L)) \cap \mathcal{C}^0([0, T]; H^{-5}(0, L))$. In the sequel, we will also need to establish the existence of solutions of (1.1) when $v \in H^{-1}(0, T)$.

Corollary 2.2. *Let $\varepsilon, T, M > 0$, $v \in H^{-1}(0, T)$ and $y_0 \in L^2(0, L)$. Then, there exists a unique solution $y \in L^2(0, T; H^{-5}(0, L)) \cap \mathcal{C}^0([0, T]; H^{-8}(0, L))$ of (1.1).*

Proof. Let $v \in H^{-1}(0, T)$. Then (see for instance [7, Theorem 1, p. 283]), there exist $v_0, v_1 \in L^2(0, T)$ such that

$$v = v_0 + v_1'. \tag{2.6}$$

Furthermore,

$$\|v\|_{H^{-1}(0,T)} = \inf_{\substack{v_0, v_1 \in L^2(0,T) \\ \text{s.t. (2.6) holds}}} \left(\|v_0\|^2_{L^2(0,T)} + \|v_1\|^2_{L^2(0,T)}\right)^{1/2}. \tag{2.7}$$

Then,

$$y = p + q_t, \tag{2.8}$$

where p and q are the solutions of

$$\begin{cases} p_t + \varepsilon p_{xxx} - M p_x = 0 & \text{in } Q, \\ p_{|x=0} = v_0(t), \qquad p_{x|x=L} = 0, \qquad p_{xx|x=L} = 0 & \text{in } (0, T), \\ p_{|t=0} = y_0 & \text{in } (0, L), \end{cases} \tag{2.9}$$

and

$$\begin{cases} q_t + \varepsilon q_{xxx} - M q_x = 0 & \text{in } Q, \\ q_{|x=0} = v_1(t), \qquad q_{x|x=L} = 0, \qquad q_{xx|x=L} = 0 & \text{in } (0, T), \\ q_{|t=0} = 0 & \text{in } (0, L), \end{cases} \tag{2.10}$$

respectively.

From Lemma 2.1 and the computations above, one deduces the desired result. □

3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by applying the *Hilbert Uniqueness Method* (HUM) (see, for instance, [14]), that is, we prove the following observability inequality:

$$\|\varphi_{|t=0}\|^2_{L^2(0,L)} \leqslant C_{\text{obs}}\|\varphi_{xx\,|x=0}\|^2_{L^2(0,T)}. \tag{3.1}$$

Here, φ is the solution of the adjoint equation

$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xxx} + M\varphi_x = 0 & \text{in } Q,\\ \varphi_{|x=0} = 0, \qquad \varphi_{x\,|x=0} = 0, \qquad (\varepsilon\varphi_{xx} - M\varphi)_{|x=L} = 0 & \text{in } (0,T),\\ \varphi_{|t=T} = \varphi_T & \text{in } (0,L), \end{cases} \tag{3.2}$$

with $\varphi_T \in L^2(0,L)$ and $C_{\text{obs}} > 0$ is a constant independent of φ. Indeed, (3.1) is equivalent to prove that for every $y_0 \in L^2(0,L)$, there exists a control $v \in L^2(0,T)$ such that $y_{|t=T} = 0$ satisfying

$$\|v\|^2_{L^2(0,T)} \leqslant \frac{C_{\text{obs}}}{\varepsilon^2}\|y_0\|^2_{L^2(0,L)},$$

where y is the solution of (1.1). Thus, once an inequality like (3.1) is established, the proof of Theorem 1.1 is finished since

$$C^{\varepsilon,0}_{\text{cost}} \leqslant \frac{C_{\text{obs}}}{\varepsilon^2}. \tag{3.3}$$

The observability inequality (3.1) is proved by means of Carleman and energy estimates, which are the goals of the following sections.

3.1. A change of unknowns

A relevant system associated to (3.2) will be

$$\begin{cases} -\phi_t - \varepsilon\phi_{xxx} + M\phi_x = 0 & \text{in } Q,\\ \phi_{x\,|x=0} = 0, \qquad \phi_{xx\,|x=0} = 0, \qquad \phi_{|x=L} = 0 & \text{in } (0,T). \end{cases} \tag{3.4}$$

Notice that (3.4) comes from

$$\phi := \varepsilon\varphi_{xx} - M\varphi. \tag{3.5}$$

Furthermore, we notice that from (3.5) and the boundary conditions on $x = 0$ in (3.2), for every $t \in (0,T)$ we have the following initial value ordinary differential equation

$$\begin{cases} \varphi_{xx} - M\varepsilon^{-1}\varphi = \varepsilon^{-1}\phi & \text{in } (0,L),\\ \varphi_{|x=0} = 0, \qquad \varphi_{x\,|x=0} = 0. \end{cases}$$

We can actually find an explicit formula for φ in terms of ϕ, but we need to distinguish the cases $M > 0$ and $M < 0$:

3.1.1. Case $M > 0$

It is not difficult to show that the solution is given by

$$\varphi(t,x) = \frac{1}{\varepsilon^{1/2}M^{1/2}}\int_0^x \sinh\big(M^{1/2}\varepsilon^{-1/2}(x-s)\big)\phi(t,s)\,\mathrm{d}s.$$

Thus,

$$\big|\varphi(t,x)\big| \leqslant \frac{1}{\varepsilon^{1/2}M^{1/2}}\left(\int_0^L \sinh^2\big(M^{1/2}\varepsilon^{-1/2}(1-s)\big)\,\mathrm{d}s\right)^{1/2}\left(\int_0^L \big|\phi(t,s)\big|^2\,\mathrm{d}s\right)^{1/2}.$$

Taking the $L^2(0,L)$-norm in this inequality, we get

$$\int_0^L \big|\varphi(t,x)\big|^2\,\mathrm{d}x \leqslant \frac{C}{\varepsilon^{1/2}M^{3/2}}\exp\big(CM^{1/2}\varepsilon^{-1/2}\big)\int_0^L \big|\phi(t,x)\big|^2\,\mathrm{d}x. \tag{3.6}$$

Moreover, since

$$\varphi_x(t,x) = \frac{1}{\varepsilon}\int_0^x \cosh\big(M^{1/2}\varepsilon^{-1/2}(x-s)\big)\phi(t,s)\,\mathrm{d}s,$$

same computations show that

$$\int_0^L \big|\varphi_x(t,x)\big|^2\,\mathrm{d}x \leqslant \frac{C}{\varepsilon^{3/2}M^{1/2}}\exp\big(CM^{1/2}\varepsilon^{-1/2}\big)\int_0^L \big|\phi(t,x)\big|^2\,\mathrm{d}x. \tag{3.7}$$

Using directly (3.5) and (3.6), we obtain

$$\int_0^L \big|\varphi_{xx}(t,x)\big|^2\,\mathrm{d}x \leqslant \left(\frac{2}{\varepsilon^2} + \frac{CM^{1/2}}{\varepsilon^{5/2}}\exp\big(CM^{1/2}\varepsilon^{-1/2}\big)\right)\int_0^L \big|\phi(t,x)\big|^2\,\mathrm{d}x. \tag{3.8}$$

3.1.2. Case $M < 0$

In this case, φ is given by

$$\varphi(t,x) = \frac{1}{\varepsilon^{1/2}|M|^{1/2}}\int_0^x \sin\big(|M|^{1/2}\varepsilon^{-1/2}(x-s)\big)\phi(s)\,\mathrm{d}s.$$

The same computations as for the case $M > 0$ show that

$$\int_0^L \big|\varphi(t,x)\big|^2\,\mathrm{d}x \leqslant \frac{C}{\varepsilon|M|}\int_0^L \big|\phi(t,x)\big|^2\,\mathrm{d}x, \tag{3.9}$$

$$\int_0^L \big|\varphi_x(t,x)\big|^2\,\mathrm{d}x \leqslant \frac{C}{\varepsilon^2}\int_0^L \big|\phi(t,x)\big|^2\,\mathrm{d}x \tag{3.10}$$

and

$$\int_0^L \big|\varphi_{xx}(t,x)\big|^2\,\mathrm{d}x \leqslant \left(\frac{2}{\varepsilon^2} + \frac{C|M|}{\varepsilon^3}\right)\int_0^L \big|\phi(t,x)\big|^2\,\mathrm{d}x. \tag{3.11}$$

3.2. Carleman estimates

To establish the Carleman estimate, we introduce some weight functions. Let

$$\alpha(t,x) = \frac{-\frac{x^2}{L^2} + \frac{4x}{L} + 1}{t^m(T-t)^m},$$

where $m \geqslant 1/2$. Notice that there exist positive constants C_0 and C_1 that do not depend on T such that

$$C_0 \leqslant T^{2m}\alpha, \qquad C_0\alpha \leqslant \alpha_x \leqslant C_1\alpha, \qquad C_0\alpha \leqslant -\alpha_{xx} \leqslant C_1\alpha, \tag{3.12}$$

and

$$|\alpha_t| + |\alpha_{xt}| + |\alpha_{xxt}| \leqslant C_1 T \alpha^{1+1/m}, \qquad |\alpha_{tt}| \leqslant C_1 T^2 \alpha^{1+2/m}, \tag{3.13}$$

for every $(t,x) \in Q$. This kind of weights has been introduced in [8] and widely used in the literature (in particular, in [9,10,12]).

We now are in position to present our Carleman inequality whose proof is given at the end of the paper (Appendix A).

Proposition 3.1. *Let $T, \varepsilon > 0$, $M \in \mathbb{R}$ and $m = 1/2$. There exists a positive constant C independent of T, ε and M such that, for any solution ϕ of (3.4), we have*

$$\iint_Q e^{-2s\alpha}\left(s^5\alpha^5|\phi|^2 + s^3\alpha^3|\phi_x|^2 + s\alpha|\phi_{xx}|^2\right) dx\, dt \leqslant Cs^5 \int_0^T e^{-2s\alpha_{|x=0}} \alpha^5_{|x=0} |\phi_{|x=0}|^2\, dt, \tag{3.14}$$

for all $s \geqslant C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$.

Furthermore, we can deduce from Proposition 3.1 and (3.6)–(3.11) a Carleman estimate for the solutions of (3.2).

Proposition 3.2. *Let $T, \varepsilon > 0$, $M \in \mathbb{R}\backslash\{0\}$ and $m = 1/2$. There exists a positive constant C independent of T, ε and M such that, for any solution φ of (3.2), we have*

$$\begin{aligned} &\iint_Q e^{-2s\alpha_{|x=L}}\left(s^5\alpha^5_{|x=0}|\varphi|^2 + s^3\alpha^3_{|x=0}|\varphi_x|^2 + s\alpha_{|x=0}|\varphi_{xx}|^2\right) dx\, dt \\ &\quad \leqslant \bar{C}\exp\left(C|M|^{1/2}\varepsilon^{-1/2}\right)s^5 \int_0^T e^{-2s\alpha_{|x=0}}\alpha^5_{|x=0}|\varphi_{xx\,|x=0}|^2\, dt, \end{aligned} \tag{3.15}$$

for all $s \geqslant C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$ and $\bar{C}$ depends at most polynomially on ε^{-1}, ε, $|M|^{-1}$ and $|M|$.

Remark 3.3. The lack of homogeneous Dirichlet condition on $x = L$ plays an important role in the choice of the power m of the weight function to prove (3.15). Indeed, a similar inequality was proved in [12] where $m \geqslant 1$ was needed to estimate a trace term on $x = L$. From this inequality one can deduce that the cost of the null controllability is bounded by $\exp(C\varepsilon^{-1})$.

Here, by means of the change of variable (3.5), which satisfies $\phi_{|x=L} = 0$, we manage to take the optimal power $m = 1/2$ as in [9,10].

It would be interesting to know if a Carleman estimate can be obtained for the solutions of (3.2) for $m = 1/2$ without using this change of variables.

3.3. *Dissipation estimates*

To prove (3.1), we will combine (3.14) with a dissipation estimate. For $M < 0$, it is easy to check that

$$\int_0^L |\phi_{|t=t_1}|^2 \,\mathrm{d}x \leqslant \int_0^L |\phi_{|t=t_2}|^2 \,\mathrm{d}x, \tag{3.16}$$

for every $0 \leqslant t_1 \leqslant t_2 \leqslant T$. For $M > 0$ we can prove the following result.

