Control, observability and application to the control of the Hirota-Satsuma system

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May 23th, 2022

Outline

PDE control system

A control system governed by a partial differential equation can be formulated

$$\begin{cases} y'(t) = f(t, y(t), v(t)), & t > 0 \\ y(0) = y_0, \end{cases}$$

- $y(t)$ is the state of the system.
- $v(t)$ is the control.
- Controllability problem: Given y_0 and $T > 0$, find $v(t)$ driving $y(t)$ to a target y_1 at time T, that is, $y(T) = y_1$.
- Controllability types
  - Exact.
  - Null: $y(T) = 0$.
  - Approximate: $y(T)$ close to y_1.
  - Local: y_0 close to y_1.

PDE control

Example: Heat equation

Consider a regular open $\Omega \subset \mathbb{R}^N$ and $\omega \subset \Omega$ (control domain)

$$\begin{cases} y_t - \Delta y = v\mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ y = 0 & x \in \partial\Omega, \\ y(0) = y_0 & x \in \Omega. \end{cases}$$

- $y = y(x,t)$: Temperature distribution.
- $v = v(x,t)$: Control supported in ω.

Question: Given $T > 0$ and $y_1 = y_1(x)$, is there v such that $y(T) = y_1$?

PDE control

Answer: In general, the answer is no due to the *regularizing effect*.

• It seems natural to consider the notion of control to the trajectories: Consider a solution of

$$\begin{cases} \bar{y}_t - \Delta \bar{y} = 0 & (x,t) \in \Omega \times (0,T), \\ \bar{y} = 0 & x \in \partial\Omega, \\ \bar{y}(0) = \bar{y}_0 & x \in \Omega, \end{cases}$$

We look for a control v such that $y(T) = \bar{y}(T)$.

• By linearity (taking $\widetilde{y} := y - \bar{y}$), this is equivalent to the null controllability:

$$y(T) = 0.$$

Therefore, we concentrate in this case.

Duality Method: Hilbert Uniqueness Method (HUM)

Construction of the control:

- We multiply $y_t - \Delta y = v \mathbb{1}_\omega$ by φ solution to the (adjoint) equation

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & (x,t) \in \Omega \times (0,T), \\ \varphi = 0 & x \in \partial\Omega, \\ \varphi(T) = \varphi_T \in L^2(\Omega) & x \in \Omega, \end{cases}$$

and integrate in $\Omega \times (0,T)$:

$$\int_\Omega y(T)\varphi_T \, \mathrm{d}x = \iint_{\omega\times(0,T)} v\varphi \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega y_0 \varphi(0) \, \mathrm{d}x, \quad \forall \varphi_T \in L^2(\Omega).$$

- v is a control such that $y(T) = 0$ if and only if

$$\iint_{\omega\times(0,T)} v\varphi \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega y_0 \varphi(0) \, \mathrm{d}x = 0, \quad \forall \varphi_T \in L^2(\Omega).$$

Observability inequality

The previous condition can be seen as an optimality condition for

$$J(\varphi_T) = \frac{1}{2} \iint\limits_{\omega\times(0,T)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega y_0 \varphi(0)\,\mathrm{d}x.$$

- J convex, continuous and coercive if there exists $C > 0$ such that

$$\int_\Omega |\varphi(0)|^2 \,\mathrm{d}x \leq C \iint\limits_{\omega\times(0,T)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

This is known as observability inequality.

- The control is given by

$$v := \widehat{\varphi},$$

where $\widehat{\varphi}$ is the solution of the adjoint equation associated to $\widehat{\varphi}_T$, minimum of J.

- Furthermore:

$$\|v\|^2_{L^2(0,T;L^2(\omega))} \leq C\|y_0\|^2_{L^2(\Omega)}.$$

- Null controllability is equivalent to observability.

Carleman estimates

How to prove the observability inequality?

