Some controllability results for the Navier-Stokes equation with a reduced number of scalar controls

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Outline

Navier-Stokes in a bounded domain

- Ω bounded connected regular open subset of $\mathbb{R}^N$ ($N = 2$ or 3)
- $T > 0$
- $\omega \subset \Omega$ (control set), $Q := \Omega \times (0, T)$, $\Sigma := \partial\Omega \times (0, T)$

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = v\mathbb{1}_\omega, \quad \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases}$$

where $v = (v_1, v_2, v_3)$ stands for the control which acts over the set ω.

Controllability problem: Can we drive the solution of the NS system to a given state at time T by means of a control $v \in L^2(\omega \times (0, T))^N$?

Because of regularization, we cannot expect exact controllability.

Exact controllability to trajectories

Consider the uncontrolled solution to the same equation:

$$\begin{cases} \bar{y}_t - \Delta \bar{y} + (\bar{y} \cdot \nabla)\bar{y} + \nabla \bar{p} = 0, \quad \nabla \cdot \bar{y} = 0 & \text{in } Q, \\ \bar{y} = 0 & \text{on } \Sigma, \\ \bar{y}(0) = \bar{y}^0 & \text{in } \Omega. \end{cases}$$

Exact controllability to trajectories: Given an initial condition y^0, can we find v such that

$$\boxed{y(T) = \bar{y}(T)}?$$

Local exact controllability to trajectories: If $\|y^0 - \bar{y}^0\|$ is small enough, can we find v such that

$$\boxed{y(T) = \bar{y}(T)}?$$

Remark: After time T, we can "turn off" the control and follow the ideal trajectory.

Some results

Under regularity assumptions on $\bar{y}$

- [Imanuvilov 2001]

  Improvements in:

- [Fernández-Cara, Guerrero, Imanuvilov, Puel, 2004]
- [Imanuvilov, Puel, Yamamoto, 2012]: $\bar{y} \in L^{\infty}(Q)$

Method of proof

- Linearization around $\bar{y}$.
- Null controllability of the linearized problem.
- Fixed point or inverse mapping theorem for the nonlinear problem.

Reduced number of controls

Now we consider:

$$y_t - \Delta y + (y \cdot \nabla) y + \nabla p = (v_1, v_2, 0) \mathbb{1}_\omega, \quad \nabla \cdot y = 0, \quad y_{|\Sigma} = 0$$

Some results:

- First results by Fernández-Cara, Guerrero, Imanuvilov, Puel (2006): Local exact controllability to the trajectories when $\overline{\omega} \cap \partial\Omega \neq \emptyset$.
- Null component depends on this geometric assumption.
- We are interested in removing this geometric property.
- Coron, Guerrero (2009): Null controllability of the Stokes system for a general $\omega \subset \Omega$

$$y_t - \Delta y + \nabla p = (v_1, v_2, 0) \mathbb{1}_\omega, \quad \nabla \cdot y = 0, \quad y_{|\Sigma} = 0$$

Controllability, observability and Carleman estimates

We consider the adjoint system

$$-\varphi_t - \Delta\varphi + \nabla\pi = 0, \quad \nabla\cdot\varphi = 0, \quad \varphi_{|\Sigma} = 0$$

Null controllability is equivalent to the Observability inequality

$$\int_\Omega |\varphi(0)|^2\,dx \le C \iint_{\omega\times(0,T)} (|\varphi_1|^2 + |\varphi_2|^2)dx\,dt, \quad \varphi = (\varphi_1,\varphi_2,\varphi_3).$$

Important tool: Carleman estimates

$$\iint_Q \rho_1(x,t)|\varphi|^2 dx\,dt \le C \iint_{\omega\times(0,T)} \rho_2(x,t)(|\varphi_1|^2 + |\varphi_2|^2)dx\,dt$$

ρ_1, ρ_2 some positive weight functions, C independent of φ.

How to prove this type of Carleman estimate?

