On the cost of null controllability of a linear KdV equation

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November 17, 2015

Outline

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Control system

A control system governed by a partial differential equation can be formulated as

$$\begin{cases} y'(t) = f(t, y(t), v(t)), & t > 0 \\ y(0) = y_0, \end{cases}$$

- $y(t) \in \mathcal{X}$ is the state of the system.
- $v(t) \in \mathcal{U}$ is the control.
- $\mathcal{X}, \mathcal{U}$ are the state and admissible controls spaces, respectively.
- Controllability problem: Given T and y_0, find $v(t)$ driving $y(t)$ to a target y_1 at time T, that is, $y(T) = y_1$.
- Controllability types: exact, aproximate, null, local, global, to the trajectories...

Model example: Heat equation

Consider a regular open $\Omega \subset \mathbb{R}^N$ and $\omega \subset \Omega$ (control domain)

$$\begin{cases} y_t - \Delta y = v \mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ y = 0 & x \in \partial\Omega, \\ y(0) = y^0 & x \in \Omega, \end{cases}$$

$y_0 \in L^2(\Omega)$ and $\mathbb{1}_\omega(x)$ the characteristic function of ω.

- We look for $v \in L^2(\omega \times (0,T))$ such that $y(T) = 0$ and

$$\|v\|_{L^2(\omega\times(0,T))} \leq C\|y_0\|_{L^2(\Omega)}.$$

- By linearity, this is equivalent to the control to the trajectories: find $v \in L^2(\omega \times (0,T))$ such that $y(T) = \bar{y}(T)$, where $\bar{y}$ is solution of

$$\begin{cases} \bar{y}_t - \Delta \bar{y} = 0 & (x,t) \in \Omega \times (0,T), \\ \bar{y} = 0 & x \in \partial\Omega, \\ \bar{y}(0) = \bar{y}^0 & x \in \Omega. \end{cases}$$

Observability and Carleman estimates

Null controllability is equivalent to the observability inequality: There exists $C > 0$ such that

$$\int_\Omega |\varphi(0)|^2 \, \mathrm{d}x \leq C \iint_{\omega \times (0,T)} |\varphi|^2 \, \mathrm{d}x \, \mathrm{d}t$$

where φ is the solution of the adjoint equation

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & (x,t) \in \Omega \times (0,T), \\ \varphi = 0 & x \in \partial\Omega, \\ \varphi(T) = \varphi_T \in L^2(\Omega) & x \in \Omega. \end{cases}$$

Carleman estimates: They have the form

$$\iint_{\Omega \times (0,T)} \rho|\varphi|^2 \, \mathrm{d}x \, \mathrm{d}t \leq C \iint_{\omega \times (0,T)} \rho|\varphi|^2 \, \mathrm{d}x \, \mathrm{d}t$$

- $\rho = \rho(x,t)$ is a continuous and positive function.
- To obtain observability, we combine it with the energy estimate

$$\int_\Omega |\varphi(0)|^2 \, \mathrm{d}x \leq \int_\Omega |\varphi(t)|^2 \, \mathrm{d}x, \quad t \in (0,T).$$

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Outline

Introduction

An estimation of the cost of null controllability

Behavior of the cost in the vanishing dispersion limit

Final comments

The Korteweg-de Vries (KdV) equation

$$y_t + y_{xxx} + y y_x = 0 \quad x \in \mathbb{R}, t \geq 0.$$

Recreation of the first sighting of a soliton
by John Scott Russell in 1834

A linear KdV equation on a bounded domain

- $T > 0$, $M \in \mathbb{R} \setminus \{0\}$ (transport coefficient), $\varepsilon > 0$ (dispersion coefficient).

