Some controllability results for the Navier-Stokes and Boussinesq systems with a reduced number of scalar controls

Seminario de Problemas Inversos y Control
CMM - Universidad de Chile

Nicolás Carreño
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie
Paris, France
(Thesis advisor: Sergio Guerrero)

September 6th, 2012

Outline

Framework:

- Ω bounded connected regular open subset of $\mathbf{R}^N$ ($N = 2$ or 3)
- $T > 0$
- $\omega \subset \Omega$ (control set), $Q := \Omega \times (0, T)$, $\Sigma := \partial\Omega \times (0, T)$

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = v \mathbb{1}_\omega, \ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \qquad \text{(NS)}$$

where v stands for the control which acts over the set ω.

Controllability problem: Can we drive the solution of (NS) to a given state at time T by means of a control $v \in L^2(\omega \times (0, T))^N$?

Because of regularization, we cannot expect exact controllability.

Exact controllability to trajectories

Consider the uncontrolled solution to the same equation:

$$\begin{cases} \bar{y}_t - \Delta \bar{y} + (\bar{y} \cdot \nabla)\bar{y} + \nabla \bar{p} = 0, \ \nabla \cdot \bar{y} = 0 & \text{in } Q, \\ \bar{y} = 0 & \text{on } \Sigma, \\ \bar{y}(0) = \bar{y}^0 & \text{in } \Omega. \end{cases}$$

Exact controllability to trajectories: Given an initial condition y^0, can we find v such that

$$\boxed{y(T) = \bar{y}(T)}\,?$$

Local exact controllability to trajectories: If $\|y^0 - \bar{y}^0\|$ is small enough, can we find v such that

$$\boxed{y(T) = \bar{y}(T)}\,?$$

Remark: After time T, we can "turn off" the control and follow the ideal trajectory.

Some results

Under regularity assumptions on $\bar{y}$

- [Fursikov, Imanuvilov 1998, 1999]

  Improvements in:
- [Fernández-Cara, Guerrero, Imanuvilov, Puel, 2004]
- [Imanuvilov, Puel, Yamamoto, 2011]

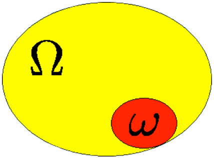
Reduced number of controls

Question: Can we find a control $v \in L^2(\omega \times (0,T))^N$ with a vanishing component, for example,

$$\boxed{v = (v_1, 0)} \text{ or } \boxed{v = (v_1, v_2, 0)} \,?$$

Some results:

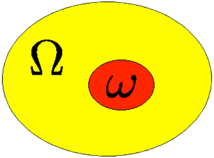
- [Fernández-Cara, Guerrero, Imanuvilov, Puel, 2006]: Local exact controllability to the trajectories when $\overline{\omega} \cap \partial\Omega \neq \emptyset$.



- Vanishing component depends on this geometric assumption.

Reduced number of controls

We are interested in removing this geometric property.



- [Coron, Guerrero, 2009]: Null controllability of the Stokes system

$$\begin{cases} y_t - \Delta y + \nabla p = (v_1, v_2, 0)\mathbb{1}_\omega, \ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases}$$

that is, $\boxed{y(T)=0}$.

Carleman estimates and controllability

Consider the Stokes system and its adjoint:

$$\begin{cases} y_t - \Delta y + \nabla p = (v_1, 0)\mathbb{1}_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \qquad \begin{cases} -\varphi_t - \Delta \varphi + \nabla \pi = 0, & \text{in } Q, \\ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T & \text{in } \Omega. \end{cases}$$

Null controllability is equivalent to the Observability inequality

$$\int_\Omega |\varphi(0)|^2 \, dx \leq C \iint_{\omega\times(0,T)} |\varphi_1|^2 dx\, dt, \quad \varphi = (\varphi_1, \varphi_2).$$

Important tool: Carleman estimates

$$\iint_Q \rho_1(x,t)|\varphi|^2 dx\, dt \leq C \iint_{\omega\times(0,T)} \rho_2(x,t)|\varphi_1|^2 dx\, dt$$

ρ_1, ρ_2 some positive weight functions, C independent of φ.

How do they prove this inequality?