Powerful tool to prove observability: Carleman estimates

$$\iint_{\Omega\times(0,T)} \rho_1|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega\times(0,T)} \rho_2|\nabla\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega\times(0,T)} \rho_3\left(|\varphi_t|^2 + |\Delta\varphi|^2\right)\mathrm{d}x\,\mathrm{d}t$$
$$\leq C \iint_{\Omega\times(0,T)} \rho_4|\varphi_t + \Delta\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + C\iint_{\omega_0\times(0,T)} \rho_1|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t$$

- $\varphi(x,t) = 0$, $x \in \partial\Omega$.
- $\rho_i = \rho_i(x,t)$ is a positive (weight) function and continuous in $\overline{\Omega} \times (0,T)$ with critical points only in $\omega_0 \subset \omega$.
- To deduce the observability, we use dissipation properties as

$$\int_\Omega |\varphi(0)|^2\,\mathrm{d}x \leq \int_\Omega |\varphi(t)|^2\,\mathrm{d}x, \quad t \in (0,T).$$

Control of a system of two equations with one control

Consider the system with one scalar control

$$\begin{cases} y_t - \Delta y = z + v\mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ z_t - \Delta z = y & (x,t) \in \Omega \times (0,T), \\ y = z = 0 & x \in \partial\Omega, \\ y(0) = y^0, \quad z(0) = z^0 & x \in \Omega. \end{cases}$$

- We look for v such that $y(T) = z(T) = 0$.
- Observability inequality: There exists $C > 0$ such that

$$\int_\Omega \left(|\varphi(0)|^2 + |\psi(0)|^2\right) \mathrm{d}x \le C \iint_{\omega\times(0,T)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t$$

where (φ, ψ) is the solution to the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = \psi & (x,t) \in \Omega \times (0,T), \\ -\psi_t - \Delta\psi = \varphi & (x,t) \in \Omega \times (0,T), \\ \varphi = \psi = 0 & x \in \partial\Omega, \\ \varphi(T) = \varphi_T, \quad \psi(T) = \psi_T & x \in \Omega. \end{cases}$$

Control of a system of two equations with one control

- The idea is to combine Carleman estimates for φ and ψ:

$$\iint_{\Omega\times(0,T)} \rho_1|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t \leq C \iint_{\Omega\times(0,T)} \rho_4|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + C \iint_{\omega_0\times(0,T)} \rho_1|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\iint_{\Omega\times(0,T)} \rho_1|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \leq C \iint_{\Omega\times(0,T)} \rho_4|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t + C \iint_{\omega_0\times(0,T)} \rho_1|\psi|^2\,\mathrm{d}x\,\mathrm{d}t$$

- Estimate the local term of ψ: $\psi = -\varphi_t - \Delta\varphi$ in $\Omega\times(0,T)$.

$$\iint_{\omega_0\times(0,T)} \rho_1|\psi|^2\,\mathrm{d}x\,\mathrm{d}t = \iint_{\omega_0\times(0,T)} \rho_1\psi(-\varphi_t - \Delta\varphi)\,\mathrm{d}x\,\mathrm{d}t$$

$$\leq \frac{1}{2C} \iint_{\omega_0\times(0,T)} \rho_1\left(|\psi|^2 + |\nabla\psi|^2 + |\Delta\psi|^2 + |\psi_t|^2\right)\mathrm{d}x\,\mathrm{d}t + C \iint_{\omega_0\times(0,T)} \rho_1|\varphi|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Generalized Hirota-Satsuma system

We are interested in control properties of the Generalized Hirota-Satsuma coupled KdV system ([Hirota-Satsuma,82])

$$\begin{cases} u_t - \frac{1}{4}u_{xxx} = 3uu_x - 6vv_x + 3w_x, \\ v_t + \frac{1}{2}v_{xxx} = -3uv_x, \\ w_t + \frac{1}{2}w_{xxx} = -3uw_x. \end{cases}$$

which generalizes the set of two coupled KdV equations ([Hirota-Satsuma,81])

$$\begin{cases} u_t - \frac{1}{4}u_{xxx} = 3uu_x - 6vv_x, \\ v_t + \frac{1}{2}v_{xxx} = -3uv_x, \end{cases}$$

describing the interaction of two long waves with different dispersion relations.