Proposition 3.4. *Let $\varepsilon, M > 0$. Then, for every pair (t_1, t_2) such that $0 \leqslant t_1 < t_2 \leqslant T$ and every solution ϕ of (3.4) the following inequality is satisfied*

$$E(\phi)(t_1) \leqslant \exp\bigl(CM(t_2 - t_1)\bigr)E(\phi)(t_2), \tag{3.17}$$

where $C > 0$ is a constant independent of M, T and ε, and

$$E(\phi)(t) := \int_0^L \bigl(\bigl|\phi(t,x)\bigr|^2 + (L-x)^3\bigl|\phi_x(t,x)\bigr|^2\bigr)\,\mathrm{d}x.$$

Proof. We proceed in two steps. First, we multiply (3.4) by $(2L-x)\phi$ and integrate in $(0, L)$. We obtain after some integration by parts

$$\begin{aligned}&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (2L-x)|\phi|^2\,\mathrm{d}x + \frac{3\varepsilon}{2}\int_0^L |\phi_x|^2\,\mathrm{d}x + \frac{\varepsilon}{2}|\phi_{x|x=L}|^2 + \frac{M}{2}\int_0^L |\phi|^2\,\mathrm{d}x\\ &\quad = LM|\phi_{|x=0}|^2.\end{aligned} \tag{3.18}$$

Next, we take the derivative with respect to x of (3.4), multiply by $\frac{(L-x)^3}{2}\phi_x$ and proceed as before. Straightforward computations lead to

$$\begin{aligned}&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{(L-x)^3}{2}|\phi_x|^2\,\mathrm{d}x + \frac{9\varepsilon}{4}\int_0^L (L-x)^2|\phi_{xx}|^2\,\mathrm{d}x\\ &\quad - \frac{3\varepsilon}{2}\int_0^L |\phi_x|^2\,\mathrm{d}x + \frac{3M}{4}\int_0^L (L-x)^2|\phi_x|^2\,\mathrm{d}x = 0.\end{aligned} \tag{3.19}$$

Adding (3.18) and (3.19) we obtain, after neglecting the positive terms on the left-hand side,

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (2L-x)|\phi|^2\,\mathrm{d}x - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{(L-x)^3}{2}|\phi_x|^2\,\mathrm{d}x \leqslant LM|\phi_{|x=0}|^2.$$

Now, notice that

$$|\phi_{|x=0}|^2 \leqslant C\|\phi\|^2_{H^1(0,\frac{L}{2})},$$

and therefore we obtain

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (2L-x)|\phi|^2\,\mathrm{d}x - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \frac{(L-x)^3}{2}|\phi_x|^2\,\mathrm{d}x$$
$$\leqslant 16CM\left(\int_0^L (2L-x)|\phi|^2\,\mathrm{d}x + \int_0^L \frac{(L-x)^3}{2}|\phi_x|^2\,\mathrm{d}x\right).$$

Estimate (3.17) can be easily deduced by integrating in (t_1, t_2). □

3.4. *Observability inequality*

In this section we combine the Carleman inequality (3.14) and the dissipation estimates obtained in the previous paragraph to finish the proof of (3.1) and therefore the proof of Theorem 1.1. Let us separate the cases $M > 0$ and $M < 0$.

3.4.1. *Case $M > 0$*

Notice that from (3.14) we have

$$\int_{T/4}^{3T/4}\int_0^L \left(|\phi|^2 + |\phi_x|^2\right)\mathrm{d}x\,\mathrm{d}t \leqslant Cs^2\left(1+T^{-2}\right)\exp(sC/T)\int_0^T |\phi_{|x=0}|^2\,\mathrm{d}t.$$

From the dissipation estimate (3.17) with $t_1 = 0$ and integrating between $T/4$ and $3T/4$ we obtain

$$\frac{T}{2}\int_0^L |\phi_{|t=0}|^2\,\mathrm{d}x \leqslant C\exp(CMT)\int_{T/4}^{3T/4}\int_0^L \left(|\phi|^2 + |\phi_x|^2\right)\mathrm{d}x\,\mathrm{d}t.$$

Combining these two inequalities and fixing $s = C(T + \varepsilon^{-1/2}T^{1/2} + M^{1/2}\varepsilon^{-1/2}T)$, we get

$$\int_0^L |\phi_{|t=0}|^2\,\mathrm{d}x$$
$$\leqslant \bar{C}\left(1+\varepsilon^{-1}\right)(1+M)\exp\left(C\left(MT + \varepsilon^{-1/2}T^{-1/2} + M^{1/2}\varepsilon^{-1/2}\right)\right)\int_0^T |\phi_{|x=0}|^2\,\mathrm{d}t, \tag{3.20}$$

where $\bar{C}$ depends at most polynomially on T^{-1} and T.

Let us now go back to φ. From (3.5), we obtain directly $\phi_{|x=0} = \varepsilon\varphi_{xx\,|x=0}$. On the other hand, from (3.6) with $t = 0$, we find

$$\int_0^L |\varphi_{|t=0}|^2\,\mathrm{d}x \leqslant \frac{C}{\varepsilon^{1/2}M^{3/2}}\exp\left(CM^{1/2}\varepsilon^{-1/2}\right)\int_0^L |\phi_{|t=0}|^2\,\mathrm{d}x.$$

These two elements combined with (3.20) give (3.1) with

$$C_{\mathrm{obs}} = \hat{C}\big(\varepsilon^{1/2} + \varepsilon^{3/2}\big)\exp\big(C\big(\varepsilon^{-1/2}T^{-1/2} + M^{1/2}\varepsilon^{-1/2} + MT\big)\big),$$

where $\hat{C}$ depends polynomially on T^{-1}, T and M^{-1}. Consequently, from (3.3), (1.4) is deduced.

3.4.2. Case $M < 0$

This case is actually simpler. From (3.14) we have

$$\int_{T/4}^{3T/4}\int_0^L |\phi|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C\exp(sC/T)\int_0^T |\phi_{|x=0}|^2\,\mathrm{d}t.$$

Using (3.16) and the same arguments as for the previous case yield

$$\int_0^L |\phi_{|t=0}|^2\,\mathrm{d}x \leqslant \frac{C}{T}\exp\big(C\big(\varepsilon^{-1/2}T^{-1/2} + |M|^{1/2}\varepsilon^{-1/2}\big)\big)\int_0^T |\phi_{|x=0}|^2\,\mathrm{d}t.$$

We recover φ as before from (3.9) instead of (3.6) and obtain (3.1) with

$$C_{\mathrm{obs}} = \varepsilon\hat{C}\exp\big(C\big(\varepsilon^{-1/2}T^{-1/2} + |M|^{1/2}\varepsilon^{-1/2}\big)\big),$$

where $\hat{C}$ depends polynomially on $|M|^{-1}$ and T^{-1}. From (3.3), this gives (1.5).

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2. For ease of comprehension, we have divided the proof in two parts presented in the following paragraphs.

4.1. Previous estimates

Here we will establish several estimates on y (solution of (1.1)) in terms of a constant depending only on T, L, ε and M times the norm of v and the norm of y_0. The principal result of this part is the following lemma.

Lemma 4.1. *For any $\delta \in (0,1)$, any $T, L, \varepsilon, M > 0$, any $y_0 \in H_n^3(0,L)$ and any $v \in H^{-1}(0,T)$, the solution y of (1.1) satisfies*

$$\|y_{xx|x=\delta L}\|^2_{L^2(0,T)} \leqslant C_{-1}(T,L,\varepsilon,M,\delta)\big(\|v\|^2_{H^{-1}(0,T)} + \|y_0\|^2_{L^2(0,L)} + \|y_0'''\|^2_{L^2(0,L)}\big), \tag{4.1}$$

where

$$C_{-1}(T,L,\varepsilon,M,\delta) := C_0(T,L,\varepsilon,M,\delta) + C_0'(T,L,\varepsilon,M,\delta) \tag{4.2}$$

with $C_0(T,L,\varepsilon,M,\delta)$ and $C_0'(T,L,\varepsilon,M,\delta)$ given by Lemma 4.2.

The proof of Lemma 4.1 relies on the following estimates for Eq. (1.1):

Lemma 4.2. *There exists $C > 0$ such that for any $\delta \in (0, 1)$, any $T, L, \varepsilon, M > 0$, any $y_0 \in H_n^3(0, L)$ and any $v \in L^2(0, T)$, the solution y of (1.1) satisfies*

$$\|y_{xx|x=\delta L}\|^2_{L^2(0,T)} \leqslant C_0(T, L, \varepsilon, M, \delta)\big(\|v\|^2_{L^2(0,T)} + \|y_0\|^2_{L^2(0,L)} + \|y_0'''\|^2_{L^2(0,L)}\big), \tag{4.3}$$

where

$$C_0(T, L, \varepsilon, M, \delta) := \frac{CTe^{CMT/L}}{L^4\delta^{10}}\left(\left(\frac{\varepsilon T}{L^3}\right)^{-1} + \left(\frac{\varepsilon T}{L^3}\right)^4 + \left(\frac{MT}{L}\right)^4\right)\left(\frac{1}{T} + \frac{1}{L} + L^5\right).$$

Furthermore, if $y_0 \equiv 0$, then

$$\|y_{txx|x=\delta L}\|^2_{L^2(0,T)} \leqslant C_0'(T, L, \varepsilon, M, \delta)\|v\|^2_{L^2(0,T)}, \tag{4.4}$$

where

$$\begin{aligned} C_0'(T, L, \varepsilon, M, \delta) := {}& \frac{Ce^{CMT/L}}{T^2L^4\delta^{22}}\left(\left(\frac{\varepsilon T}{L^3}\right)^{-1} + \left(\frac{\varepsilon T}{L^3}\right)^4 + \left(\frac{MT}{L}\right)^4\right) \\ & \times \left(\left(\frac{\varepsilon T}{L^3}\right)^{-2} + \left(\frac{\varepsilon T}{L^3}\right)^4 + \left(\frac{MT}{L}\right)^2 + \left(\frac{MT}{L}\right)^8\right). \end{aligned}$$

Proof. To start the proof, we perform the change of variables

$$\tilde{t} := \frac{t}{T} \quad \text{and} \quad \tilde{x} := \frac{x}{L}.$$

Then, $\tilde{y}(\tilde{t}, \tilde{x}) := y(t, x)$ satisfies system (1.1) for

$$(T, L, \varepsilon, M) = (1, 1, \tilde{\varepsilon}, \tilde{M}) \quad \text{and} \quad (v, y_0) = (\tilde{v}, \tilde{y}_0)$$

with

$$\tilde{\varepsilon} := \frac{\varepsilon T}{L^3}, \qquad \tilde{M} := \frac{MT}{L}, \qquad \tilde{v}(\tilde{t}) := v(t) \quad \text{and} \quad \tilde{y}_0(\tilde{x}) := y_0(x).$$

For y we will prove the following estimates:

$$\|\tilde{y}_{\tilde{x}\tilde{x}|\tilde{x}=\delta}\|^2_{L^2(0,1)} \leqslant \frac{C(1 + \tilde{\varepsilon}^4 + \tilde{M}^4)e^{C\tilde{M}}}{\delta^{10}}\big(\|\tilde{v}\|^2_{L^2(0,1)} + \|\tilde{y}_0\|^2_{L^2(0,1)} + \|\tilde{y}_0'''\|^2_{L^2(0,1)}\big) \tag{4.5}$$

and, if $y_0 \equiv 0$,

$$\|\tilde{y}_{\tilde{t}\tilde{x}\tilde{x}|\tilde{x}=\delta}\|^2_{L^2(0,1)} \leqslant \frac{C(1 + \tilde{\varepsilon}^4 + \tilde{M}^4)e^{C\tilde{M}}(\tilde{\varepsilon}^{-2} + \tilde{\varepsilon}^4 + \tilde{M}^2 + \tilde{M}^8)}{\delta^{22}}\|\tilde{v}\|^2_{L^2(0,1)} \tag{4.6}$$

from where inequalities (4.3) and (4.4) are easily deduced by going back to the original variable. Here and in the following, C denotes a positive constant independent of all parameters.

For the sake of clearness we will denote y, t and x instead of $\tilde{y}$, $\tilde{t}$ and $\tilde{x}$ until the end of the proof of Lemma 4.2.

The proof of (4.5) and (4.6) is divided in several steps:

Step 0. A previous computation.