Method introduced in [Coron, Guerrero, 2009]:

- $\nabla \cdot \varphi = 0 \Rightarrow \Delta\pi = 0$,
- Look at the equation satisfied by $\nabla\Delta\varphi_1$ and apply Carleman estimates* (doing this eliminates the pressure),
- Use $\nabla \cdot \varphi = 0$ to recover φ_2 in the LHS.

***Remark:** When applying the operator $\nabla\Delta$, we lose the boundary conditions. Special Carleman estimates are needed:

- [Fernández-Cara, González-Burgos, Guerrero, Puel, 2006]
  RHS in L^2
- [Imanuvilov, Puel, Yamamoto, 2009]
  RHS in H^{-1}

Navier-Stokes system

We deal with the local null controllability of

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = (v_1,0)\mathbb{1}_\omega, \ \nabla\cdot y = 0 & \text{in } Q,\\ y = 0 & \text{on } \Sigma,\\ y(0) = y^0 & \text{in } \Omega,\end{cases}$$

with no assumption on the control domain $\emptyset \neq \omega \subset \Omega$.

Theorem (Guerrero, C., 2011)

For every $T > 0$ and $\omega \subset \Omega$, the NS system is locally null controllable by a control $v \in L^2(\omega\times(0,T))^2$ of the form $v = (v_1, 0)$.

- We can also choose $v = (0, v_2)$.
- For $N = 3$, $v = (v_1, v_2, 0)$, $v = (v_1, 0, v_3)$ or $v = (0, v_2, v_3)$.

Method of proof

- Linearization around zero.
- Null controllability of the linearized system.
  Main tool: Carleman estimate for the adjoint system.
- Inverse mapping theorem to obtain the result for the nonlinear system.

Linear system

We deal with the null controllability of the linearized system around 0:

$$\begin{cases} y_t - \Delta y + \nabla p = f + (v_1, 0)\mathbb{1}_\omega, \; \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \qquad \text{(L)}$$

where f is taken to decrease exponentially to zero in $t = T$.

We need a suitable observability inequality for the adjoint system.

Adjoint system

Consider the nonhomogeneous adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = g,\ \nabla\cdot\varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T & \text{in } \Omega, \end{cases}$$

where $g \in L^2(Q)^2$ and $\varphi^T \in L^2(\Omega)^2$.

We want to show a Carleman estimate of the type:

$$\iint_Q \rho_1(t)|\varphi|^2 \leq C\left(\iint_Q \rho_2(t)|g|^2 + \iint_{\omega\times(0,T)} \rho_3(t)|\varphi_1|^2\right)$$

for every $\varphi = (\varphi_1, \varphi_2)$ solution of the adjoint system.

Weight functions

Let ω_0 be a nonempty open set such that $\overline{\omega_0} \subset \omega$ and $\lambda > 1$

$$\alpha(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\ell^8(t)} > 0, \; \xi(x,t) = \frac{e^{\lambda\eta(x)}}{\ell^8(t)} > 0,$$

where $\eta \in C^2(\overline{\Omega})$ and $\ell \in C^\infty([0,T])$ are s.t.

$$|\nabla \eta| > 0 \text{ in } \overline{\Omega} \setminus \omega_0, \; \eta > 0 \text{ in } \Omega \text{ and } \eta \equiv 0 \text{ on } \partial\Omega,$$

$$\ell(t) = t \quad \forall t \in [0, T/4], \; \ell(t) = T - t \quad \forall t \in [3T/4, T].$$

Existence of η : [Fursikov, Imanuvilov, 1996].

Carleman estimate for the adjoint system

Proposition: Carleman inequality

There exists a constant $C > 0$ (depending on Ω, ω, T and λ)

$$s^4 \iint_Q e^{-5s\alpha^*} (\xi^*)^4 |\varphi|^2 \leq C \left(\iint_Q e^{-3s\alpha^*} |g|^2 + s^7 \iint_{\omega \times (0,T)} e^{-2s\alpha - 3s\alpha^*} \xi^7 |\varphi_1|^2 \right)$$

for every $s \geq C$ and every $\varphi = (\varphi_1, \varphi_2)$ solution of the adjoint system.

What is different with the Stokes case?