If $\overline{\omega} \cap \partial\Omega \neq \emptyset$:

- From an estimate of the type

$$\iint_Q \rho_1(x,t)|\varphi|^2 dx\,dt \leq C \iint_{\omega\times(0,T)} \rho_2(x,t)(|\varphi_1|^2+|\varphi_2|^2+|\varphi_3|^2) dx\,dt$$

  we can use $\varphi|_\Sigma = 0$ and $\nabla \cdot \varphi = 0$ to estimate φ_3 by φ_1 and φ_2.
- This estimate was proved in [Imanuvilov, 2001], [Fernández-Cara, Guerrero, Imanuvilov, Puel, 2004] for a general $\omega \subset \Omega$.
- The main difficulty is estimating de pressure.

General case $\omega \subset \Omega$: Method introduced in [Coron, Guerrero, 2009]:

- $\nabla \cdot \varphi = 0 \Rightarrow \Delta \pi = 0$,
- Look at the (heat) equations satisfied by $\nabla \Delta \varphi_1$ and $\nabla \Delta \varphi_2$ and apply suitable Carleman estimates (doing this eliminates the pressure),
- Use $\varphi_{|\Sigma} = 0$ and $\nabla \cdot \varphi = 0$ to recover φ_3 in the LHS,
- Regularity estimates for the Stokes system to estimate the boundary terms.

Two key points:

- Avoid estimating the pressure,
- Not make appear local terms of φ_3.

Navier-Stokes system

We deal with the local null controllability of

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = (v_1, v_2, 0)\mathbb{1}_\omega, \quad \nabla\cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases}$$

with no assumption on the control domain $\emptyset \neq \omega \subset \Omega$.

Theorem (Guerrero, C., 2013, JMFM)

For every $T > 0$ and $\omega \subset \Omega$, the NS system is locally null controllable by controls $v_1, v_2 \in L^2(\omega \times (0, T))$.

- We can also choose $v = (v_1, 0, v_3)$ or $v = (0, v_2, v_3)$.

Method of proof

- Linearization around zero

$$y_t - \Delta y + \nabla p = f + (v_1, v_2, 0)\mathbb{1}_\omega, \quad \nabla \cdot y = 0, \quad y_{|\Sigma} = 0$$

- Null controllability of the linearized system (Main part of the proof).
Main tool: Carleman estimate for the adjoint system

$$-\varphi_t - \Delta\varphi + \nabla\pi = g, \quad \nabla \cdot \varphi = 0, \quad \varphi_{|\Sigma} = 0$$

There exists a constant $C > 0$ (depending on Ω, ω, T)

$$\int_Q \rho_1(t)|\varphi|^2 \leq C\left(\int_Q \rho_2(t)|g|^2 + \int_{\omega\times(0,T)} \rho_3(t)(|\varphi_1|^2 + |\varphi_2|^2)\right)$$

- Inverse mapping theorem for the nonlinear system

$$\mathcal{A} = (\quad y_t - \Delta y + (y\cdot\nabla)y + \nabla p - (v_1, v_2, 0)\mathbb{1}_\omega \quad , \quad y(0) \quad)$$

What is different with the Stokes case?

- $g \neq 0$.
- $\Delta\pi \neq 0$.
- Applying $\nabla\Delta\cdot$ is more difficult now, since g is not regular.
- We overcome this by separating φ in $\varphi^g + \varphi^i$, with φ^g not regular and φ^i very regular.

Once the Carleman estimate is proved, we can construct a solution (y, v_1, v_2) of

$$\begin{cases} y_t - \Delta y + \nabla p = f + (v_1, v_2, 0)\mathbb{1}_\omega, \quad \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases}$$

such that

$$\boxed{\iint_Q \rho_2^{-1}|y|^2 + \iint_Q \rho_3^{-1}(|v_1|^2 + |v_2|^2)\mathbb{1}_\omega < +\infty,}$$

which gives that y goes to zero at T exponentially (so do the controls), provided that

$$\iint_Q \rho_1^{-1}|f|^2 < +\infty.$$

Extension: Boussinesq system

Now we consider the Boussinesq system:

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = (v_1,0,0)\mathbb{1}_\omega + (0,0,\theta), \quad \nabla\cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta\theta + y\cdot\nabla\theta = v_0\mathbb{1}_\omega & \text{in } Q, \\ y=0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad \theta(0) = \theta^0 & \text{in } \Omega. \end{cases}$$

Theorem (C., 2013, MCRF)