Let us first perform a previous computation which will be useful in the proof. Let $\theta \in \mathcal{C}^3([0,1])$ be a positive increasing function such that $\theta(0) = \theta'(0) = \theta''(0) = 0$. After some integrations by parts we obtain, for all $h \in H^1(0,1;L^2(0,1)) \cap L^2(0,1;H^3(0,1))$,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta|h|^2\,\mathrm{d}x + \frac{3\tilde{\varepsilon}}{2}\int_0^1 \theta'|h_x|^2\,\mathrm{d}x + \frac{\tilde{M}}{2}\int_0^1 \theta'|h|^2\,\mathrm{d}x \\
&\quad + \tilde{\varepsilon}\theta(1)h_{xx|x=1}h_{|x=1} - \tilde{\varepsilon}\theta'(1)h_{x|x=1}h_{|x=1} - \frac{\tilde{\varepsilon}}{2}\theta(1)|h_{x|x=1}|^2 \\
&\quad - \frac{\tilde{M}}{2}\theta(1)|h_{|x=1}|^2 + \frac{\tilde{\varepsilon}}{2}\theta''(1)|h_{|x=1}|^2 \\
&= \frac{\tilde{\varepsilon}}{2}\int_0^1 \theta'''|h|^2\,\mathrm{d}x + \int_0^1 \theta(h_t + \tilde{\varepsilon}h_{xxx} - \tilde{M}h_x)h\,\mathrm{d}x.
\end{aligned} \tag{4.7}$$

Step 1. Estimate of y_x.

In this paragraph we lift the boundary and initial conditions. Let

$$z(t,x) := \int_0^t y(s,x)\,\mathrm{d}s - (1-x)^6\int_0^t \tilde{v}(s)\,\mathrm{d}s, \quad (t,x) \in (0,1)^2. \tag{4.8}$$

Then, z satisfies

$$\begin{cases}
z_t + \tilde{\varepsilon}z_{xxx} - \tilde{M}z_x = f & \text{in } (0,1)^2, \\
z_{|x=0} = 0, \qquad z_{x|x=1} = 0, \qquad z_{xx|x=1} = 0 & \text{in } (0,1), \\
z_{|t=0} = 0 & \text{in } (0,1),
\end{cases} \tag{4.9}$$

where

$$f(t,x) := -(1-x)^6\tilde{v}(t) + \big(120(1-x)^3\tilde{\varepsilon} - 6(1-x)^5\tilde{M}\big)\int_0^t \tilde{v}(s)\,\mathrm{d}t + \tilde{y}_0. \tag{4.10}$$

We multiply the equation in (4.9) by $(1+x)z$ and integrate in space. After integration by parts we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (1+x)|z|^2\,\mathrm{d}x + \frac{3\tilde{\varepsilon}}{2}\int_0^1 |z_x|^2\,\mathrm{d}x \leqslant \tilde{M}|z_{|x=1}|^2 + \int_0^1 (1+x)fz\,\mathrm{d}x.$$

From (4.7) with $h = z_x$, $\theta = \frac{x^3}{4}$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{x^3}{4}|z_x|^2\,\mathrm{d}x + \frac{9\tilde{\varepsilon}}{8}\int_0^1 x^2|z_{xx}|^2\,\mathrm{d}x \leqslant \frac{3\tilde{\varepsilon}}{4}\int_0^1 |z_x|^2\,\mathrm{d}x + \frac{1}{4}\int_0^1 x^3 f_x z_x\,\mathrm{d}x.$$

Adding these inequalities, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (1+x)|z|^2\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{x^3}{4}|z_x|^2\,\mathrm{d}x + \frac{3\tilde{\varepsilon}}{4}\int_0^1 |z_x|^2\,\mathrm{d}x + \frac{9\tilde{\varepsilon}}{8}\int_0^1 x^2|z_{xx}|^2\,\mathrm{d}x\\ &\quad\leqslant \tilde{M}|z_{|x=1}|^2 + \int_0^1 (1+x) f z\,\mathrm{d}x + \frac{1}{4}\int_0^1 x^3 f_x z_x\,\mathrm{d}x.\end{aligned} \tag{4.11}$$

Now, notice that we can estimate the trace term in the following way:

$$\begin{aligned}\tilde{M}|z_{|x=1}|^2 &= \tilde{M}\int_0^1 \partial_x\big(x^{3/2}|z|^2\big)\,\mathrm{d}x = 2\tilde{M}\int_0^1 x^{3/2} z_x z\,\mathrm{d}x + \frac{3\tilde{M}}{2}\int_0^1 x^{1/2}|z|^2\,\mathrm{d}x\\ &\leqslant C\tilde{M}\int_0^1 \big((1+x)|z|^2 + x^3|z_x|^2\big)\,\mathrm{d}x.\end{aligned}$$

Then, we have:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left((1+x)|z|^2 + \frac{x^3}{4}|z_x|^2\right)\mathrm{d}x + \frac{3\tilde{\varepsilon}}{4}\int_0^1 |z_x|^2\,\mathrm{d}x + \frac{9\tilde{\varepsilon}}{4}\int_0^1 x^2|z_{xx}|^2\,\mathrm{d}x\\ &\quad\leqslant C(1+\tilde{M})\int_0^1 \left((1+x)|z|^2 + \frac{x^3}{4}|z_x|^2\right)\mathrm{d}x + C\int_0^1 \big(|f|^2 + |f_x|^2\big)\,\mathrm{d}x.\end{aligned}$$

From Gronwall's Lemma, we obtain in particular

$$\tilde{\varepsilon}\int_0^1\!\!\int_0^1 x^2|z_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C\mathrm{e}^{C\tilde{M}}\int_0^1\!\!\int_0^1 \big(|f|^2 + |f_x|^2\big)\,\mathrm{d}x\,\mathrm{d}t. \tag{4.12}$$

Now, we use (4.7) with $h = z_{xx}$ and $\theta = x^5$:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 x^5|z_{xx}|^2\,\mathrm{d}x + \frac{15\tilde{\varepsilon}}{2}\int_0^1 x^4|z_{xxx}|^2\,\mathrm{d}x\\ &\quad\leqslant 30\tilde{\varepsilon}\int_0^1 x^2|z_{xx}|^2\,\mathrm{d}x + \frac{\tilde{\varepsilon}}{2}|z_{xxx|x=1}|^2 + \int_0^1 x^5 f_{xx} z_{xx}\,\mathrm{d}x.\end{aligned} \tag{4.13}$$

To gain more derivatives of z and estimate the boundary term on z_{xxx}, we use again (4.7) with $h = z_{xxx}$ and $\theta = ax^7$ ($a := \frac{1}{28}$). Notice that from (4.9) and $f_{x|x=1} = 0$, we have that $z_{4x|x=1} = 0$. Thus:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 ax^7|z_{xxx}|^2\,\mathrm{d}x + \frac{21\tilde{\varepsilon}a}{2}\int_0^1 x^6|z_{4x}|^2\,\mathrm{d}x + a\tilde{\varepsilon}z_{5x\,|x=1}z_{xxx\,|x=1}$$
$$-\frac{a\tilde{M}}{2}|z_{xxx\,|x=1}|^2 + \frac{3\tilde{\varepsilon}}{4}|z_{xxx\,|x=1}|^2$$
$$\leqslant \frac{15\tilde{\varepsilon}}{4}\int_0^1 x^4|z_{xxx}|^2\,\mathrm{d}x + a\int_0^1 x^7 f_{xxx}z_{xxx}\,\mathrm{d}x.$$

Again from (4.9) and $f_{xx\,|x=1} = 0$, we have the relation $\tilde{\varepsilon}z_{5x\,|x=1} = \tilde{M}z_{xxx\,|x=1}$. Using it in this last inequality, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 ax^7|z_{xxx}|^2\,\mathrm{d}x + \frac{21\tilde{\varepsilon}a}{2}\int_0^1 x^6|z_{4x}|^2\,\mathrm{d}x + \frac{3\tilde{\varepsilon}}{4}|z_{xxx\,|x=1}|^2$$
$$\leqslant \frac{15\tilde{\varepsilon}}{4}\int_0^1 x^4|z_{xxx}|^2\,\mathrm{d}x + a\int_0^1 x^7 f_{xxx}z_{xxx}\,\mathrm{d}x. \tag{4.14}$$

Adding (4.13) and (4.14), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \big(x^5|z_{xx}|^2 + ax^7|z_{xxx}|^2\big)\,\mathrm{d}x + \frac{15\tilde{\varepsilon}}{4}\int_0^1 x^4|z_{xxx}|^2\,\mathrm{d}x + \frac{21\tilde{\varepsilon}a}{2}\int_0^1 x^6|z_{4x}|^2\,\mathrm{d}x + \frac{\tilde{\varepsilon}}{4}|z_{xxx\,|x=1}|^2$$
$$\leqslant \frac{1}{2}\int_0^1 \big(x^5|z_{xx}|^2 + ax^7|z_{xxx}|^2\big)\,\mathrm{d}x + C\int_0^1 |f_{xxx}|^2\,\mathrm{d}x + 30\tilde{\varepsilon}\int_0^1 x^2|z_{xx}|^2\,\mathrm{d}x.$$

Using Gronwall's Lemma and (4.12), we find

$$\int_0^1\int_0^1 x^5|z_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t + \tilde{\varepsilon}\int_0^1\int_0^1 \big(x^4|z_{xxx}|^2 + x^6|z_{4x}|^2\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C\mathrm{e}^{C\tilde{M}}\int_0^1\int_0^1 \big(|f|^2 + |f_{xxx}|^2\big)\,\mathrm{d}x\,\mathrm{d}t. \tag{4.15}$$

Finally, we go back to y. From (4.8), we get

$$y_x = -\tilde{\varepsilon}z_{4x} + \tilde{M}z_{xx} + f_x - 6(1-x)^5\tilde{v}(t). \tag{4.16}$$

Then, using (4.15), we have

$$\tilde{\varepsilon}\int_0^1\int_0^1 x^6|y_x|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C\big(\tilde{\varepsilon}^2 + \tilde{\varepsilon}\tilde{M}^2 + \tilde{\varepsilon}\big)\mathrm{e}^{C\tilde{M}}\int_0^1\int_0^1 \big(|f|^2 + |f_{xxx}|^2\big)\,\mathrm{d}x\,\mathrm{d}t + C\tilde{\varepsilon}\int_0^1 |\tilde{v}|^2\,\mathrm{d}t,$$

which, from the definition of f (see (4.10)) yields

$$\tilde{\varepsilon}\int_0^1\int_0^1 x^6|y_x|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C\tilde{\varepsilon}\big(1+\tilde{\varepsilon}^4+\tilde{M}^4\big)\mathrm{e}^{C\tilde{M}}\bigg(\int_0^1 |\tilde{v}|^2\,\mathrm{d}t+\int_0^1\big(|\tilde{y}_0|^2+\big|\tilde{y}_0'''\big|^2\big)\,\mathrm{d}x\bigg). \tag{4.17}$$

For this last inequality we have used Young's inequality.

Step 2. Estimates of y_{xx} and y_{xxx}, and conclusion of (4.5).

First, we use (4.7) with $h = y_x$ and $\theta = x^9$. We have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 x^9|y_x|^2\,\mathrm{d}x+\frac{27\tilde{\varepsilon}}{2}\int_0^1 x^8|y_{xx}|^2\,\mathrm{d}x \leqslant 252\tilde{\varepsilon}\int_0^1 x^6|y_x|^2\,\mathrm{d}x.$$

Integrating in time and thanks to (4.17) we find

$$\tilde{\varepsilon}\int_0^1\int_0^1 x^8|y_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C\tilde{\varepsilon}\big(\tilde{\varepsilon}^{-1}+\tilde{\varepsilon}^4+\tilde{M}^4\big)\mathrm{e}^{C\tilde{M}}\bigg(\int_0^1 |\tilde{v}|^2\,\mathrm{d}t+\int_0^1\big(|\tilde{y}_0|^2+\big|\tilde{y}_0'''\big|^2\big)\,\mathrm{d}x\bigg). \tag{4.18}$$

Next, integrating by parts, we get

$$\tilde{\varepsilon}\int_0^1 x^{10}|y_{xxx}|^2\,\mathrm{d}x = -\tilde{\varepsilon}\int_0^1 x^{10}y_{4x}y_{xx}\,\mathrm{d}x+45\tilde{\varepsilon}\int_0^1 x^8|y_{xx}|^2\,\mathrm{d}x.$$

Then, by Young's inequality and (4.18):

$$\begin{aligned}\tilde{\varepsilon}\int_0^1 x^{10}|y_{xxx}|^2\,\mathrm{d}x &= \lambda\tilde{\varepsilon}\int_0^1 x^{12}|y_{4x}|^2\,\mathrm{d}x\\ &\quad + C_\lambda\tilde{\varepsilon}\big(\tilde{\varepsilon}^{-1}+\tilde{\varepsilon}^4+\tilde{M}^4\big)\mathrm{e}^{C\tilde{M}}\bigg(\int_0^1 |\tilde{v}|^2\,\mathrm{d}t+\int_0^1\big(|\tilde{y}_0|^2+\big|\tilde{y}_0'''\big|^2\big)\,\mathrm{d}x\bigg),\end{aligned} \tag{4.19}$$

for every $\tilde{\varepsilon} > 0$ and $\lambda > 0$ is to be chosen later on.