- $g \neq 0$.
- $\Delta\pi \neq 0$.
- We consider de Stokes systems:

$$\begin{cases} -w_t - \Delta w + \nabla \pi_w = \rho(t) g, \ \nabla \cdot w = 0 \text{ in } Q, \\ w = 0 \text{ on } \Sigma, \ w(T) = 0 \text{ in } \Omega, \end{cases}$$

$$\begin{cases} -z_t - \Delta z + \nabla \pi_z = -\rho'(t) \varphi, \ \nabla \cdot z = 0 \text{ in } Q, \\ z = 0 \text{ on } \Sigma, \ z(T) = 0 \text{ in } \Omega, \end{cases}$$

where $\boxed{\rho(t) = e^{-3/2s\alpha^*}}$ and $\boxed{\rho(t)\,\varphi = w + z}$.

- Now $\Delta\pi_z = 0$ and we can apply the previous method to z.
- Regularity estimates for w.

- $-(\nabla\Delta z_1)_t - \Delta(\nabla\Delta z_1) = -\rho'(t)\nabla\Delta\varphi_1$. No boundary conditions.
- We apply a Carleman inequality with nonhomogeneous boundary conditions [Imanuvilov, Puel, Yamamoto, 2009].
- Parabolic and elliptic Carleman estimates to obtain the local term in z_1.
- Regularity estimates for Stokes to eliminate the boundary terms.

Null controllability of the linear system

We need weights that do not vanish at $t = 0$.
Let

$$\tilde{\ell}(t) = \begin{cases} \|\ell\|_\infty & 0 \le t \le T/2, \\ \ell(t) & T/2 < t \le T. \end{cases}$$

We define β and γ as α and ξ.

$$\|\varphi(0)\|^2_{L^2(\Omega)^2} + \iint_Q e^{-5s\beta^*} (\gamma^*)^4 |\varphi|^2 \le C \left(\iint_Q e^{-3s\beta^*} |g|^2 + \iint_{\omega\times(0,T)} e^{-2s\widehat{\beta}-3s\beta^*} \widehat{\gamma}^7 |\varphi_1|^2 \right).$$

This is proved using classical energy estimates for Stokes and the previous Carleman inequality.

Recall the linear system:

$$\begin{cases} y_t - \Delta y + \nabla p = f + (v_1, 0)\mathbb{1}_\omega, \ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega. \end{cases}$$

If

$$\iint_Q e^{5s\beta^*}(\gamma^*)^{-4}|f|^2 < +\infty,$$

then we can prove that there exists a control v_1 such that $\boxed{y(T)=0}$. Furthermore,

$$\boxed{\iint_Q e^{3s\beta^*}|y|^2 + \iint_Q e^{2s\widehat{\beta}+3s\beta^*}\widehat{\gamma}^{-7}|v_1|^2\mathbb{1}_\omega < +\infty,}$$

which gives that y goes to zero at T exponentially (so does the control).

Controllability of the NS system

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = (v_1, 0)\mathbb{1}_\omega,\ \nabla\cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega. \end{cases}$$

We consider the operator:

$$\mathcal{A}(y,p,v_1) = (\, y_t - \Delta y + (y\cdot\nabla)y + \nabla p - (v_1,0)\mathbb{1}_\omega,\ y(0)\,)$$

Of class C^1 between special espaces (where in particular $\boxed{y(T)=0}$).

$$\mathcal{A}'(0,0,0)(y,p,v_1) = (y_t - \Delta y + \nabla p - (v_1,0)\mathbb{1}_\omega, y(0)\,)$$

is surjective by the null controllability of the linear system.
Inverse mapping theorem around (0,0,0) gives the result for NS, i.e., there exists $\delta > 0$ such that if $\|y^0\| < \delta$, then there exists (y,p,v_1) such that

$$\boxed{\mathcal{A}(y,p,v_1) = (0,y^0)}.$$

Extension: Boussinesq system

Now we consider the Boussinesq system:

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = v 1_\omega + \theta\, e_3,\ \nabla \cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta \theta + y \cdot \nabla \theta = v_0 1_\omega & \text{in } Q, \\ y = 0,\ \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0,\ \theta(0) = \theta^0 & \text{in } \Omega. \end{cases}$$

Goal: To find a control $v \in L^2(\omega \times (0,T))^3$ of the form $v = (v_1, 0, 0)$, and $v_0 \in L^2(\omega \times (0,T))$ such that

$$\boxed{y(T) = 0 \text{ and } \theta(T) = \bar{\theta}(T)}$$

where

$$\begin{cases} \nabla \bar{p} = \bar{\theta}\, e_3 & \text{in } Q, \\ \bar{\theta}_t - \Delta \bar{\theta} = 0 & \text{in } Q, \\ \bar{\theta} = 0 \text{ on } \Sigma, \bar{\theta}(0) = \bar{\theta}^0 & \text{in } \Omega, \end{cases}$$

i.e., local controllability to the trajectory $(0, \bar{p}, \bar{\theta})$.