Most works have focused on the existence of soliton solutions and conserved quantities. Up to our knowledge, this is the first control related work on this system.

Control of the Generalized Hirota-Satsuma system
Joint work with Eduardo Cerpa (PUC) and Emmanuelle Crépeau (LJK)

We propose to study the following control system posed in a bounded domain $(0, L)$:

$$(GHSS)\begin{cases} u_t - \frac{1}{4}u_{xxx} = 3uu_x - 6vv_x + 3w_x, & (t,x) \in (0,T)\times(0,L),\\ v_t + \frac{1}{2}v_{xxx} = -3uv_x + p\mathbb{1}_\gamma, & (t,x) \in (0,T)\times(0,L),\\ w_t + \frac{1}{2}w_{xxx} = -3uw_x + q\mathbb{1}_\omega, & (t,x) \in (0,T)\times(0,L), \end{cases}$$

with boundary conditions

$$\begin{cases} u(t,0) = u(t,L) = 0,\ u_x(t,0) = 0, & t \in (0,T),\\ v(t,0) = v(t,L) = 0,\ v_x(t,L) = 0, & t \in (0,T),\\ w(t,0) = w(t,L) = 0,\ w_x(t,L) = 0, & t \in (0,T), \end{cases}$$

and the initial conditions

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad w(0,x) = w_0(x), \quad x \in (0,L).$$

Here, $p = p(t,x)$ and $q = q(t,x)$ are distributed controls acting on $\gamma \subset (0,L)$ and $\omega \in (0,L)$, respectively.

With this set of boundary and initial conditions, $(GHSS)$ is well posed at least for small data.

$$(GHSS)\begin{cases} u_t - \frac{1}{4}u_{xxx} = 3uu_x - 6vv_x + 3w_x, & (t,x)\in(0,T)\times(0,L),\\ v_t + \frac{1}{2}v_{xxx} = -3uv_x + p\mathbb{1}_\gamma, & (t,x)\in(0,T)\times(0,L),\\ w_t + \frac{1}{2}w_{xxx} = -3uw_x + q\mathbb{1}_\omega, & (t,x)\in(0,T)\times(0,L).\end{cases}$$

Some remarks:

- We seek to control 3 equations with only 2 controls.
- We act on the first equation through the coupling w_x.
- $\gamma\subset(0,L)$ is an arbitrary open set, but ω "touches" the boundary ($\omega=(a,L)$, with $a\in(0,L)$).

Our result establishes the local null controllability of $(GHSS)$:

Theorem (C.-Cerpa-Crépeau, 2020)

Let $\gamma\subset(0,L)$ and $\omega=(a,L)$, with $a\in(0,L)$. Assume that $(u_0,v_0,w_0)\in[L^2(0,L)]^3$. Then, for every $T>0$ there exists $\delta>0$ such that if $\|(u_0,v_0,w_0)\|_{[L^2(0,L)]^3}<\delta$, there are controls $p\in L^2(0,T;L^2(\gamma))$ and $q\in L^2(0,T;L^2(\omega))$ such that the solutions $u,v,w\in C([0,T];L^2(0,L))\cap L^2(0,T;H^1(0,L))$ of (GHSS) satisfies

$$u(T,x)=v(T,x)=w(T,x)=0 \quad \textit{in } (0,L).$$

General strategy

The strategy consists in 3 main steps:

1. Linearization around the origin:

$$(L)\begin{cases} u_t - \frac{1}{4}u_{xxx} = f_1 + 3w_x, & (t,x)\in(0,T)\times(0,L),\\ v_t + \frac{1}{2}v_{xxx} = f_2 + p\mathbb{1}_\gamma, & (t,x)\in(0,T)\times(0,L),\\ w_t + \frac{1}{2}w_{xxx} = f_3 + q\mathbb{1}_\omega, & (t,x)\in(0,T)\times(0,L),\\ +\text{b.c.} + \text{i.c.} \end{cases}$$

   where $f_1 = 3uu_x - 6vv_x$, $f_2 = -3uv_x$ and $f_3 = -3uw_x$.