Now, similar computations made to obtain (4.14) (taking $h = y_{xxx}$ and $\theta = x^{13}$ in (4.7)) yield

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L x^{13}|y_{xxx}|^2\,\mathrm{d}x+\frac{39\tilde{\varepsilon}}{2}\int_0^1 x^{12}|y_{4x}|^2\,\mathrm{d}x+78\tilde{\varepsilon}|y_{xxx|x=1}|^2 \leqslant 858\tilde{\varepsilon}\int_0^1 x^{10}|y_{xxx}|^2\,\mathrm{d}x.$$

We take $\lambda = \frac{39}{858\times 4}$ in (4.19) and we deduce

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 x^{13}|y_{xxx}|^2\,\mathrm{d}x+\frac{39\tilde{\varepsilon}}{4}\int_0^1 x^{12}|y_{4x}|^2\,\mathrm{d}x+78\tilde{\varepsilon}|y_{xxx|x=1}|^2\\ &\quad\leqslant C\tilde{\varepsilon}\big(\tilde{\varepsilon}^{-1}+\tilde{\varepsilon}^4+\tilde{M}^4\big)\mathrm{e}^{C\tilde{M}}\bigg(\int_0^1 |\tilde{v}|^2\,\mathrm{d}t+\int_0^1\big(|\tilde{y}_0|^2+\big|\tilde{y}_0'''\big|^2\big)\,\mathrm{d}x\bigg).\end{aligned}$$

Integrating in time between 0 and 1, together with (4.18) and (4.19), we obtain

$$\tilde{\varepsilon}\int_0^1\int_0^1 \left(x^8|y_{xx}|^2 + x^{10}|y_{xxx}|^2 + x^{12}|y_{4x}|^2\right) \mathrm{d}x\, \mathrm{d}t + \tilde{\varepsilon}\int_0^1 |y_{xxx\,|x=1}|^2\, \mathrm{d}t \leqslant C\tilde{\varepsilon}\left(\tilde{\varepsilon}^{-1} + \tilde{\varepsilon}^4 + \tilde{M}^4\right)\mathrm{e}^{C\tilde{M}}\left(\int_0^1 |\tilde{v}|^2\, \mathrm{d}t + \int_0^1 \left(|\tilde{y}_0|^2 + \left|\tilde{y}_0'''\right|^2\right) \mathrm{d}x\right). \tag{4.20}$$

To conclude, notice that for any $\delta > 0$ we have

$$\int_0^1 |y_{xx\,|x=\delta}|^2\, \mathrm{d}t = -2\int_0^1\int_\delta^1 y_{xx}y_{xxx}\, \mathrm{d}x\, \mathrm{d}t \leqslant \int_0^1\int_\delta^1 \left(\frac{x^8}{\delta^8}|y_{xx}|^2 + \frac{x^{10}}{\delta^{10}}|y_{xxx}|^2\right) \mathrm{d}x\, \mathrm{d}t.$$

Combining this with (4.20), we obtain (4.5).

Step 3. Estimates of y_{5x} and y_{6x}, and conclusion of (4.6).

From here on, we consider $y_0 \equiv 0$. We take $h = y_{6x}$ and $\theta = x^{22}$ in (4.7):

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 x^{22}|y_{6x}|^2\, \mathrm{d}x + 33\tilde{\varepsilon}\int_0^1 x^{21}|y_{7x}|^2\, \mathrm{d}x + \tilde{\varepsilon}\, y_{8x\,|x=1}y_{6x\,|x=1} - 22\tilde{\varepsilon}\, y_{7x\,|x=1}y_{6x\,|x=1} - \frac{\tilde{M}}{2}|y_{6x\,|x=1}|^2 \leqslant 4620\tilde{\varepsilon}\int_0^1 x^{19}|y_{6x}|^2\, \mathrm{d}x + \frac{\tilde{\varepsilon}}{2}|y_{7x\,|x=1}|^2.$$

We recall that we already had found that $y_{4x\,|x=1} = 0$ and $y_{5x\,|x=1} = \tilde{M}\tilde{\varepsilon}^{-1}y_{xxx\,|x=1}$. With the help of these identities, the equation and the boundary conditions in (1.1), we find that

$$y_{6x\,|x=1} = -\tilde{\varepsilon}^{-1}y_{txxx\,|x=1}, \qquad y_{7x\,|x=1} = \tilde{M}^2\tilde{\varepsilon}^{-2}y_{xxx\,|x=1},$$

and

$$y_{8x\,|x=1} = \tilde{M}\tilde{\varepsilon}^{-1}y_{6x\,|x=1} - \tilde{\varepsilon}^{-1}y_{t5x\,|x=1} = -2\tilde{M}\tilde{\varepsilon}^{-2}y_{txxx\,|x=1}.$$

Using these identities in the previous inequality we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 x^{22}|y_{6x}|^2\, \mathrm{d}x + 33\tilde{\varepsilon}\int_0^1 x^{21}|y_{7x}|^2\, \mathrm{d}x + \frac{3\tilde{M}\tilde{\varepsilon}^{-2}}{2}|y_{txxx\,|x=1}|^2 + 11\tilde{M}^2\tilde{\varepsilon}^{-2}\frac{\mathrm{d}}{\mathrm{d}t}|y_{xxx\,|x=1}|^2 \leqslant 4620\tilde{\varepsilon}\int_0^1 x^{19}|y_{6x}|^2\, \mathrm{d}x + \frac{\tilde{M}^4\tilde{\varepsilon}^{-3}}{2}|y_{xxx\,|x=1}|^2. \tag{4.21}$$

Let us estimate the first term in the right-hand side. First, notice that we have

$$\tilde{\varepsilon}\int_0^1 x^{17}|y_{5x}|^2\, \mathrm{d}x = -\tilde{\varepsilon}\int_0^1 x^{17}y_{4x}y_{6x}\, \mathrm{d}x + 136\tilde{\varepsilon}\int_0^1 x^{15}|y_{4x}|^2\, \mathrm{d}x \leqslant C\int_0^1 x^{22}|y_{6x}|^2\, \mathrm{d}x + C\left(\tilde{\varepsilon} + \tilde{\varepsilon}^2\right)\int_0^1 x^{12}|y_{4x}|^2\, \mathrm{d}x. \tag{4.22}$$

Now, again by integration by parts we get

$$\tilde{\varepsilon}\int_0^1 x^{19}|y_{6x}|^2\,\mathrm{d}x = \tilde{\varepsilon}y_{5x\,|x=1}y_{6x\,|x=1} - 171\tilde{\varepsilon}|y_{5x\,|x=1}|^2 + 171\tilde{\varepsilon}\int_0^1 x^{17}|y_{5x}|^2\,\mathrm{d}x - \tilde{\varepsilon}\int_0^1 x^{19}y_{5x}y_{7x}\,\mathrm{d}x. \tag{4.23}$$

Combining (4.22)–(4.23) with (4.21) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 x^{22}|y_{6x}|^2\,\mathrm{d}x + \frac{33\tilde{\varepsilon}}{2}\int_0^1 x^{21}|y_{7x}|^2\,\mathrm{d}x + \int_0^1 x^{17}|y_{5x}|^2\,\mathrm{d}x\\ &\quad + 2310\tilde{M}\tilde{\varepsilon}^{-1}\frac{\mathrm{d}}{\mathrm{d}t}|y_{xxx\,|x=1}|^2 + 11\tilde{M}^2\tilde{\varepsilon}^{-2}\frac{\mathrm{d}}{\mathrm{d}t}|y_{xxx\,|x=1}|^2\\ &\leqslant C\int_0^1 x^{22}|y_{6x}|^2\,\mathrm{d}x + C\big(1+\tilde{\varepsilon}^2+\tilde{M}^4\tilde{\varepsilon}^{-3}\big)\bigg(\int_0^1 x^{12}|y_{4x}|^2\,\mathrm{d}x + |y_{xxx\,|x=1}|^2\bigg).\end{aligned}$$

Using (4.20) to estimate the last term and applying Gronwall's Lemma, we get

$$\begin{aligned}&\int_0^1\int_0^1 \big(x^{17}|y_{5x}|^2 + x^{22}|y_{6x}|^2\big)\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant C\big(\tilde{\varepsilon}^{-1}+\tilde{\varepsilon}^4+\tilde{M}^4\big)\mathrm{e}^{C\tilde{M}}\big(1+\tilde{\varepsilon}^2+\tilde{M}^4\tilde{\varepsilon}^{-3}\big)\int_0^1 |v|^2\,\mathrm{d}t.\end{aligned} \tag{4.24}$$

Finally, using the equation in (1.1), we obtain from (4.20) and (4.24):

$$\begin{aligned}&\int_0^1\int_0^1 \big(x^{17}|y_{txx}|^2 + x^{22}|y_{txxx}|^2\big)\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant C\big(\tilde{\varepsilon}^{-1}+\tilde{\varepsilon}^4+\tilde{M}^4\big)\mathrm{e}^{C\tilde{M}}\big(\tilde{\varepsilon}^2+\tilde{\varepsilon}^4+\tilde{M}^4\tilde{\varepsilon}^{-1}+\tilde{M}^2\big)\int_0^1 |v|^2\,\mathrm{d}t.\end{aligned}$$

From here, we obtain (4.6) as in the end of Step 2. □

Proof of Lemma 4.1. As in the proof of Corollary 2.2, we write $v = v_0 + v_1'$, for some $v_0, v_1 \in L^2(0,T)$, and $y = p + q_t$, where p and q are the solutions of (2.9) and (2.10), respectively. From Lemma 4.2, using (4.3) for p and (4.4) for q, we have from (2.8) that

$$\begin{aligned}&\|y_{xx\,|x=\delta L}\|^2_{L^2(0,T)}\\ &\quad\leqslant \big(C_0(T,L,\varepsilon,M,\delta) + C_0'(T,L,\varepsilon,M,\delta)\big)\\ &\qquad\times\big(\|v_0\|^2_{L^2(0,T)} + \|v_1\|^2_{L^2(0,T)} + \|y_0\|^2_{L^2(0,L)} + \|y_0'''\|^2_{L^2(0,L)}\big),\end{aligned}$$

for every $v_0, v_1 \in L^2(0,T)$ such that (2.6) is satisfied. Therefore, from (2.6)–(2.7) we obtain (4.1). □

Finally, let us state another technical result whose proof is given in Appendix B.

Lemma 4.3. *Let* $0 < L_1 < L$. *For any* $h \in H^3_n(L_1, L)$, *there exists* $\tilde{h} \in H^3_n(0, L)$ *such that*

$$\tilde{h}_{|(L_1,L)} = h \quad \textit{and} \quad \|\tilde{h}\|^2_{H^3_n(0,L)} \leqslant C_1(L_1, L)\|h\|^2_{H^3_n(L_1,L)},$$

where

$$C_1(L_1, L) := CL_1\big(L_1^5 + (L - L_1)^5 + (L - L_1)^{-1}\big) + 1. \tag{4.25}$$

4.2. Auxiliary problem and conclusion

Now, we introduce the following auxiliary control problem:

$$\begin{cases} w_t + \varepsilon w_{xxx} - M w_x = 0 & \text{in } (0, T) \times (L_0, L), \\ w_{xx|x=L_0} = u(t), \quad w_{x|x=L} = 0, \quad w_{xx|x=L} = 0 & \text{in } (0, T), \\ w_{|t=0} = w_0 & \text{in } (L_0, L). \end{cases} \tag{4.26}$$

For this problem, we define the cost associated to the null controllability as follows:

$$K^\varepsilon_{\text{cost}} := \sup_{\substack{w_0 \in H^3_n(L_0,L) \\ w_0 \neq 0}} \min_{\substack{u \in L^2(0,T) \\ w_{|t=T}=0}} \frac{\|u\|^2_{L^2(0,T)}}{\|w_0\|^2_{H^3_n(L_0,L)}}. \tag{4.27}$$

We find now a lower bound of $K^\varepsilon_{\text{cost}}$.