Theorem (C., 2011)

For every $T > 0$ and $\omega \subset \Omega$, the Boussinesq system is locally controllable to the trajectory $(0, \bar{p}, \bar{\theta})$ by controls $v_0 \in L^2(\omega \times (0,T))$ and $v \in L^2(\omega \times (0,T))^3$ of the form $v = (v_1, 0, 0)$.

- We can also choose $v = (0, v_2, 0)$.
- For $N = 2$, $v \equiv 0$: No control is needed in the fluid equation.

Linearized system around $(0, \bar{p}, \bar{\theta})$:

$$\begin{cases} y_t - \Delta y + \nabla p = f + (v_1, 0, 0)\mathbb{1}_\omega + \theta\, e_3, \nabla \cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta\theta + y \cdot \nabla\bar{\theta} = f_0 + v_0\mathbb{1}_\omega & \text{in } Q, \\ y = 0,\ \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0,\ \theta(0) = \theta^0 & \text{in } \Omega, \end{cases}$$

where f and f_0 will be taken to decrease exponentially to zero in T.

The (nonhomogeneous) adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = g - \psi\nabla\bar{\theta},\ \nabla \cdot \varphi = 0 & \text{in } Q, \\ -\psi_t - \Delta\psi = g_0 + \varphi_3 & \text{in } Q, \\ \varphi = 0,\ \psi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T,\ \psi(T) = \psi^T & \text{in } \Omega, \end{cases}$$

where $g \in L^2(Q)^3$, $g_0 \in L^2(Q)$, $\varphi^T \in L^2(\Omega)^3$ and $\psi^T \in L^2(\Omega)$.

We prove a Carleman estimate of the type

$$\iint_Q \rho_1(t)(|\varphi|^2 + |\psi|^2) \leq C \left(\iint_Q \rho_2(t)(|g|^2 + |g_0|^2) + \iint_{\omega\times(0,T)} \rho_3(t)(|\varphi_1|^2 + |\psi|^2) \right)$$

for every $(\varphi, \psi) = (\varphi_1, \varphi_2, \varphi_3, \psi)$ solution of the adjoint system.

How do we prove it?

- With the previous method, we obtain local terms of φ_1 and φ_3.
- We eliminate φ_3 using the equation.

$$\varphi_3 = -\psi_t - \Delta\psi - g_0.$$

- For ψ, we use the classical Carleman for the heat equation.

Insensitizing controls for Navier-Stokes system

We consider the problem of insensitizing controls for the NS system:

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = f + v 1\!\!1_{\omega},\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 + \tau \widehat{y}^0 & \text{in } \Omega, \end{cases} \qquad \text{(S)}$$

where τ is a small constant and $\|\widehat{y}^0\|_{L^2(\Omega)^N} = 1$. Both are unknown.
Insensitizing control problem: To find a control $v \in L^2(\omega \times (0,T))^N$ such that the functional (Sentinel)

$$J(y) = \iint_{\mathcal{O} \times (0,T)} |y|^2 \, dx \, dt, \ \mathcal{O} \subset \Omega \ \text{(Observation set)}$$

is not affected by the uncertainty of the initial data, that is,

$$\left. \frac{\partial J(y)}{\partial \tau} \right|_{\tau=0} = 0, \ \forall \widehat{y}^0 \in L^2(\Omega)^N \text{ s.t. } \|\widehat{y}^0\|_{L^2(\Omega)^N} = 1.$$

Some previous works

- Heat equation: [Bodart, Fabre, 1995], [de Teresa, 2000]
- Gradient as Sentinel: [Guerrero, 2007]
- Stokes: [Guerrero, 2007]
- Navier-Stokes: [Gueye, 2010]

We are interested in controls with one vanishing component.