2. (Main step) A null control result for (L) for any given f_1, f_2 and f_3 (and i.c.).
3. Local inversion argument to recover the (local) null controllability result for the nonlinear system.

   We consider $\mathcal{F} : \mathcal{B}_1 \to \mathcal{B}_2$, where $\mathcal{B}_1, \mathcal{B}_2$ are suitable weighted (Banach) spaces.

$$\begin{aligned}\mathcal{F}(u,v,w,p,q) :=\Big(&u_t - \tfrac{1}{4}u_{xxx} - 3uu_x + 6vv_x - 3w_x, u(0,\cdot),\\ &v_t + \tfrac{1}{2}v_{xxx} + 3uv_x - p\mathbb{1}_\gamma, v(0,\cdot),\\ &w_t + \tfrac{1}{2}w_{xxx} + 3uw_x - q\mathbb{1}_\omega, w(0,\cdot)\Big).\end{aligned}$$

   We prove that $\mathcal{F}$ is of class C^1 and the surjectivity of

$$\mathcal{F}'(0) = \left(u_t - \tfrac{1}{4}u_{xxx} - 3w_x, u(0,\cdot), v_t + \tfrac{1}{2}v_{xxx} - p\mathbb{1}_\gamma, v(0,\cdot), w_t + \tfrac{1}{2}w_{xxx} - q\mathbb{1}_\omega, w(0,\cdot)\right)$$

   which is given by the null controllability of (L).

Null controllability of the linearized system (Step 2)

Some remarks on:

$$
(L)\begin{cases}
u_t - \frac{1}{4}u_{xxx} = f_1 + 3w_x, & (t,x)\in(0,T)\times(0,L),\\
v_t + \frac{1}{2}v_{xxx} = f_2 + p\mathbb{1}_\gamma, & (t,x)\in(0,T)\times(0,L),\\
w_t + \frac{1}{2}w_{xxx} = f_3 + q\mathbb{1}_\omega, & (t,x)\in(0,T)\times(0,L),\\
+\text{b.c.} + \text{i.c.}
\end{cases}
$$

- The control p cannot be removed with this approach.

- By duality, the null controllability of (L) comes from an observability inequality for the adjoint system

$$
\begin{cases}
-\phi_t + \frac{1}{4}\phi_{xxx} = g_1, & (t,x)\in(0,T)\times(0,L),\\
-\psi_t - \frac{1}{2}\psi_{xxx} = g_2, & (t,x)\in(0,T)\times(0,L),\\
-\eta_t - \frac{1}{2}\eta_{xxx} = g_3 - 3\phi_x, & (t,x)\in(0,T)\times(0,L),\\
+\text{b.c.} + \text{i.c.}
\end{cases}
$$

More precisely, we prove

$$
\int_0^T\int_0^L \rho_1\left(|\phi|^2+|\psi|^2+|\eta|^2\right)\mathrm{d}x\,\mathrm{d}t \le C\int_0^T\int_\gamma \rho_2|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + C\int_0^T\int_\omega \rho_3|\eta|^2\,\mathrm{d}x\,\mathrm{d}t
$$
$$
+ C\int_0^T \rho_4\left(\|g_1\|^2_{H_0^1} + \|g_2\|^2_{H_0^1} + \|g_3\|^2_{H_0^1}\right)\mathrm{d}t
$$

where ρ_i are weight functions coming from Carleman estimates.