Lemma 4.4. *Let* $T, L, \varepsilon, M > 0$ *and* $L_0 < L$. *Then*

$$K^\varepsilon_{\text{cost}} \geqslant \frac{\sinh^2((L - L_0)M^{1/2}\varepsilon^{-1/2})}{\varepsilon T M(L - L_0)}. \tag{4.28}$$

Proof. We remark that $K^\varepsilon_{\text{obs}}$ is the smallest constant such that

$$\sup_{h \in H^3_n(L_0,L)} \frac{(\int_{L_0}^L \varphi_{|t=0} h \, \mathrm{d}x)^2}{\|h\|^2_{H^3_n(L_0,L)}} \leqslant \varepsilon^2 K^\varepsilon_{\text{cost}} \int_0^T |\varphi_{|x=L_0}|^2 \, \mathrm{d}t, \tag{4.29}$$

for any $\varphi_T \in L^2(L_0, L)$, where φ is the solution of

$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xxx} + M\varphi_x = 0 & \text{in } (0, T) \times (L_0, L), \\ \varphi_{x|x=L_0} = 0, \quad (\varepsilon\varphi_{xx} - M\varphi)_{|x=L_0} = 0, \quad (\varepsilon\varphi_{xx} - M\varphi)_{|x=L} = 0 & \text{in } (0, T), \\ \varphi_{|t=T} = \varphi_T & \text{in } (L_0, L). \end{cases} \tag{4.30}$$

Now, let

$$\psi(x) := \cosh\big((x - L_0)M^{1/2}\varepsilon^{-1/2}\big).$$

It is straightforward to check that ψ satisfies the partial differential equation and the boundary conditions in (4.30). Furthermore, since $h \equiv 1$ belongs to $H^3_n(L_0, L)$, we have

$$\sup_{h\in H^3_n(L_0,L)} \frac{(\int_{L_0}^L \psi h \,\mathrm{d}x)^2}{\|h\|^2_{H^3_n(0,L)}} \geqslant \frac{1}{L-L_0}\left(\int_{L_0}^L \psi \,\mathrm{d}x\right)^2 = \frac{\varepsilon \sinh^2((L-L_0)M^{1/2}\varepsilon^{-1/2})}{M(L-L_0)}.$$

On the other hand,

$$\int_0^T |\psi_{|x=L_0}|^2 \,\mathrm{d}t = T.$$

Consequently, from (4.29) we deduce that (4.28) holds. □

Conclusion of the proof of Theorem 1.2. We argue by contradiction, i.e., we suppose that for any $y_0 \in H^3_n(0, L)$, there exists $v \in H^{-1}(0, T)$ such that the solution of (1.1) satisfies $y_{|t=T} = 0$ and

$$\|v\|^2_{H^{-1}(0,T)} \leqslant \left(\frac{(1-\gamma)\bar{K}(T, L, \varepsilon, M, \delta)}{C_{-1}(T, L, \varepsilon, M, \delta)C_1(\delta L, L)} - 1\right)\|y_0\|^2_{H^3_n(0,L)}. \tag{4.31}$$

Let $w_0 \in H^3_n(L_0, L)$ where $0 < L_0 < L$. We choose y_0 to be the extension of w_0 given by Lemma 4.3. In particular, we have

$$\|y_0\|^2_{H^3_n(0,L)} \leqslant C_1(L_0, L)\|w_0\|^2_{H^3_n(L_0,L)}. \tag{4.32}$$

Observe that y solves (4.26) with $u := y_{xx|x=L_0}$ and $w_0 := y_{0|(L_0,L)}$. Let us take $L_0 := \delta L$. Then, using Lemma 4.1 we have

$$\|u\|^2_{L^2(0,T)} \leqslant C_{-1}(T, L, \varepsilon, M, \delta)\big(\|v\|^2_{H^{-1}(0,T)} + \|y_0\|^2_{H^3_n(0,L)}\big).$$

From (4.31) and (4.32):

$$\|u\|^2_{L^2(0,T)} \leqslant (1-\gamma)\bar{K}(T, L, \varepsilon, M, \delta)\|w_0\|^2_{H^3_n(\delta L,L)}$$

which implies that

$$K^\varepsilon_{\mathrm{cost}} \leqslant (1-\gamma)\bar{K}(T, L, \varepsilon, M, \delta).$$

This and (4.28) show that $\gamma \leqslant 0$, which is a contradiction. □

5. Proof of Theorem 1.5

The proof of Theorem 1.5 relies on finding a particular solution $\widehat{\varphi}$ of (3.2) such that $\|\widehat{\varphi}_{xx|x=0}\|_{L^2(0,T)}$ decays exponentially as $\varepsilon \to 0^+$ and $\|\widehat{\varphi}_{|t=0}\|_{L^2(0,L)}$ behaves like a constant in ε. The proof we perform here is inspired by [10, Theorem 1.4]. The main difference with respect to [10] is that the boundary condition on $x = L$ is not homogeneous.

Let $M < 0$. In this case, we can look at (3.2) as

$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xxx} - |M|\varphi_x = 0 & \text{in } Q, \\ \varphi_{|x=0} = 0, \qquad \varphi_{x|x=0} = 0, \qquad (\varepsilon\varphi_{xx} + |M|\varphi)_{|x=1} = 0 & \text{in } (0,T), \\ \varphi_{|t=T} = \varphi_T & \text{in } (0,L). \end{cases} \tag{5.1}$$

First, notice that we have the dissipation estimate:

$$\|\varphi_{|t=t_1}\|_{L^2(0,L)} \leqslant \|\varphi_{|t=t_2}\|_{L^2(0,L)} \quad \text{for every } 0 \leqslant t_1 \leqslant t_2 \leqslant T. \tag{5.2}$$

Now, we choose $R > 0$ such that

$$0 < 7R < L - |M|T, \tag{5.3}$$

and a nonnegative function $\widehat{\varphi}_T \in \mathcal{C}_0^\infty(0,L)$ such that

$$\mathrm{Supp}(\widehat{\varphi}_T) \subset (L-2R, L-R) \quad \text{and} \quad \|\widehat{\varphi}_T\|_{L^2(0,L)} = 1. \tag{5.4}$$

Let $\widehat{\varphi}$ be the solution of (5.1) associated to $\widehat{\varphi}_T$ as initial condition. We will prove that

$$\|\widehat{\varphi}_{|t=0}\|_{L^2(0,L)} \geqslant c > 0 \tag{5.5}$$

and

$$\|\widehat{\varphi}_{xx|x=0}\|_{L^2(0,T)} \leqslant C(\varepsilon)\exp\left[-\frac{R^{3/2}}{3^{3/2}\varepsilon^{1/2}T^{1/2}}\right]\|\widehat{\varphi}_T\|_{L^2(0,L)}, \tag{5.6}$$

where $C(\varepsilon) > 0$ depends on ε^{-1} at most polynomially.

Let us explain how (5.5) and (5.6) allow us to conclude Theorem 1.5. Let $v \in L^2(0,T)$ be a control which drives the solution y of (1.1) from y_0 to 0 (we know such a v exists by Theorem 1.1 and [12]). We multiply (1.1) by $\widehat{\varphi}$ and integrate by parts to get

$$-\int_0^L y_0\widehat{\varphi}_{|t=0}\,\mathrm{d}x = \varepsilon\int_0^T v\widehat{\varphi}_{xx|x=0}\,\mathrm{d}t \leqslant \varepsilon\|v\|_{L^2(0,T)}\|\widehat{\varphi}_{xx|x=0}\|_{L^2(0,T)}.$$

Setting $y_0 := -\widehat{\varphi}_{|t=0}$ and using (5.4), (5.5) and (5.6) in this last inequality we obtain (1.8).

Proof of (5.5). Let us define $\theta(t,x) := \widehat{\varphi}_T(x + |M|(T-t))$ in Q. It follows from (5.3) and (5.4) that $\theta(t,\cdot) \in \mathcal{C}_0^\infty(0,L)$ for all $t \in [0,T]$.

We multiply (5.1) by θ and after integration by parts we obtain

$$\int_0^L \widehat{\varphi}_T \theta_{|t=T}\,\mathrm{d}x = \int_0^L \widehat{\varphi}_{|t=0}\theta_{|t=0}\,\mathrm{d}x + \varepsilon \iint_Q \widehat{\varphi}\theta_{xxx}\,\mathrm{d}x\,\mathrm{d}t. \tag{5.7}$$

From (5.2) with $t_2 = T$, we find

$$\|\widehat{\varphi}\|_{L^2(Q)} \leqslant T^{1/2}\|\widehat{\varphi}_T\|_{L^2(0,L)}.$$

On the other hand, from the definition of θ, (5.3) and (5.4), it is easy to see that

$$\|\theta_{|t=0}\|_{L^2(0,L)} = \|\widehat{\varphi}_T\|_{L^2(0,L)}$$

and

$$\|\theta_{xxx}\|_{L^2(Q)} \leqslant T^{1/2}R^{1/2}\|\widehat{\varphi}_T'''\|_{L^\infty(0,L)}.$$

Using these elements in (5.7), together with Young's inequality, we obtain (5.5) for ε small enough depending on T and R. □

Proof of (5.6). The objective is to prove that

$$\|\widehat{\varphi}\|_{L^2(0,T;H^3(0,R))} \leqslant C(\varepsilon)\exp\left[-\frac{R^{3/2}}{3^{3/2}\varepsilon^{1/2}T^{1/2}}\right]\|\widehat{\varphi}_T\|_{L^2(0,L)}, \tag{5.8}$$

from where (5.6) will readily follow. This is done by proving the estimate

$$\|\widehat{\varphi}(t)\|_{L^2(0,3R)} \leqslant C\exp\left[-\frac{R^{3/2}}{3^{3/2}\varepsilon^{1/2}T^{1/2}}\right]\|\widehat{\varphi}_T\|_{L^2(0,L)}, \tag{5.9}$$

and then applying an internal regularity result proved in [10] to conclude.

We consider a cut-off function $\gamma \in C^\infty(\mathbb{R})$ such that

$$\gamma \geqslant 0, \qquad \gamma' \leqslant 0, \qquad \gamma = 1 \quad \text{in } (-\infty, L-3R), \qquad \gamma = 0 \quad \text{in } (L-2R, +\infty). \tag{5.10}$$

We set $\beta(t,x) := -|M|(T-t) - x$ and multiply (5.1) by $\gamma(-\beta)\mathrm{e}^{r\beta}\widehat{\varphi}$, where $r > 0$ is to be chosen later on. We perform several integrations by parts, but observe that from (5.3) and (5.10), we have that $\gamma(-\beta(t,x)) = 0$ for all $(t,x) \in [0,T]\times[L-2R, L]$, so there are no boundary terms. We get, after neglecting the positive terms,

$$\begin{aligned}
&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \gamma(-\beta)\mathrm{e}^{r\beta}|\widehat{\varphi}|^2\,\mathrm{d}x - \frac{\varepsilon r^3}{2}\int_0^L \gamma(-\beta)\mathrm{e}^{r\beta}|\widehat{\varphi}|^2\,\mathrm{d}x \\
&\quad \leqslant \varepsilon C(r)\big(\|\gamma'\|_\infty + \|\gamma''\|_\infty + \|\gamma'''\|_\infty\big)\int_{L-3R-|M|(T-t)}^{L-2R-|M|(T-t)} \mathrm{e}^{r\beta}|\widehat{\varphi}|^2\,\mathrm{d}x,
\end{aligned}$$

where $C(r)$ is a polynomial function of degree 2 in r and we have used (5.10) to restrict the limits in the integral. Multiplying by $\exp(-\varepsilon r^3(T-t))$ and using that β is decreasing, we have

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\bigl(-\varepsilon r^3(T-t)\bigr)\int_0^L \gamma(-\beta)\mathrm{e}^{r\beta}|\widehat{\varphi}|^2\,\mathrm{d}x\right)$$
$$\leqslant \varepsilon C(r)\exp\bigl(-\varepsilon r^3(T-t)+r(3R-L)\bigr)\int_0^L |\widehat{\varphi}|^2\,\mathrm{d}x.$$

By (5.2), we obtain

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\bigl(-\varepsilon r^3(T-t)\bigr)\int_0^L \gamma(-\beta)\mathrm{e}^{r\beta}|\widehat{\varphi}|^2\,\mathrm{d}x\right)\leqslant \varepsilon C(r)\exp\bigl(r(3R-L)\bigr)\int_0^L |\widehat{\varphi}_T|^2\,\mathrm{d}x.$$

Integrating in (t,T), we get:

$$\int_0^L \gamma(-\beta)\mathrm{e}^{r\beta}|\widehat{\varphi}|^2\,\mathrm{d}x \leqslant \varepsilon C(r)T\exp\bigl(\varepsilon r^3(T-t)+r(3R-L)\bigr)\int_0^L |\widehat{\varphi}_T|^2\,\mathrm{d}x,$$

where we have used the fact that $\gamma(s)\widehat{\varphi}_T(s)=0$ for all $s\in\mathbb{R}$.