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } (0,T) \times (0,L), \\ y_{|x=0} = v(t), \quad y_{x|x=L} = 0, \quad y_{xx|x=L} = 0 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,L). \end{cases}$$

- Controllability has been studied by Guilleron (2014) and Cerpa, Rivas, Zhang (2013).
- We are interested in the behavior of the cost of null controllability with respect to ε.

$$C^{\varepsilon}_{cost} := \sup_{y_0 \in L^2(0,L)} \left\{ \min_{v \in L^2(0,T)} \frac{\|v\|_{L^2(0,T)}}{\|y_0\|_{L^2(0,L)}} : y_{|t=0} = y_0, y_{|t=T} = 0 \text{ in } (0,L) \right\}.$$

- C^{ε}_{cost} is the best constant such that

$$\|v\|_{L^2(0,T)} \le C \|y_0\|_{L^2(0,L)}.$$

Examples

- Heat equation:

$$\begin{cases} y_t - \varepsilon y_{xx} - M y_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = v(t), \quad y_{|x=L} = 0 & \text{in } (0,T). \end{cases}$$

  Coron, Guerrero (2005): $C^{\varepsilon}_{cost} \leq C_0 \exp\left(C(T,M)\varepsilon^{-1}\right)$.

- (Classic) KdV equation:

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = v(t), \quad y_{|x=L} = 0, \quad y_{x|x=L} = 0 & \text{in } (0,T). \end{cases}$$

  Glass, Guerrero (2009): $C^{\varepsilon}_{cost} \leq C_0 \exp\left(C(T,M)\varepsilon^{-1/2}\right)$.

- (Our) KdV equation:

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = v(t), \quad y_{x|x=L} = 0, \quad y_{xx|x=L} = 0 & \text{in } (0,T). \end{cases}$$

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An estimate of the cost of null controllability

Theorem

Let $T > 0$, $M \in \mathbb{R}$ and $\varepsilon > 0$ be fixed. Then,

$$C^{\varepsilon}_{cost} \le C_0 \exp\left(C(\varepsilon^{-1/2}T^{-1/2} + M^{1/2}\varepsilon^{-1/2} + MT)\right), \quad \text{if } M > 0, \text{ and}$$

$$C^{\varepsilon}_{cost} \le C_0 \exp\left(C(\varepsilon^{-1/2}T^{-1/2} + |M|^{1/2}\varepsilon^{-1/2})\right), \quad \text{if } M < 0,$$

where $C > 0$ is a constant independent of T, M and ε, and $C_0 > 0$ depends polynomially on ε^{-1}, T^{-1} and $|M|^{-1}$.

- In particular, if ε is small enough

$$C^{\varepsilon}_{cost} \le C_0 \exp\left(C(T, M)\varepsilon^{-1/2}\right).$$

Duality argument

- The proof is based on an observability inequality

$$\|\varphi_{|t=0}\|_{L^2(0,L)} \leq C_{obs} \|\varphi_{xx|x=0}\|_{L^2(0,T)},$$

where φ satisfies (adjoint equation)

$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xxx} + M\varphi_x = 0 & \text{in } (0,T)\times(0,L), \\ \varphi_{|x=0} = 0, \quad \varphi_{x|x=0} = 0, \quad (\varepsilon\varphi_{xx} - M\varphi)_{|x=L} = 0 & \text{in } (0,T). \end{cases}$$

- We consider the function $\phi := \varepsilon\varphi_{xx} - M\varphi$, which solves

$$\begin{cases} -\phi_t - \varepsilon\phi_{xxx} + M\phi_x = 0 & \text{in } (0,T)\times(0,L), \\ \phi_{x|x=0} = 0, \quad \phi_{xx|x=0} = 0, \quad \phi_{|x=L} = 0 & \text{in } (0,T) \end{cases}$$

and we prove (Carleman estimate)

$$\int_0^T \int_0^L e^{-2s\alpha}|\phi|^2 \leq C_0 \int_0^T e^{-2s\alpha}|\phi_{|x=0}|^2, \quad \alpha = \frac{p(x)}{t^{1/2}(T-t)^{1/2}}.$$

- We recover φ from ϕ and $\varphi_{|x=0} = \varphi_{x|x=0} = 0$ (O.D.E.).