For every $T>0$ and $\omega\subset\Omega$, the Boussinesq system is locally controllable to the trajectory $(0,\bar{p},\bar{\theta})$ by controls $v_0, v_1 \in L^2(\omega\times(0,T))$, where

$$\begin{cases} \nabla\bar{p} = \bar{\theta}\, e_3 & \text{in } Q, \\ \bar{\theta}_t - \Delta\bar{\theta} = 0 & \text{in } Q, \\ \bar{\theta} = 0 \text{ on } \Sigma, \bar{\theta}(0) = \bar{\theta}^0 & \text{in } \Omega. \end{cases}$$

- We can also choose $v = (0, v_2, 0)$.
- For $N=2$, $v\equiv 0$: No control is needed in the fluid equation.

Insensitizing controls for Navier-Stokes system

We consider the problem of insensitizing controls for the NS system:

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = f + v\mathbb{1}_\omega, \quad \nabla\cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 + \tau\widehat{y}^0 & \text{in } \Omega, \end{cases}$$

where τ is a small constant and $\|\widehat{y}^0\|_{L^2(\Omega)^3} = 1$. Both are unknown.
Insensitizing control problem: To find a control $v \in L^2(\omega\times(0,T))^3$ such that the functional (Sentinel)

$$J(y) = \iint_{\mathcal{O}\times(0,T)} |y|^2\,dx\,dt,\ \mathcal{O}\subset\Omega \text{ (Observation set)}$$

is not affected by the uncertainty of the initial data, that is,

$$\left.\frac{\partial J(y)}{\partial\tau}\right|_{\tau=0} = 0,\ \forall\widehat{y}^0 \in L^2(\Omega)^3 \text{ s.t. } \|\widehat{y}^0\|_{L^2(\Omega)^3} = 1.$$

Some previous works

- Heat equation: [Bodart, Fabre, 1995], [de Teresa, 2000], [Bodart, González-Burgos, Pérez-García, 2002]
- Quasi-Geostrophic ocean model: [Fernández-Cara, García, Osses, 2005]
- Stokes: [Guerrero, 2007]
- Navier-Stokes: [Gueye, 2013]

We are interested in controls with one vanishing component.

A cascade Navier-Stokes system

The previous condition is equivalent to the following null controllability problem: To find a control $v = (v_1, v_2, 0)$ such that $z(0) = 0$, where

$$\begin{cases} w_t - \Delta w + (w \cdot \nabla) w + \nabla p^0 = f + (v_1, v_2, 0) 1\!\!1_\omega, & \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + (z \cdot \nabla^t w) - (w \cdot \nabla) z + \nabla q = w 1\!\!1_{\mathcal{O}}, & \nabla \cdot z = 0 & \text{in } Q, \\ w = z = 0 & & \text{on } \Sigma, \\ w(0) = y^0, \quad z(T) = 0 & & \text{in } \Omega. \end{cases}$$

Theorem (Gueye, C., 2014, JMPA)

Assume $y^0 = 0$ and $\mathcal{O} \cap \omega \neq \emptyset$. There exists $\delta > 0$ such that if $\|e^{K/t^{10}} f\|_{L^2(Q)^3} < \delta$, there exist a controls $v_1, v_2 \in L^2(\omega \times (0, T))$ such that $z(0) = 0$.

- We can also choose: $v = (v_1, 0, v_3)$ or $v = (0, v_2, v_3)$.

Same strategy as before

Null controllability of the linearized system around 0:

$$\begin{cases} w_t - \Delta w + \nabla p^0 = f^0 + (v_1, v_2, 0)\mathbb{1}_\omega, & \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + \nabla q = f^1 + w\mathbb{1}_{\mathcal{O}}, & \nabla \cdot z = 0 & \text{in } Q, \\ w = z = 0 & & \text{on } \Sigma, \\ w(0) = 0, \quad z(T) = 0 & & \text{in } \Omega, \end{cases}$$

where f^0 and f^1 decrease exponentially to zero at $t = 0$.

- The control acts on z through w.