Now, notice that $\gamma(-\beta(t,x))=1$ for all $(t,x)\in[0,T]\times[0,3R]$ thanks to (5.3), so we have

$$\exp\Bigl(-r\bigl(|M|(T-t)\bigr)\int_0^{3R}|\widehat{\varphi}|^2\,\mathrm{d}x \leqslant \varepsilon C(r)\exp\bigl(\varepsilon r^3(T-t)-r(L-6R)\bigr)\Bigr)\int_0^L |\widehat{\varphi}_T|^2\,\mathrm{d}x,$$

and thus

$$\int_0^{3R}|\widehat{\varphi}|^2\,\mathrm{d}x \leqslant \varepsilon C(r)\exp\bigl(\varepsilon r^3T-r\bigl(L-|M|T-6R\bigr)\bigr)\int_0^L |\widehat{\varphi}_T|^2\,\mathrm{d}x.$$

Again from (5.3), we obtain

$$\int_0^{3R}|\widehat{\varphi}|^2\,\mathrm{d}x \leqslant \varepsilon C(r)\exp\bigl(\varepsilon r^3T-Rr\bigr)\int_0^L |\widehat{\varphi}_T|^2\,\mathrm{d}x.$$

We finish the proof of (5.9) by choosing $r>0$ such that it minimises the expression inside the exponential, that is,

$$r:=\frac{R^{1/2}}{3^{1/2}\varepsilon^{1/2}T^{1/2}}.$$

To prove (5.8), we will use the following lemma, which corresponds to Proposition 3.3 in [10].

Lemma 5.1. *Let $\varepsilon\in(0,1]$ and $M\in\mathbb{R}$. Consider a solution w of*

$$\begin{cases} -w_t-\varepsilon w_{xxx}+Mw_x=0 & \text{in } Q,\\ w_{|x=0}=0,\quad w_{x|x=0}=0,\quad w_{|x=L}=u(t) & \text{in } (0,T),\\ w_{|t=T}=w_T & \text{in } (0,L), \end{cases}\tag{5.11}$$

for some $u \in L^2(0,T)$ *and* $w_T \in H^3(0,L) \cap H_0^2(0,L)$. *Then,* $w \in L^2(0,T;H^4(0,L/2))$, *with the estimate*

$$\|w\|_{L^2(0,T;H^4(0,L/2))} \leqslant C(\varepsilon)\big(\|w_T\|_{H^3(0,L)} + \|u\|_{L^2(0,T)}\big), \tag{5.12}$$

for some constant $C(\varepsilon)$ *depending at most polynomially in* ε^{-1} *and* $|M|$.

Let $w := \widehat{\varphi}_{|[0,2R]}$ and apply Lemma 5.1 with $(0,2R)$ and $(0,R)$ instead of $(0,L)$ and $(0,L/2)$, respectively. Notice that with this setting we have $w_T = 0$ and $u = \widehat{\varphi}_{|x=2R} \in L^2(0,T)$. Thus, from (5.12) we have

$$\|\widehat{\varphi}\|_{L^2(0,T;H^4(0,R))} \leqslant C(\varepsilon)\|\widehat{\varphi}_x\|_{L^2(0,T;L^2(0,2R))}. \tag{5.13}$$

Now, we estimate the term in the right-hand side in a slightly larger interval. To do this, we multiply (5.1) by $(3R-x)^3\widehat{\varphi}$ and integrate in $(0,3R)$. We obtain

$$\begin{aligned} &-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{3R}(3R-x)^3|\widehat{\varphi}|^2\,\mathrm{d}x + \frac{9\varepsilon}{2}\int_0^{3R}(3R-x)^2|\widehat{\varphi}_x|^2\,\mathrm{d}x \\ &\quad = 3\varepsilon\int_0^{3R}|\widehat{\varphi}|^2\,\mathrm{d}x + \frac{3|M|}{2}\int_0^{3R}(3R-x)^2|\widehat{\varphi}|^2\,\mathrm{d}x. \end{aligned}$$

Since $\widehat{\varphi}_T = 0$ in $(0,3R)$, we get by integrating between 0 and T

$$\big\|(3R-\cdot)\widehat{\varphi}_x\big\|_{L^2(0,T;L^2(0,3R))} \leqslant \frac{C}{\varepsilon^{1/2}}\|\widehat{\varphi}\|_{L^2(0,T;L^2(0,3R))}. \tag{5.14}$$

Combining this with (5.13) and (5.9), we obtain (5.8). □

The proof of Theorem 1.5 is complete.

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Appendix A. Proof of Proposition 3.1

We now follow the steps of [9] and [10]. Let $\psi := \mathrm{e}^{-s\alpha}\phi$. Using Eq. (3.4), we get

$$L_1\psi + L_2\psi = L_3\psi,$$

where we have denoted

$$L_1\psi := \varepsilon\psi_{xxx} + \psi_t + 3\varepsilon s^2\alpha_x^2\psi_x - M\psi_x,$$
$$L_2\psi := \left(\varepsilon s^3\alpha_x^3 + s\alpha_t - Ms\alpha_x\right)\psi + 3\varepsilon s\alpha_{xx}\psi_x + 3\varepsilon s\alpha_x\psi_{xx}$$

and

$$L_3\psi := -3\varepsilon s^2\alpha_x\alpha_{xx}\psi.$$

Notice that we have the following boundary values for ψ:

$$\psi_{|x=L} = 0, \tag{A.1}$$
$$\psi_{x\,|x=0} = -s\alpha_{x\,|x=0}\psi_{|x=0}, \tag{A.2}$$
$$\psi_{xx\,|x=0} = \left(s^2\alpha_x^2 - s\alpha_{xx}\right)_{|x=0}\psi_{|x=0}. \tag{A.3}$$

Taking the L^2-norm we have

$$\|L_1\psi\|^2_{L^2(Q)} + \|L_2\psi\|^2_{L^2(Q)} + 2(L_1\psi, L_2\psi)_{L^2(Q)} = \|L_3\psi\|^2_{L^2(Q)}. \tag{A.4}$$

In the following, our efforts will be devoted to computing the double product in the previous equation. Let us denote by $(L_i\psi)_j$ the jth term of $L_i\psi$.

Computing $((L_1\psi)_1, L_2\psi)_{L^2(Q)}$.

For the first term, we integrate by parts twice in space:

$$\begin{aligned}
&\big((L_1\psi)_1, (L_2\psi)_1\big)_{L^2(Q)}\\
&\quad= -\frac{1}{2}\varepsilon\iint_Q\left(\varepsilon s^3\alpha_x^3 + s\alpha_t - Ms\alpha_x\right)\partial_x|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad-\varepsilon\iint_Q\left(3\varepsilon s^3\alpha_x^2\alpha_{xx} + s\alpha_{xt} - Ms\alpha_{xx}\right)\psi\psi_{xx}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad-\varepsilon\int_0^T\left(\varepsilon s^3\alpha_x^3 + s\alpha_t - Ms\alpha_x\right)_{|x=0}\psi_{|x=0}\psi_{xx\,|x=0}\,\mathrm{d}t\\
&\quad= \frac{3}{2}\varepsilon\iint_Q\left(3\varepsilon s^3\alpha_x^2\alpha_{xx} + s\alpha_{xt} - Ms\alpha_{xx}\right)|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad-\frac{1}{2}\varepsilon\int_0^T\left(\varepsilon s^3\alpha_x^3 + s\alpha_t - Ms\alpha_x\right)_{|x=L}|\psi_{x\,|x=L}|^2\,\mathrm{d}t\\
&\qquad-3\varepsilon^2s^3\iint_Q\alpha_{xx}^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\varepsilon\int_0^T\left(6\varepsilon s^3\alpha_x\alpha_{xx}^2 + s\alpha_{xxt}\right)_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t\\
&\qquad+\frac{1}{2}\varepsilon\int_0^T\left(\varepsilon s^3\alpha_x^3 + s\alpha_t - Ms\alpha_x\right)_{|x=0}|\psi_{x\,|x=0}|^2\,\mathrm{d}t
\end{aligned}$$

$$+\varepsilon\int_0^T\big(3\varepsilon s^3\alpha_x^2\alpha_{xx}+s\alpha_{xt}-Ms\alpha_{xx}\big)_{|x=0}\psi_{|x=0}\psi_{x\,|x=0}\,\mathrm{d}t$$
$$-\varepsilon\int_0^T\big(\varepsilon s^3\alpha_x^3+s\alpha_t-Ms\alpha_x\big)_{|x=0}\psi_{|x=0}\psi_{xx\,|x=0}\,\mathrm{d}t.$$

Using the properties (3.12), (3.13), (A.2) and (A.3), we obtain

$$\begin{aligned}
&\big((L_1\psi)_1,(L_2\psi)_1\big)_{L^2(Q)}\\
&\geqslant \frac{9}{2}\varepsilon^2 s^3\iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t - C\varepsilon s\big(T+|M|T^2\big)\iint_Q \alpha^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\varepsilon^2\frac{s^3}{2}\int_0^T\alpha^3_{x\,|x=L}|\psi_{x\,|x=L}|^2\,\mathrm{d}t - C\varepsilon s\big(T+|M|T^2\big)\int_0^T\alpha^3_{x\,|x=L}|\psi_{x\,|x=L}|^2\,\mathrm{d}t\\
&\quad-C\varepsilon^2 s^3T^2\iint_Q\alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t - C\varepsilon\big(\varepsilon s^5+\varepsilon s^4T+s^3\big(T+\varepsilon T^2+|M|T^2\big)\\
&\quad+s^2\big(T^2+|M|T^3\big)+sT^3\big)\int_0^T\alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t.
\end{aligned}$$

We integrate by parts again in space in the second term, and using the boundary values for $\psi_{x\,|x=0}$ and $\psi_{xx\,|x=0}$:

$$\begin{aligned}
&\big((L_1\psi)_1,(L_2\psi)_2\big)_{L^2(Q)}\\
&=-3\varepsilon^2 s\iint_Q\alpha_{xx}|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t+3\varepsilon^2 s\int_0^T\alpha_{xx\,|x=L}\psi_{x\,|x=L}\psi_{xx\,|x=L}\,\mathrm{d}t\\
&\quad-3\varepsilon^2 s\int_0^T\alpha_{xx\,|x=0}\psi_{x\,|x=0}\psi_{xx\,|x=0}\,\mathrm{d}t\\
&\geqslant-3\varepsilon^2 s\iint_Q\alpha_{xx}|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t-\frac{\varepsilon^2 s}{2}\int_0^T\alpha_{x\,|x=L}|\psi_{xx\,|x=L}|^2\,\mathrm{d}t\\
&\quad-C\varepsilon^2 sT^2\int_0^T\alpha^3_{x\,|x=L}|\psi_{x\,|x=L}|^2\,\mathrm{d}t-C\varepsilon^2\big(s^4T+s^3T^2\big)\int_0^T\alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t.
\end{aligned}$$

Here, we have used the boundary values (A.2), (A.3), the properties in (3.12) and Young's inequality. For the third term, we proceed in a similar manner:

$$\begin{aligned}
&\big((L_1\psi)_1,(L_2\psi)_3\big)_{L^2(Q)}\\
&=-\frac{3}{2}\varepsilon^2 s\iint_Q\alpha_{xx}|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{3}{2}\varepsilon^2 s\int_0^T\alpha_{x\,|x=L}|\psi_{xx\,|x=L}|^2\,\mathrm{d}t\\
&\quad-\frac{3}{2}\varepsilon^2 s\int_0^T\alpha_{x\,|x=0}|\psi_{xx\,|x=0}|^2\,\mathrm{d}t
\end{aligned}$$

$$\geqslant -\frac{3}{2}\varepsilon^2 s \iint_Q \alpha_{xx}|\psi_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{3}{2}\varepsilon^2 s \int_0^T \alpha_{x|x=L}|\psi_{xx|x=L}|^2 \,\mathrm{d}t - C\varepsilon^2\big(s^5 + s^3T^2\big) \int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2 \,\mathrm{d}t.$$