Outline

Behavior of the cost in the vanishing dispersion limit

- We are now interested in the behavior of C^{ε}_{cost} as $\varepsilon \to 0^+$.
- Consider the transport equation ($\varepsilon = 0$)

$$\begin{aligned} y_t - M y_x &= 0 \quad \text{in } (0,T)\times(0,L), \\ y_{|t=0} &= y_0 \quad \text{in } (0,L) \end{aligned}$$

with controls:

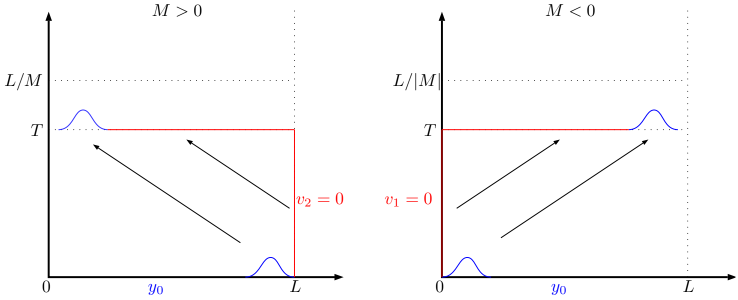
$$\begin{aligned} y_{|x=0} &= v_1(t) \quad \text{if } M < 0, \\ y_{|x=L} &= v_2(t) \quad \text{if } M > 0. \end{aligned}$$

- The transport equation is controllable if only if $T \geq L/|M|$.

On the controllability of the transport equation

$$y_t - My_x = 0 \text{ in } (0,T)\times(0,L)$$

- $T < L/|M|$



- $C_{cost} = +\infty$
- Then, it is natural to expect for KdV: $\lim_{\varepsilon\to 0^+} C^{\varepsilon}_{cost} = +\infty$

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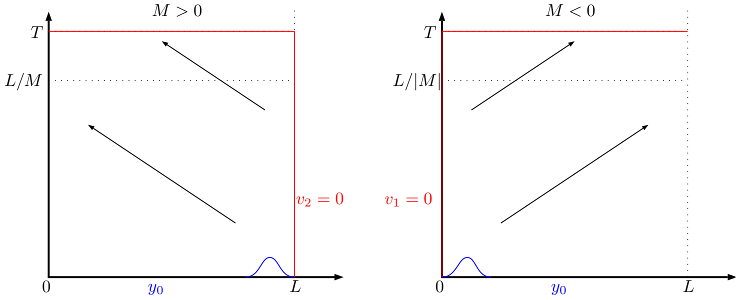

- $C_{cost} = 0$
- Then, it is natural to expect for KdV: $\lim_{\varepsilon\to 0^+} C^{\varepsilon}_{cost} = 0$

Some results

- For the heat equation:

$$\begin{cases} y_t - \varepsilon y_{xx} - M y_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = v(t), \quad y_{|x=L} = 0 & \text{in } (0,T). \end{cases}$$

Coron, Guerrero (2005) proved

1. $T < L/|M|$: $C^{\varepsilon}_{cost} \geq \exp(C\varepsilon^{-1})$ if $M \neq 0$.
2. $T \geq KL/|M|$: $C^{\varepsilon}_{cost} \leq \exp(-C\varepsilon^{-1})$ if $K > 0$ large (uniform contr.).

- For the classic KdV equation:

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = v(t), \quad y_{|x=L} = 0, \quad y_{x|x=L} = 0 & \text{in } (0,T), \end{cases}$$

Glass, Guerrero (2009) proved

1. $T < L/|M|$: $C^{\varepsilon}_{cost} \geq \exp(C\varepsilon^{-1/2})$ if $M \neq 0$.
2. $T \geq KL/M$: $C^{\varepsilon}_{cost} \leq \exp(-C\varepsilon^{-1/2})$ if $M > 0$, $K > 0$ large (u. c.).

Is it possible to obtain uniform controllability with respect to $\varepsilon \to 0^+$?

- $C^{\varepsilon}_{cost} \leq C_\varepsilon \exp(-C(T,M)\varepsilon^{-1/2})$, T large?
- A possible strategy is to combine an observability inequality:

$$\|\varphi_{|t=T/2}\|_{L^2(0,L)} \leq C_\varepsilon \exp\left(C\varepsilon^{-1/2}\right)\|\varphi_{xx|x=0}\|_{L^2(0,T)}$$

with an exponential dissipation estimate (T large enough):

$$\|\varphi_{|t=0}\|_{L^2(0,L)} \leq C_\varepsilon \exp\left(-CT\varepsilon^{-1/2}\right)\|\varphi_{|t=T/2}\|_{L^2(0,L)}.$$

  - Observability OK.
  - But, dissipation is not possible.