A cascade Navier-Stokes system

The previous condition is equivalent to the following null controllability problem: To find a control $v = (v_1, 0)$ such that $z(0) = 0$, where

$$\begin{cases} w_t - \Delta w + (w \cdot \nabla)w + \nabla p^0 = f + (v_1, 0)\mathbb{1}_\omega, \ \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + (z \cdot \nabla^t w) - (w \cdot \nabla)z + \nabla q = w\mathbb{1}_{\mathcal{O}}, \ \nabla \cdot z = 0 & \text{in } Q, \\ w = z = 0 & \text{on } \Sigma, \\ w(0) = y^0, \ z(T) = 0 & \text{in } \Omega. \end{cases}$$

Theorem (Gueye, C., 2012)

Assume $y^0 = 0$ and $\mathcal{O} \cap \omega \neq \emptyset$. There exists $\delta > 0$ such that if $\|e^{K/t^{10}} f\|_{L^2(Q)^2} < \delta$, there exists a control $v_1 \in L^2(\omega \times (0, T))$ such that $z(0) = 0$.

- We can also choose $v = (0, v_2)$.
- For $N = 3$: $v = (v_1, v_2, 0)$, $v = (v_1, 0, v_3)$ or $v = (0, v_2, v_3)$.

Same strategy as before

Null controllability of the linearized system around 0:

$$\begin{cases} w_t - \Delta w + \nabla p^0 = f^0 + (v_1, 0)\mathbb{1}_\omega,\ \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + \nabla q = f^1 + w\mathbb{1}_{\mathcal{O}},\ \nabla \cdot z = 0 & \text{in } Q, \\ w = z = 0 & \text{on } \Sigma, \\ w(0) = 0,\ z(T) = 0 & \text{in } \Omega, \end{cases}$$

where f^0 and f^1 decrease exponentially to zero at $t = 0$.

- The control acts on z through w.

As before, we want to show an estimate of the form

$$\iint_Q \rho_1(t)(|\varphi|^2 + |\psi|^2) \leq C \left(\iint_Q \rho_2(t)(|g^0|^2 + |g^1|^2 + |\nabla g^1|^2) + \iint_{\omega\times(0,T)} \rho_3(t)|\varphi_1|^2 \right)$$

where $(\varphi, \psi) = (\varphi_1, \varphi_2, \psi)$ is the solution of the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = g^0 + \psi 1\!\!1_{\mathcal{O}}, & \text{in } Q, \\ \psi_t - \Delta\psi + \nabla\kappa = g^1, & \text{in } Q, \\ \nabla\cdot\varphi = \nabla\cdot\psi = 0, \text{ in } Q, \ \varphi = \psi = 0 \text{ on } \Sigma & \\ \varphi(T) = 0, \ \psi(0) = \psi^0 & \text{in } \Omega. \end{cases}$$

- $g^1 \in L^2(0, T; H_0^1(\Omega)^2)$ with $\nabla \cdot g^1 = 0$.

Idea of proof

- **Main idea:** To combine Carleman inequalities for ψ and φ, and estimate the local term in ψ by local term φ_1.
- For φ, we use the Carleman for NS.
- Because of the pressure term, we need a Carleman for ψ with local term in $\Delta\psi_1$.
- Need to apply the operator $\nabla\nabla\Delta$ to the equation satisfied by ψ_1. More regularity needed for ψ (and g^1).
- Use the equation

$$\Delta\psi_1 = -\Delta\varphi_{1,t} - \Delta^2\varphi_1 + \partial_1\nabla\cdot g^0 - \Delta g_1^0$$

to eliminate the local term $\Delta\psi_1$.

Final comments

- What about controllability to trajectories?

$$-\varphi_t - \Delta\varphi + \bar{y} \cdot D\varphi + \nabla\pi = g.$$

  Terms in φ_2 that we do not know how to estimate.
- What about two vanishing components, e.g., $v = (v_1, 0, 0)$?.
  [P. Lissy, 2012]: Return method.
- Other boundary conditions: Navier-slip.
- Insensitizing controls for Boussinesq system.
- Inverse problems? Observations in one less direction?

Thank you for your attention