Some controllability results with a reduced number of controls
Quelques résultats de contrôlabilité avec un nombre réduit de contrôles

Matinée Contrôle

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March 21th, 2013

Outline

Local null controllability of Navier-Stokes

- Ω bounded connected regular open subset of $\mathbb{R}^3$
- $T > 0$
- $\omega \subset \Omega$ (control set), $Q := \Omega \times (0, T)$, $\Sigma := \partial\Omega \times (0, T)$

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = (v_1, v_2, 0)\mathbb{1}_\omega, \ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases}$$

where v_1 and v_2 stand for the controls which act over the set ω.

Local null controllability problem:

If $\|y^0\|$ is small enough, can we find controls v_1 and v_2 in $L^2(\omega \times (0, T))$ such that $y(T) = 0$?

Some results

- First results by Fernández-Cara, Guerrero, Imanuvilov, Puel (2006) when $\overline{\omega} \cap \partial\Omega \neq \emptyset$ (We are interested in removing this geometric condition)
- Coron, Guerrero (2009) Null controllability of the Stokes system for a general $\omega \subset \Omega$

$$y_t - \Delta y + \nabla p = (v_1, v_2, 0)\mathbb{1}_\omega, \ \nabla \cdot y = 0, \ y|_\Sigma = 0$$

- Recently, Lissy (2012), Local null controllability of Navier-Stokes with $(v_1, 0, 0)$ (Return method)

Our result

Theorem (C., Guerrero) Local null controllability for general ω

For every $T > 0$ and $\omega \subset \Omega$, the N-S system is locally null controllable by a control $(v_1, v_2, 0) \in L^2(\omega \times (0, T))^3$ (or $(v_1, 0, v_3)$, or $(0, v_2, v_3)$).

Method of proof

- Linearization around zero

$$y_t - \Delta y + \nabla p = f + (v_1, v_2, 0)\mathbb{1}_\omega,\ \nabla \cdot y = 0,\ y|_\Sigma = 0$$

- Null controllability of the linearized system (Main part of the proof).
  Main tool: Carleman estimate for the adjoint system

$$-\varphi_t - \Delta\varphi + \nabla\pi = g,\ \nabla \cdot \varphi = 0,\ \varphi|_\Sigma = 0$$

  There exists a constant $C > 0$ (depending on Ω, ω, T)

$$\int_Q \rho_1(t)|\varphi|^2 \leq C\left(\int_Q \rho_2(t)|g|^2 + \int_{\omega\times(0,T)} \rho_3(t)(|\varphi_1|^2 + |\varphi_2|^2)\right)$$

- Inverse mapping theorem for the nonlinear system

$$\mathcal{A} = (y_t - \Delta y + (y\cdot\nabla)y + \nabla p - (v_1, v_2, 0)\mathbb{1}_\omega, y(0))$$

Boussinesq system

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = (v_1, 0, 0)\mathbb{1}_\omega + (0,0,\theta)\,,\ \nabla\cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta\theta + y\cdot\nabla\theta = v_0\mathbb{1}_\omega & \text{in } Q, \\ y = 0,\ \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0,\ \theta(0) = \theta^0 & \text{in } \Omega, \end{cases}$$

- y : Velocity, θ : Temperature
- Same method applied to N-S to control with $(v_1, 0, v_3)$ and v_0
- We use θ to control the third equation and set $v_3 \equiv 0$

Theorem

For every $T > 0$ and $\omega \subset \Omega$, the Boussinesq system is locally null controllable by controls $v_0 \in L^2(\omega \times (0, T))$ and $(v_1, 0, 0) \in L^2(\omega \times (0, T))^3$ (or $(0, v_2, 0)$).

A coupled Navier-Stokes system

We consider the following null controllability problem: To find a control $v = (v_1, v_2, 0)$ such that $z(0) = 0$, where

$$
\begin{cases}
w_t - \Delta w + (w \cdot \nabla) w + \nabla p^0 = f + (v_1, v_2, 0)\mathbb{1}_\omega, \; \nabla \cdot w = 0 & \text{in } Q, \\
-z_t - \Delta z + (z \cdot \nabla^t) w - (w \cdot \nabla) z + \nabla q = w \mathbb{1}_{\mathcal{O}}, \; \nabla \cdot z = 0 & \text{in } Q, \\
w = z = 0 & \text{on } \Sigma, \\
w(0) = y^0, \; z(T) = 0 & \text{in } \Omega.
\end{cases}
$$

- z is controlled by $w\mathbb{1}_{\mathcal{O}}$
- Application to insensitizing controls for Navier-Stokes

Theorem (joint work with M. Gueye)

Assume $y^0 = 0$, $\|e^{K/t^{10}} f\|_{L^2(Q)^3} < \infty$ and $\mathcal{O} \cap \omega \neq \emptyset$. The previous system is null controllable by a control $(v_1, v_2, 0) \in L^2(\omega \times (0,T))^3$ (or $(v_1, 0, v_3)$, or $(0, v_2, v_3)$).

Perspectives and some references

- Local exact controllability to the trajectories for Navier-Stokes

$$-\varphi_t - \Delta\varphi + \bar{y} \cdot D\varphi + \nabla\pi = g$$

- Boundary controllability with one vanishing component (taking the trace of an extended controlled solution does not work)
- No control in the heat equation for the Boussinesq system (i.e., $v_0 \equiv 0$)

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