Sketch of the proof of the observability inequality:

$$\begin{cases} -\phi_t + \frac{1}{4}\phi_{xxx} = g_1, & (t,x) \in (0,T) \times (0,L), \\ -\psi_t - \frac{1}{2}\psi_{xxx} = g_2, & (t,x) \in (0,T) \times (0,L), \\ -\eta_t - \frac{1}{2}\eta_{xxx} = g_3 - 3\phi_x, & (t,x) \in (0,T) \times (0,L), \\ +\text{b.c.} + \text{i.c.} \end{cases}$$

• Carleman estimate for the KdV equation [Capistrano-Filho et al., 2015], [Araruna et al., 2016]:

$$\int_0^T\int_0^L \rho\left(|\varphi|^2+|\varphi_x|^2+|\varphi_{xx}|^2\right)dxdt \leq C\left(\int_0^T\int_0^L \rho|\varphi_t+\nu\varphi_{xxx}|^2dxdt+\int_0^T\int_{\omega_0} \rho\left(|\varphi|^2+|\varphi_{xx}|^2\right)dxdt\right),$$

where $\nu \neq 0$ and $\varphi = \phi, \psi, \eta$.

• Local term of φ_{xx} is estimated using regularity of the solutions.

• We obtain local terms like:

$$C\int_0^T\int_\gamma \rho|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + C\int_0^T\int_\omega \rho|\eta|^2\,\mathrm{d}x\,\mathrm{d}t + C\int_0^T\int_\omega \rho|\phi|^2\,\mathrm{d}x\,\mathrm{d}t$$

• Taking $\omega = (a, L)$ and since $\phi(t, L) = 0$:

$$\int_0^T\int_\omega \rho|\phi|^2\,\mathrm{d}x\,\mathrm{d}t \leq C\int_0^T\int_\omega \rho|\phi_x|^2\,\mathrm{d}x\,\mathrm{d}t \quad \text{(Poincaré's inequality)}$$

and we can use the coupling of the third equation: $3\phi_x = g_3 + \eta_t + \frac{1}{2}\eta_{xxx}$.

Final comments

$$
(GHSS)\begin{cases} u_t - \frac{1}{4}u_{xxx} = 3uu_x - 6vv_x + 3w_x, & (t,x)\in(0,T)\times(0,L),\\ v_t + \frac{1}{2}v_{xxx} = -3uv_x + p\mathbb{1}_\gamma, & (t,x)\in(0,T)\times(0,L),\\ w_t + \frac{1}{2}w_{xxx} = -3uw_x + q\mathbb{1}_\omega, & (t,x)\in(0,T)\times(0,L),\end{cases}
$$

- Due to our strategy, it is not possible to control $(GHSS)$ without the control p, since in the linearized system v is independent of u and w.

- A possible approach to controllability with only one control is the use of nonlinear arguments as Coron's Return Method.

- The assumption on ω touching the boundary of $(0,L)$ is only technical. In fact, an improved Carleman estimate proved in [Bárcena-Petisto, Guerrero, Pazoto, 2021] can be adapted to this case and consider an arbitrary $\omega \subset (0,L)$.

- For the Hirota-Satsuma system

$$
\begin{cases} u_t - \frac{1}{4}u_{xxx} = 3uu_x - 6vv_x + p\mathbb{1}_\omega, & (t,x)\in(0,T)\times(0,L),\\ v_t + \frac{1}{2}v_{xxx} = -3uv_x, & (t,x)\in(0,T)\times(0,L),\end{cases}
$$

a local exact controllability to a stationary solution has been proven by [Cerpa, Cifuentes, 21]:

$$
u(T,x) = 2R(x), v(T,x) = R(x), \quad \text{in } (0,L).
$$

Note: The linearized system around $(2R,R)$ is coupled through $R_x u$.

Work in progress
Joint work with Emmanuelle Crépeau (LJK)

Insensitizing control problem for the Korteweg-de Vries (KdV) equation:

Consider the equation

$$\begin{cases} y_t + y_{xxx} + y_x + yy_x = f + u\mathbb{1}_\omega, & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = y(t,L) = 0, \ y_x(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x) + \tau \hat{y}_0(x), & x \in (0,L), \end{cases}$$

where $f = (t,x)$ is an external force, $u = u(t,x)$ is the distributed control acting on $\omega \subset (0,L)$, and $\tau \hat{y}_0(x)$ is an unknown perturbation of the initial data.