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- Observability OK.
- But, dissipation is not possible.

Non-uniform controllability result for arbitrary $T > 0$ and $M > 0$

Theorem[1]

Let $T, L, M > 0$ and $\delta \in (0,1)$. Then, there exists $\varepsilon_0 > 0$ such that

$$C^{\varepsilon}_{cost} \geq C \exp\big((1-\delta)LM^{1/2}\varepsilon^{-1/2}\big), \quad \forall \varepsilon \in (0,\varepsilon_0)$$

where C depends polynomially on ε^{-1} and ε.

[1] C., Guerrero. On the non-uniform null controllability of a linear KdV equation. *Asymptot. Anal.*, 2015.

An auxiliary problem

Find $u \in L^2(0,T)$ such that:

$$\begin{cases} w_t + \varepsilon w_{xxx} - M w_x = 0 & \text{in } (0,T)\times(\delta L, L), \\ w_{xx|x=\delta L} = u(t), \quad w_{x|x=L} = 0, \quad w_{xx|x=L} = 0 & \text{in } (0,T), \\ w_{|t=0} = w_0, \quad w_{|t=T} = 0 & \text{in } (\delta L, L). \end{cases}$$

We define its cost: $K^{\varepsilon}_{cost} := \sup_{\substack{w_0 \in H^3_n(\delta L, L) \\ w_0 \neq 0}} \min_{\substack{u \in L^2(0,T) \\ w_{|t=T}=0}} \frac{\|u\|_{L^2(0,T)}}{\|w_0\|_{H^3_n(\delta L, L)}}$.

- We prove that $K^{\varepsilon}_{cost} \geq C \sinh\left((1-\delta) L M^{1/2} \varepsilon^{-1/2}\right)$.
- By setting $u := y_{xx|x=\delta L}$, we can prove that $K^{\varepsilon}_{cost} \lesssim C^{\varepsilon}_{cost}$.

Particular solution for the adjoint equation

The adjoint equation is given by

$$\begin{cases} -\psi_t - \varepsilon\psi_{xxx} + M\psi_x = 0 & \text{in } (0,T)\times(\delta L, L), \\ \psi_{x|x=\delta L} = (\varepsilon\psi_{xx} - M\psi)_{|x=\delta L} = (\varepsilon\psi_{xx} - M\psi)_{|x=L} = 0 & \text{in } (0,T), \\ \psi_{|t=T} = \psi_T & \text{in } (\delta L, L). \end{cases}$$

- $\displaystyle \sup_{h\in H^3_n(\delta L,L)} \frac{\int_{\delta L}^{L} \psi_{|t=0} h}{\|h\|_{H^3_n(\delta L,L)}} \leq \varepsilon K^{\varepsilon}_{cost} \|\psi_{|x=\delta L}\|_{L^2(0,T)}$ (observability ineq.).
- $\hat{\psi}(x) := \cosh\left((x-\delta L) M^{1/2} \varepsilon^{-1/2}\right)$ is a solution.

An explosion result of the cost when $M < 0$

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = v(t), \quad y_{x|x=L} = 0, \quad y_{xx|x=L} = 0 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,L). \end{cases}$$

Theorem

Let $M < 0$. Then, for every $T < L/|M|$ there exist $C > 0$ (independent of ε) and $\varepsilon_0 > 0$ such that

$$C_{cost}^{\varepsilon} \geq \exp\left(C\varepsilon^{-1/2}\right), \quad \forall \varepsilon \in (0,\varepsilon_0).$$

- The idea is to construct a particular y_0 such that

$$\|v\|_{L^2(0,T)} \geq \exp\left(C\varepsilon^{-1/2}\right)\|y_0\|_{L^2(0,L)}$$

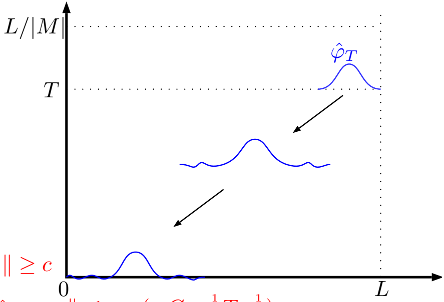
for every v driving y to zero.