Putting together these computations, we obtain

$$\begin{aligned}
&\big((L_1\psi)_1, L_2\psi\big)_{L^2(Q)} \\
&\geqslant \frac{9}{2}\varepsilon^2 s^3 \iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t - \frac{9}{2}\varepsilon^2 s \iint_Q \alpha_{xx}|\psi_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \varepsilon^2 s \int_0^T \alpha_{x|x=L}|\psi_{xx|x=L}|^2 \,\mathrm{d}t - \varepsilon^2 \frac{s^3}{2} \int_0^T \alpha^3_{x|x=L}|\psi_{x|x=L}|^2 \,\mathrm{d}t \\
&\quad - C\varepsilon s\big(T + |M|T^2\big) \iint_Q \alpha^3|\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t - C\varepsilon^2 s^3 T^2 \iint_Q \alpha^5|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - C\varepsilon s\big(T + \varepsilon T^2 + |M|T^2\big) \int_0^T \alpha^3_{x|x=L}|\psi_{x|x=L}|^2 \,\mathrm{d}t \\
&\quad - C\varepsilon\big(\varepsilon s^5 + \varepsilon s^4 T + s^3\big(T + \varepsilon T^2 + |M|T^2\big) \\
&\quad + s^2 T\big(T + |M|T^2\big) + sT^3\big) \int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2 \,\mathrm{d}t. \qquad \text{(A.5)}
\end{aligned}$$

Computing $((L_1\psi)_2, L_2\psi)_{L^2(Q)}$.
For the first term, we integrate by parts in time:

$$\begin{aligned}
\big((L_1\psi)_2, (L_2\psi)_1\big)_{L^2(Q)} &= -\frac{1}{2} \iint_Q \big(3\varepsilon s^3 \alpha_x^2 \alpha_{xt} + s\alpha_{tt} - Ms\alpha_{xt}\big)|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\geqslant -C\big(\varepsilon s^3 + s\big(T + |M|T^2\big)\big)T \iint_Q \alpha^5|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The second terms gives:

$$\big((L_1\psi)_2, (L_2\psi)_2\big)_{L^2(Q)} = 3\varepsilon s \iint_Q \alpha_{xx}\psi_x\psi_t \,\mathrm{d}x\,\mathrm{d}t.$$

In the third term, we integrate by parts first in space and then in time. We obtain

$$\begin{aligned}
\big((L_1\psi)_2, (L_2\psi)_3\big)_{L^2(Q)} &= -\frac{3}{2}\varepsilon s \iint_Q \alpha_x \partial_t|\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t - 3\varepsilon s \iint_Q \alpha_{xx}\psi_x\psi_t \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - 3\varepsilon s \int_0^T \alpha_{x|x=0}\psi_{x|x=0}\psi_{t|x=0} \,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{2}\varepsilon s \iint_Q \alpha_{xt}|\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - 3\varepsilon s \iint_Q \alpha_{xx}\psi_x\psi_t \,\mathrm{d}x\,\mathrm{d}t - 3\varepsilon s^2 \int_0^T \alpha_{x\,|x=0}\alpha_{xt\,|x=0}|\psi_{|x=0}|^2 \,\mathrm{d}t \\
&\geqslant -3\varepsilon s \iint_Q \alpha_{xx}\psi_x\psi_t \,\mathrm{d}x\,\mathrm{d}t - C\varepsilon s T \iint_Q \alpha^3|\psi_x|^2 \\
&\quad - C\varepsilon s^2 T^2 \int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2 \,\mathrm{d}t,
\end{aligned}
$$

where we have used the boundary value (A.2).

Putting together this inequalities, we have

$$
\begin{aligned}
\big((L_1\psi)_2, L_2\psi\big)_{L^2(Q)} &\geqslant -C\big(\varepsilon s^3 + s\big(T + |M|T^2\big)\big)T \iint_Q \alpha^5|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - C\varepsilon s T \iint_Q \alpha^3|\psi_x|^2 - C\varepsilon s^2 T^2 \int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2 \,\mathrm{d}t. \qquad \text{(A.6)}
\end{aligned}
$$

Computing $((L_1\psi)_3, L_2\psi)_{L^2(Q)}$.

We integrate by parts in space and the properties (3.12)–(3.13) to treat the first term:

$$
\begin{aligned}
&\big((L_1\psi)_3, (L_2\psi)_1\big)_{L^2(Q)} \\
&\quad = -\frac{1}{2}\varepsilon \iint_Q \big(15\varepsilon s^5\alpha_x^4\alpha_{xx} + 6s^3\alpha_x\alpha_{xx}\alpha_t + 3s^3\alpha_x^2\alpha_{xt} - 9Ms^3\alpha_x^2\alpha_{xx}\big)|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \frac{1}{2}\varepsilon \int_0^T \big(3\varepsilon s^5\alpha_x^5 + 3s^3\alpha_x^2\alpha_t + 3Ms^3\alpha_x^3\big)_{|x=0}|\psi_{|x=0}|^2 \,\mathrm{d}t \\
&\quad \geqslant \frac{15}{2}C_0^5\varepsilon^2 s^5 \iint_Q \alpha^5|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t - C\varepsilon s^3\big(T + |M|T^2\big) \iint_Q \alpha^5|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - C\varepsilon\big(\varepsilon s^5 + s^3\big(T + |M|T^2\big)\big) \int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2 \,\mathrm{d}t.
\end{aligned}
$$

The second term simply gives:

$$
\big((L_1\psi)_3, (L_2\psi)_2\big)_{L^2(Q)} = 9\varepsilon^2 s^3 \iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t.
$$

As for the third term, we integrate by parts in space once and use the boundary value (A.2). We obtain:

$$
\begin{aligned}
&\big((L_1\psi)_3, (L_2\psi)_3\big)_{L^2(Q)} \\
&\quad = -\frac{27}{2}\varepsilon^2 s^3 \iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{9}{2}\varepsilon^2 s^3 \int_0^T \alpha^3_{x\,|x=L}|\psi_{x\,|x=L}|^2 \,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{9}{2}\varepsilon^2 s^3 \int_0^T \alpha^3_{x\,|x=0} |\psi_{x\,|x=0}|^2 \,\mathrm{d}t \\
&= -\frac{27}{2}\varepsilon^2 s^3 \iint_Q \alpha_x^2 \alpha_{xx} |\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{9}{2}\varepsilon^2 s^3 \int_0^T \alpha^3_{x\,|x=L} |\psi_{x\,|x=L}|^2 \,\mathrm{d}t - \frac{9}{2}\varepsilon^2 s^5 \int_0^T \alpha^5_{x\,|x=0} |\psi_{|x=0}|^2 \,\mathrm{d}t.
\end{aligned}
$$

Putting together these estimates, we get:

$$
\begin{aligned}
&\big((L_1\psi)_3, L_2\psi\big)_{L^2(Q)} \\
&\geqslant -\frac{9}{2}\varepsilon^2 s^3 \iint_Q \alpha_x^2 \alpha_{xx} |\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{15}{2} C_0^5 \varepsilon^2 s^5 \iint_Q \alpha^5 |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{9}{2}\varepsilon^2 s^3 \int_0^T \alpha^3_{x\,|x=L} |\psi_{x\,|x=L}|^2 \,\mathrm{d}t - C\varepsilon s^3\big(T + |M|T^2\big) \iint_Q \alpha^5 |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - C\varepsilon\big(\varepsilon s^5 + s^3\big(T + |M|T^2\big)\big) \int_0^T \alpha^5_{|x=0} |\psi_{|x=0}|^2 \,\mathrm{d}t. \qquad \text{(A.7)}
\end{aligned}
$$

Computing $((L_1\psi)_4, L_2\psi)_{L^2(Q)}$.
For the first term, we integrate by parts in space:

$$
\begin{aligned}
\big((L_1\psi)_4, (L_2\psi)_1\big)_{L^2(Q)} &= \frac{M}{2} \iint_Q \big(3\varepsilon s^3 \alpha_x^2 \alpha_{xx} + s\alpha_{xt} - Ms\alpha_{xx}\big) |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{M}{2} \int_0^T \big(\varepsilon s^3 \alpha_x^3 + s\alpha_t - Ms\alpha_x\big)_{|x=0} |\psi_{|x=0}|^2 \,\mathrm{d}t \\
&\geqslant -C\big(\varepsilon s^3 + s\big(T + |M|T^2\big)\big)|M|T^2 \iint_Q \alpha^5 |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - C\big(\varepsilon s^3 + s\big(T + |M|T^2\big)\big)|M|T^2 \int_0^T \alpha^5_{|x=0} |\psi_{|x=0}|^2 \,\mathrm{d}t.
\end{aligned}
$$

The second term gives directly:

$$
\big((L_1\psi)_4, (L_2\psi)_2\big)_{L^2(Q)} \geqslant -C|M|\varepsilon s T^2 \iint_Q \alpha^3 |\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t.
$$

The third and final term gives, after integration by parts:

$$
\begin{aligned}
&\big((L_1\psi)_4, (L_2\psi)_3\big)_{L^2(Q)} \\
&= \frac{3}{2} M\varepsilon s \iint_Q \alpha_{xx} |\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t - \frac{3}{2} M\varepsilon s \int_0^T \alpha_{x\,|x=L} |\psi_{x\,|x=L}|^2 \,\mathrm{d}t
\end{aligned}
$$

$$+\frac{3}{2}M\varepsilon s\int_0^T \alpha_{x|x=0}|\psi_{x|x=0}|^2\,\mathrm{d}t$$
$$\geqslant -C|M|\varepsilon sT^2\iint_Q \alpha^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$-C|M|\varepsilon sT^2\int_0^T \alpha^3_{x|x=L}|\psi_{x|x=L}|^2\,\mathrm{d}t - C|M|\varepsilon s^3T^2\int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t.$$

Putting together these expressions, we obtain:

$$\big((L_1\psi)_4, L_2\psi\big)_{L^2(Q)}$$
$$\geqslant -C\big(\varepsilon s^3 + s\big(T+|M|T^2\big)\big)|M|T^2\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$-C|M|\varepsilon sT^2\iint_Q \alpha^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t - C|M|\varepsilon sT^2\int_0^T \alpha^3_{x|x=L}|\psi_{x|x=L}|^2\,\mathrm{d}t$$
$$-C\big(\varepsilon s^3 + s\big(T+|M|T^2\big)\big)|M|T^2\int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t. \tag{A.8}$$

The entire product $(L_1\psi, L_2\psi)_{L^2(Q)}$.

Adding inequalities (A.5)–(A.8), we find four positive terms, namely:

$$A_1 := \frac{15}{2}C_0^5\varepsilon^2 s^5\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t, \qquad A_2 := \frac{9}{2}C_0\varepsilon^2 s\iint_Q \alpha|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t,$$
$$A_3 := 4\varepsilon^2 s^3\int_0^T \alpha^3_{x|x=L}|\psi_{x|x=L}|^2\,\mathrm{d}t, \qquad A_4 := \varepsilon^2 s\int_0^T \alpha_{x|x=L}|\psi_{xx|x=L}|^2\,\mathrm{d}t.$$

In the following, we explain how to estimate the nonpositive integrals coming from the addition of (A.5)–(A.8) in terms of A_i.

Let us start with the terms concerning $|\psi|^2$ in Q. We can easily check that they can all be bounded by

$$C\big(s^3\big(\varepsilon T + \varepsilon^2T^2 + |M|\varepsilon T^2\big) + s\big(T^2 + |M|T^3 + |M|^2T^4\big)\big)\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t,$$

which by taking $s \geqslant C(T + T^{1/2}\varepsilon^{-1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$ can be absorbed by A_1.

The integral of $|\psi_{x|x=L}|^2$, can be bounded by

$$C\varepsilon s\big(T + \varepsilon T^2 + |M|T^2\big)\int_0^T \alpha^3_{x|x=L}|\psi_{x|x=L}|^2\,\mathrm{d}t.$$

Taking $s \geqslant C(T + T^{1/2}\varepsilon^{-1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$, this term can be absorbed by A_3.