Let $\mathcal{O} \subset (0,L)$. We define the observation functional:

$$J(y) = \frac{1}{2} \int_0^T \int_{\mathcal{O}} |y|^2 \, dx dt.$$

We look for a control $u \in L^2(0,T; L^2(\omega))$ such that

$$\left. \frac{\partial J(y)}{\partial \tau} \right|_{\tau=0} = 0, \quad \forall \hat{y}_0 \in L^2(Q), \|\hat{y}_0\|_{L^2(0,L)} = 1. \tag{1}$$

Work in progress

The existence of a control u such that condition (1) holds is equivalent to the following (partial) control problem (linear case): Find a control u such that $z(0,x)=0$, where

$$\begin{cases} w_t + w_{xxx} + w_x = u\mathbb{1}_\omega, & (t,x)\in(0,T)\times(0,L), \\ -z_t - z_{xxx} - z_x = w\,\mathbb{1}_{\mathcal{O}}, & (t,x)\in(0,T)\times(0,L), \\ +\text{b.c.} \\ w(0,x)=y_0(x),\ z(T,x)=0, & x\in(0,L), \end{cases}$$

Adjoint system:

$$\begin{cases} -\varphi_t - \varphi_{xxx} - \varphi_x = \psi\,\mathbb{1}_{\mathcal{O}}, & (t,x)\in(0,T)\times(0,L), \\ \psi_t + \psi_{xxx} + \psi_x = 0, & (t,x)\in(0,T)\times(0,L), \\ +\text{b.c.} \\ \varphi(T,x)=0,\ \psi(0,x)=\psi_0, & x\in(0,L), \end{cases}$$

We (need to) prove an observability inequality of the type

$$\iint_Q e^{-C_1/t}|\varphi|^2\,dxdt \le C\int_0^T\!\!\int_\omega e^{-C_2/t}|\varphi|^2\,dxdt.$$

But: we do not have enough regularity to estimate the local term of φ_{xx}.

Work in progress

We propose the observation functional:

$$J_1(y) = \frac{1}{2}\int_0^T\int_{\mathcal{O}} |y|^2 \theta \, dx dt,$$

where $\theta \in C_0^\infty(\mathcal{O};[0,1])$ such that $\theta(x) = 1$ for all $x \in \mathcal{O}_0$, for some $\mathcal{O}_0 \subset \mathcal{O}$.

Adjoint system:

$$\begin{cases} -\varphi_t - \varphi_{xxx} - \varphi_x = \psi\,\theta, & (t,x) \in (0,T)\times(0,L), \\ \psi_t + \psi_{xxx} + \psi_x = 0, & (t,x) \in (0,T)\times(0,L), \\ +\text{b.c.} & \\ \varphi(T,x) = 0,\ \psi(0,x) = \psi_0, & x \in (0,L), \end{cases}$$

Work in progress

Also, we consider the case:

$$J(y) = \frac{1}{2}\int_0^T\int_{\mathcal{O}} |y_x|^2 \theta \, dxdt,$$

where $\theta \in C_0^\infty(\mathcal{O};[0,1])$ such that $\theta(x) = 1$ for all $x \in \mathcal{O}_0$, for some $\mathcal{O}_0 \subset \mathcal{O}$.

Adjoint system:

$$\begin{cases} -\varphi_t - \varphi_{xxx} - \varphi_x = -(\psi_x\,\theta)_x, & (t,x) \in (0,T)\times(0,L), \\ \psi_t + \psi_{xxx} + \psi_x = 0, & (t,x) \in (0,T)\times(0,L), \\ +\text{b.c.} & \\ \varphi(T,x) = 0,\ \psi(0,x) = \psi_0, & x \in (0,L), \end{cases}$$

Same as before, we do not have enough regularity to estimate the local term of φ_{xx}. Hence, we propose the functional:

$$J_2(y) = \frac{1}{2}\int_0^T\int_{\mathcal{O}} e^{-c/t(T-t)}|y|^2\theta \, dxdt,$$

that would help to get more regularity for $(\psi_x\,\theta)_x$.

Thank you