Idea of proof

We construct a particular solution $\hat{\varphi}$ of

$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xxx} + M\varphi_x = 0 & \text{in } (0,T)\times(0,L), \\ \varphi_{|x=0} = 0, \quad \varphi_{x|x=0} = 0, \quad (\varepsilon\varphi_{xx} - M\varphi)_{|x=L} = 0 & \text{in } (0,T), \\ \varphi_{|t=T} = \hat{\varphi}_T & \text{in } (0,L), \end{cases}$$

where $0 \leq \hat{\varphi}_T \in \mathcal{C}_0^\infty(0,L)$, $\|\hat{\varphi}_T\|_{L^2(0,L)} = 1$.



We have that:

$$\int_0^L y_0 \widehat{\varphi}_{|t=0}\,\mathrm{d}x = -\int_0^T v(t)\widehat{\varphi}_{xx|x=0}\,\mathrm{d}t.$$

Take $y_0 := \widehat{\varphi}_{|t=0}$.

$\|\hat{\varphi}_{|t=0}\| \geq c$

$\|\hat{\varphi}_{xx|x=0}\| \leq \exp(-C\varepsilon^{-\frac{1}{2}}T^{-\frac{1}{2}})$

A uniform null controllability result[2]

If T large enough and

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = v_0(t), \quad y_{x|x=L} = v_1(t), \quad y_{xx|x=L} = v_2(t) & \text{in } (0,T), \\ y_{|t=0} = y_0, \quad y_{|t=T} = 0 & \text{in } (0,L). \end{cases}$$

- We can prove, with $v_1(t) = 0$:

$$\|v_0\|_{L^2(0,T)} + \|v_2\|_{L^2(0,T)} \leq C_\varepsilon \exp\left(-C(T,M)\varepsilon^{-1/2}\right)\|y_0\|_{L^2(0,\delta L)}$$

- Also, we can prove, with $v_0(t) = 0$:

$$\|v_1\|_{L^2(0,T)} + \|v_2\|_{L^2(0,T)} \leq C_0 \exp\left(-C(T,M)\varepsilon^{-1/2}\right)\|y_0\|_{L^2(0,\delta L)}.$$

  - y_0 supported in $(0,\delta L)$, $\delta \in (0,1)$.

[2] C., Guerrero. Uniform null controllability of a linear KdV equation using two controls. *Preprint*.

Outline

Summary

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = v(t), \quad y_{x|x=L} = 0, \quad y_{xx|x=L} = 0 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,L). \end{cases}$$

• We prove that there exists y_0 such that for every v driving y to 0

$$\|v\|_{L^2(0,T)} \geq \exp\left(C\varepsilon^{-1/2}\right)\|y_0\|_{L^2(0,L)}, \quad \varepsilon \text{ small},$$

in two cases:

- $M > 0$, $T > 0$.
- $M < 0$, $T < L/|M|$.

• If we allow to control $y_{xx|x=L}$ and y_0 supported in $(0, \delta L)$, the controls remain bounded with respect to ε if T is large enough.

Open problem

$$
\begin{cases}
y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } (0,T)\times(0,L), \\
y_{|x=0} = v_0(t), \quad y_{x|x=L} = v_1(t), \quad y_{xx|x=L} = 0 & \text{in } (0,T), \\
y_{|t=0} = y_0, \quad y_{|t=T} = 0 & \text{in } (0,L).
\end{cases}
$$

$$
\|v_0\|_{L^2(0,T)} + \|v_1\|_{L^2(0,T)} \leq C_0 \exp\left(-C\varepsilon^{-1/2}\right)\|y_0\|_{L^2(0,L)}?
$$

or

$$
\|v_0\|_{L^2(0,T)} + \|v_1\|_{L^2(0,T)} \geq C_0 \exp\left(C\varepsilon^{-1/2}\right)\|y_0\|_{L^2(0,L)}?
$$

Thank you for your attention