As before, we need a Carleman estimate for the solutions $(\varphi, \psi) = (\varphi_1, \varphi_2, \varphi_3, \psi_1, \psi_2, \psi_3)$ of the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = g^0 + \psi 1\!\!1_{\mathcal{O}}, & \nabla\cdot\varphi = 0 & \text{in } Q, \\ \psi_t - \Delta\psi + \nabla\kappa = g^1, & \nabla\cdot\psi = 0 & \text{in } Q, \\ \varphi = \psi = 0 & & \text{on } \Sigma \\ \varphi(T) = 0, \quad \psi(0) = \psi^0 & & \text{in } \Omega, \end{cases}$$

with only local terms of φ_1 and φ_2.

Idea of proof

- Carleman for ψ_1 and ψ_2.
- Carleman for φ_1 and φ_2.
- Estimate local terms on ψ_1 and ψ_2 using

$$\Delta\psi_i = -\Delta\varphi_{i,t} - \Delta^2\varphi_i + \partial_i\nabla\cdot g^0 - \Delta g_i^0 \quad \text{in } \omega\cap\mathcal{O},\ i=1,2.$$

Insensitizing controls for the Boussinesq system (Joint work with M. Gueye and S. Guerrero)

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = f + (v_1,0,0)\mathbb{1}_\omega + (0,0,\theta), \quad \nabla\cdot y = 0 & \text{in } Q,\\ \theta_t - \Delta\theta + y\cdot\nabla\theta = f_0 + v_0\mathbb{1}_\omega & \text{in } Q,\\ y = 0, \quad \theta = 0 & \text{on } \Sigma,\\ y(0) = y^0 + \tau\widehat{y}_0, \quad \theta(0) = \theta^0 + \tau\widehat{\theta}_0 & \text{in } \Omega. \end{cases}$$

We look to insensitize the functional

$$J_\tau(y,\theta) := \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} \left(|y|^2 + |\theta|^2\right) dx\, dt,$$

$$\left.\frac{\partial J_\tau(y,\theta)}{\partial\tau}\right|_{\tau=0} = 0 \quad \forall\,(\widehat{y}_0,\widehat{\theta}_0) \in L^2(\Omega)^4 \text{ s.t. } \|\widehat{y}_0\|_{L^2(\Omega)^3} = \|\widehat{\theta}_0\|_{L^2(\Omega)} = 1.$$

The associated linear system:

$$\begin{cases} w_t - \Delta w + \nabla p_0 = f^w + (v_1, 0, 0)\,\mathbb{1}_\omega + (0,0,r), & \nabla\cdot w = 0 & \text{in } Q,\\ -z_t - \Delta z + \nabla p_1 = f^z + w\mathbb{1}_{\mathcal{O}}, & \nabla\cdot z = 0 & \text{in } Q,\\ r_t - \Delta r = f^r + v_0\,\mathbb{1}_\omega & & \text{in } Q,\\ -q_t - \Delta q = f^q + z_3 + r\mathbb{1}_{\mathcal{O}} & & \text{in } Q, \end{cases}$$

where we want to prove $z(0) = 0$ and $q(0) = 0$. We need a Carleman estimate for

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_\varphi = g^\varphi + \psi\,\mathbb{1}_{\mathcal{O}}, & \nabla\cdot\varphi = 0 & \text{in } Q,\\ \psi_t - \Delta\psi + \nabla\pi_\psi = g^\psi + (0,0,\sigma), & \nabla\cdot\psi = 0 & \text{in } Q,\\ -\phi_t - \Delta\phi = g^\phi + \varphi_3 + \sigma\,\mathbb{1}_{\mathcal{O}} & & \text{in } Q,\\ \sigma_t - \Delta\sigma = g^\sigma & & \text{in } Q, \end{cases}$$

with only local terms of φ_1 and ϕ.

Perspectives, open problems

- What about controllability to trajectories for Navier-Stokes? Adjoint equation:

$$-\varphi_t - \Delta\varphi + \bar{y} \cdot D\varphi + \nabla\pi = g.$$

  Problem: The components of φ are mixed.
- What about two vanishing components, e.g., $v = (v_1, 0, 0)$?. [Coron, Lissy, 2013]: Return method.
- Boundary controllability with one vanishing component (taking the trace of an extended controlled solution does not work).
- Boussinesq system: No control on the heat equation.

Some references

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