Furthermore, taking $s \geqslant C(T + T^{1/2}\varepsilon^{-1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$ shows that all the integrals concerning $|\psi_{|x=0}|^2$ can be estimated by

$$C\varepsilon^2 s^5 \int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t. \tag{A.9}$$

Finally, let us treat the terms containing $|\psi_x|^2$ in Q. They can be estimated by

$$C\varepsilon s\big(T + |M|T^2\big)\iint_Q \alpha^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Now, similarly as the previous steps, integration by parts in space shows that:

$$\begin{aligned}
&C\varepsilon s\big(T + |M|T^2\big)\iint_Q \alpha^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t\\
&= \frac{3}{2}C\varepsilon s\big(T + |M|T^2\big)\iint_Q \big(2\alpha\alpha_x^2 + \alpha^2\alpha_{xx}\big)|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \frac{3}{2}C\varepsilon s\big(T + |M|T^2\big)\int_0^T \alpha^2_{|x=0}\alpha_{x\,|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t\\
&\quad - C\varepsilon s\big(T + |M|T^2\big)\iint_Q \alpha^3\psi\psi_{xx}\,\mathrm{d}x\,\mathrm{d}t - C\varepsilon s\big(T + |M|T^2\big)\int_0^T \alpha^3_{|x=0}\psi_{|x=0}\psi_{x\,|x=0}\,\mathrm{d}t\\
&\geqslant -C\big(\varepsilon sT^2\big(T + |M|T^2\big) + \varepsilon^{1/2}s^2\big(T^{3/2} + |M|^{3/2}T^3\big)\big)\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - C\varepsilon^{3/2}\big(T^{1/2} + |M|^{1/2}T\big)\iint_Q \alpha|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - C\varepsilon\big(sT^2\big(T + |M|T^2\big) + s^2T\big(T + |M|T^2\big)\big)\int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t.
\end{aligned} \tag{A.10}$$

Notice that here we have used (A.2) and Young's inequality. By taking $s \geqslant C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$, the first two integrals can be absorbed by A_1 and A_2, respectively, and the last one can be estimated by (A.9).

Finally, all these estimations give

$$\begin{aligned}
&(L_1\psi, L_2\psi)_{L^2(Q)}\\
&\geqslant C\varepsilon^2 s^5\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + C\varepsilon^2 s\iint_Q \alpha|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \varepsilon^2 s^3\int_0^T \alpha^3_{x\,|x=L}|\psi_{x\,|x=L}|^2\,\mathrm{d}t + C\varepsilon^2 s\int_0^T \alpha_{x\,|x=L}|\psi_{xx\,|x=L}|^2\,\mathrm{d}t - C\varepsilon^2 s^5\int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t,
\end{aligned}$$

for every $s \geqslant C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$.

Coming back to (A.4), and together with the fact that

$$\|L_3\psi\|^2_{L^2(Q)} \leqslant C\varepsilon^2 s^4 T \iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t,$$

we obtain

$$\begin{aligned}
&\varepsilon^2 s^5 \iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s \iint_Q \alpha|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s^3 \int_0^T \alpha^3_{x|x=L}|\psi_{x|x=L}|^2\,\mathrm{d}t \\
&\quad + \varepsilon^2 s \int_0^T \alpha_{x|x=L}|\psi_{xx|x=L}|^2\,\mathrm{d}t \\
&\leqslant C\varepsilon^2 s^5 \int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t,
\end{aligned} \tag{A.11}$$

for every $s \geqslant C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$.

Coming back to the original variable.

Let us now go back to the original variable ϕ. First, we point out that the same computations made in (A.10), show that

$$\begin{aligned}
\varepsilon^2 s^3 \iint_Q \alpha^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t &\leqslant C\varepsilon^2 s^5 \iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + C\varepsilon^2 s \iint_Q \alpha|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\varepsilon^2 s^5 \int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t,
\end{aligned}$$

as long as $s \geqslant CT$. This means that we can add this term to the left-hand side of (A.11), and together with $\psi = \mathrm{e}^{-s\alpha}\phi$, we have directly from (A.11) that

$$\begin{aligned}
&\varepsilon^2 s^5 \iint_Q \mathrm{e}^{-2s\alpha}\alpha^5|\phi|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s^3 \iint_Q \alpha^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s \iint_Q \alpha|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \varepsilon^2 s^3 \int_0^T \alpha^3_{x|x=L}|\psi_{x|x=L}|^2\,\mathrm{d}t + \varepsilon^2 s \int_0^T \alpha_{x|x=L}|\psi_{xx|x=L}|^2\,\mathrm{d}t \\
&\leqslant C\varepsilon^2 s^5 \int_0^T \mathrm{e}^{-2s\alpha_{|x=0}}\alpha^5_{|x=0}|\phi_{|x=0}|^2\,\mathrm{d}t,
\end{aligned} \tag{A.12}$$

for every $s \geqslant C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$.

Now, from $\psi = \mathrm{e}^{-s\alpha}\phi$, we find that

$$s^{3/2}\mathrm{e}^{-s\alpha}\alpha^{3/2}\phi_x = s^{3/2}\alpha^{3/2}\psi_x + s^{5/2}\mathrm{e}^{-s\alpha}\alpha^{3/2}\alpha_x\phi,$$

and taking the $L^2(Q)$-norm, we see that we can add

$$\varepsilon^2 s^3 \iint_Q \mathrm{e}^{-2s\alpha}\alpha^3|\phi_x|^2\,\mathrm{d}x\,\mathrm{d}t$$

to the left-hand side of (A.12). Similarly, from

$$s^{1/2}e^{-s\alpha}\alpha^{1/2}\phi_{xx} = s^{1/2}\alpha^{1/2}\psi_{xx} + s^{3/2}e^{-s\alpha}\alpha^{1/2}\alpha_{xx}\phi + 2s^{3/2}\alpha^{1/2}\alpha_x\psi_x + s^{5/2}e^{-s\alpha}\alpha^{1/2}\alpha_x^2\phi,$$

we can add

$$\varepsilon^2 s \iint_Q e^{-2s\alpha}\alpha|\phi_{xx}|^2\,dx\,dt$$

to the left-hand side of (A.12) if $s \geqslant CT$. Finally, using (A.1), we can add to the left-hand side of (A.12) the respective boundary integrals and obtain

$$\begin{aligned}
&\varepsilon^2 s^5 \iint_Q e^{-2s\alpha}\alpha^5|\phi|^2\,dx\,dt + \varepsilon^2 s^3 \iint_Q e^{-2s\alpha}\alpha^3|\phi_x|^2\,dx\,dt + \varepsilon^2 s \iint_Q e^{-2s\alpha}\alpha|\phi_{xx}|^2\,dx\,dt \\
&\quad + \varepsilon^2 s^3 \int_0^T e^{-2s\alpha_{|x=L}}\alpha^3_{x|x=L}|\phi_{x|x=L}|^2\,dt + \varepsilon^2 s \int_0^T e^{-2s\alpha_{|x=L}}\alpha_{x|x=L}|\phi_{xx|x=L}|^2\,dt \\
&\leqslant C\varepsilon^2 s^5 \int_0^T e^{-2s\alpha_{|x=0}}\alpha^5_{|x=0}|\phi_{|x=0}|^2\,dt,
\end{aligned} \tag{A.13}$$

for every $s \geqslant C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$. The proof of Proposition 3.1 is complete.

Appendix B. Proof of Lemma 4.3

We define $\tilde{h}$ by:

$$\tilde{h}(x) := p(x), \quad x \in [0, L_1], \qquad \tilde{h}(x) := h(x), \quad x \in (L_1, L],$$

where $p(x) = ax^2 + bx + c$ is given by

$$a := \frac{h''(L_1)}{2}, \qquad b := h'(L_1) - L_1 h''(L_1) \quad \text{and} \quad c := h(L_1) - L_1 h'(L_1) + L_1^2\frac{h''(L_1)}{2}.$$

It is easy to check that $\tilde{h} \in H_n^3(0, L)$ and

$$\|\tilde{h}\|^2_{H_n^3(0,L)} = \|p\|^2_{L^2(0,L_1)} + \|h\|^2_{H_n^3(L_1,L)}. \tag{B.1}$$

Let us estimate the term concerning p. Notice that

$$h(L_1) = -\frac{1}{L-L_1}\int_{L_1}^L \big((L-x)h\big)'\,dx = \frac{1}{L-L_1}\int_{L_1}^L h\,dx - \frac{1}{L-L_1}\int_{L_1}^L (L-x)h'\,dx.$$

Taking the square and using Cauchy–Schwarz inequality we get

$$\begin{aligned} \left|h(L_1)\right|^2 &\leqslant \frac{2}{L-L_1}\int_{L_1}^{L} |h|^2\,\mathrm{d}x + \frac{2(L-L_1)}{3}\int_{L_1}^{L} \left|h'\right|^2\mathrm{d}x \\ &\leqslant \frac{2}{L-L_1}\int_{L_1}^{L} |h|^2\,\mathrm{d}x + \frac{(L-L_1)^5}{6}\int_{L_1}^{L} \left|h'''\right|^2\mathrm{d}x, \end{aligned} \tag{B.2}$$

where we have used that $h'(L) = h''(L) = 0$ in the last inequality.

Now, using that $h'(L) = 0$, we have

$$h'(L_1) = -\int_{L_1}^{L} h''\,\mathrm{d}x \leqslant (L-L_1)^{1/2}\left(\int_{L_1}^{L} \left|h''\right|^2\mathrm{d}x\right)^{1/2}$$

and therefore

$$\left|h'(L_1)\right|^2 \leqslant (L-L_1)\int_{L_1}^{L} \left|h''\right|^2\mathrm{d}x \leqslant \frac{(L-L_1)^3}{2}\int_{L_1}^{L} \left|h'''\right|^2\mathrm{d}x. \tag{B.3}$$

Similarly, we get

$$\left|h''(L_1)\right|^2 \leqslant (L-L_1)\int_{L_1}^{L} \left|h'''\right|^2\mathrm{d}x. \tag{B.4}$$

From (B.2)–(B.4) we obtain

$$\begin{aligned} \|p\|^2_{L^2(0,L_1)} &\leqslant CL_1\left(\left|h''(L_1)\right|^2 L_1^4 + \left|h'(L_1)\right|^2 L_1^2 + \left|h(L_1)\right|^2\right) \\ &\leqslant CL_1\left((L-L_1)L_1^4 + (L-L_1)^3 L_1^2 + (L-L_1)^5 + (L-L_1)^{-1}\right)\|h\|^2_{H^3_n(L_1,L)}. \end{aligned}$$

Combining this with (B.1) and using Young's inequality, we obtain the desired result.

References

[1] E. Cerpa and E. Crépeau, Boundary controllability for the nonlinear Korteweg–de Vries equation on any critical domain, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**(2) (2009), 457–475.

[2] E. Cerpa, I. Rivas and B.-Y. Zhang, Boundary controllability of the Korteweg–de Vries equation on a bounded domain, *SIAM J. Control Optim.* **51**(4) (2013), 2976–3010.

[3] T. Colin and J.-M. Ghidaglia, Un problème mixte pour l'équation de Korteweg–de Vries sur un intervalle borné, *C. R. Acad. Sci. Paris Sér. I Math.* **324**(5) (1997), 599–603.

[4] T. Colin and J.-M. Ghidaglia, An initial-boundary value problem for the Korteweg–de Vries equation posed on a finite interval, *Adv. Differential Equations* **6**(12) (2001), 1463–1492.

[5] J.-M. Coron, *Control and Nonlinearity*, Mathematical Surveys and Monographs, Vol. 136, American Mathematical Society, Providence, RI, 2007.

[6] J.-M. Coron and S. Guerrero, Singular optimal control: A linear 1-D parabolic–hyperbolic example, *Asymptot. Anal.* **44**(3–4) (2005), 237–257.

[7] L. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, Providence, RI, 1998.

[8] A.V. Fursikov and O.Yu. Imanuvilov, *Controllability of Evolution Equations*, Lecture Notes Series, Vol. 34, Global Analysis Research Center, Research Institute of Mathematics, Seoul National Univ., Seoul, 1996.
[9] O. Glass and S. Guerrero, Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit, *Asymptot. Anal.* **60**(1,2) (2008), 61–100.
[10] O. Glass and S. Guerrero, Uniform controllability of a transport equation in zero diffusion–dispersion limit, *Math. Models Methods Appl. Sci.* **19**(9) (2009), 1567–1601.
[11] S. Guerrero and G. Lebeau, Singular optimal control for a transport–diffusion equation, *Comm. Partial Differential Equations* **32**(10–12) (2007), 1813–1836.
[12] J.-P. Guilleron, Null controllability of a linear KdV equation on an interval with special boundary conditions, *Math. Control Signals Syst.* **26**(3) (2014), 375–401.
[13] E. Kramer, I. Rivas and B.-Y. Zhang, Well-posedness of a class of non-homogeneous boundary value problems of the Korteweg–de Vries equation on a finite domain, *ESAIM Control Optim. Calc. Var.* **19**(2) (2013), 358–384.
[14] J.-L. Lions, Exact controllability, stabilization and perturbations for distributed systems, *SIAM Rev.* **30**(1) (1988), 1–68.
[15] L. Rosier, Exact boundary controllability for the Korteweg–de Vries equation on a bounded domain, *ESAIM Control Optim. Calc. Var.* **2** (1997), 33–55.
[16] L. Rosier, Control of the surface of a fluid by a wavemaker, *ESAIM Control Optim. Calc. Var.* **10**(3) (2004), 346–380.