# Some controllability results for the Navier-Stokes equation with a reduced number of scalar controls

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# Outline

# Navier-Stokes in a bounded domain

- $\Omega$ bounded connected regular open subset of $\mathbf{R}^N$ ($N = 2$ or $3$)
- $T > 0$
- $\omega \subset \Omega$ (control set), $Q := \Omega \times (0, T)$, $\Sigma := \partial\Omega \times (0, T)$

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = v \mathbb{1}_\omega, \ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \qquad \text{(NS)}$$

where $v$ stands for the control which acts over the set $\omega$.

**Controllability problem:** Can we drive the solution of (NS) to a given state at time $T$ by means of a control $v \in L^2(\omega \times (0, T))^N$?

Because of regularization, we cannot expect exact controllability.

# Exact controllability to trajectories

Consider the uncontrolled solution to the same equation:

$$\begin{cases} \bar{y}_t - \Delta \bar{y} + (\bar{y} \cdot \nabla)\bar{y} + \nabla \bar{p} = 0,\ \nabla \cdot \bar{y} = 0 & \text{in } Q, \\ \bar{y} = 0 & \text{on } \Sigma, \\ \bar{y}(0) = \bar{y}^0 & \text{in } \Omega. \end{cases}$$

Exact controllability to trajectories: Given an initial condition $y^0$, can we find $v$ such that

$$\boxed{y(T) = \bar{y}(T)}\ ?$$

Local exact controllability to trajectories: If $\|y^0 - \bar{y}^0\|$ is small enough, can we find $v$ such that

$$\boxed{y(T) = \bar{y}(T)}\ ?$$

**Remark:** After time $T$, we can "turn off" the control and follow the ideal trajectory.

# Some results

Under regularity assumptions on $\bar{y}$

- [Imanuvilov 2001]

  Improvements in:

- [Fernández-Cara, Guerrero, Imanuvilov, Puel, 2004]
- [Imanuvilov, Puel, Yamamoto, 2012]: $\bar{y} \in L^\infty(Q)$

**Method of proof**

- Linearization around $\bar{y}$.
- Null controllability of the linearized problem.
- Fixed point or inverse mapping theorem for the nonlinear problem.

# Reduced number of controls

Now we consider:

$$y_t - \Delta y + (y \cdot \nabla) y + \nabla p = (v_1, v_2, 0) \mathbb{1}_\omega, \ \nabla \cdot y = 0, \ y|_\Sigma = 0$$

**Some results:**

- First results by Fernández-Cara, Guerrero, Imanuvilov, Puel (2006): Local exact controllability to the trajectories when $\overline{\omega} \cap \partial\Omega \neq \emptyset$.
- Null component depends on this geometric assumption.
- We are interested in removing this geometric property.
- Coron, Guerrero (2009): Null controllability of the Stokes system for a general $\omega \subset \Omega$

$$y_t - \Delta y + \nabla p = (v_1, v_2, 0) \mathbb{1}_\omega, \ \nabla \cdot y = 0, \ y|_\Sigma = 0$$

# Controllability, observability and Carleman estimates

We consider the adjoint system

$$-\varphi_t - \Delta\varphi + \nabla\pi = 0,\ \nabla\cdot\varphi = 0,\ \varphi|_\Sigma = 0$$

Null controllability is equivalent to the Observability inequality

$$\int_\Omega |\varphi(0)|^2\,dx \le C \iint_{\omega\times(0,T)} (|\varphi_1|^2 + |\varphi_2|^2)dx\,dt, \quad \varphi = (\varphi_1, \varphi_2, \varphi_3).$$

**Important tool:** Carleman estimates

$$\iint_Q \rho_1(x,t)|\varphi|^2 dx\,dt \le C \iint_{\omega\times(0,T)} \rho_2(x,t)(|\varphi_1|^2 + |\varphi_2|^2)dx\,dt$$

$\rho_1$, $\rho_2$ some positive weight functions, $C$ independent of $\varphi$.

## How to prove this type of Carleman estimate?

If $\overline{\omega} \cap \partial\Omega \neq \emptyset$:

- From an estimate of the type

$$\iint_Q \rho_1(x,t)|\varphi|^2 dx\, dt \leq C \iint_{\omega\times(0,T)} \rho_2(x,t)|\varphi|^2 dx\, dt$$

  we can use $\varphi|_\Sigma = 0$ and $\nabla \cdot \varphi = 0$ to estimate $\varphi_3$ by $\varphi_1$ and $\varphi_2$.
- This estimate was proved in [Imanuvilov, 2001], [Fernández-Cara, Guerrero, Imanuvilov, Puel, 2004] for a general $\omega \subset \Omega$.
- The main difficulty is estimating de pressure.

General case $\omega \subset \Omega$ : Method introduced in [Coron, Guerrero, 2009]:

- $\nabla \cdot \varphi = 0 \Rightarrow \Delta \pi = 0$,
- Look at the (heat) equations satisfied by $\nabla \Delta \varphi_1$ and $\nabla \Delta \varphi_2$ and apply suitable Carleman estimates (doing this eliminates the pressure),
- Use $\varphi|_\Sigma = 0$ and $\nabla \cdot \varphi = 0$ to recover $\varphi_3$ in the LHS,
- Regularity estimates for the Stokes system to estimate the boundary terms.

# Navier-Stokes system

We deal with the local null controllability of

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = (v_1, v_2, 0)\mathbb{1}_\omega, \ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases}$$

with no assumption on the control domain $\emptyset \neq \omega \subset \Omega$.

**Theorem (Guerrero, C., 2013, JMFM)**

For every $T > 0$ and $\omega \subset \Omega$, the NS system is locally null controllable by controls $v_1, v_2 \in L^2(\omega \times (0, T))$.

- We can also choose $v = (v_1, 0, v_3)$ or $v = (0, v_2, v_3)$.

# Method of proof

- Linearization around zero

$$y_t - \Delta y + \nabla p = f + (v_1, v_2, 0)\mathbb{1}_\omega, \; \nabla \cdot y = 0, \; y|_\Sigma = 0$$

- Null controllability of the linearized system (Main part of the proof).
  **Main tool:** Carleman estimate for the adjoint system

$$-\varphi_t - \Delta\varphi + \nabla\pi = g, \; \nabla \cdot \varphi = 0, \; \varphi|_\Sigma = 0$$

  There exists a constant $C > 0$ (depending on $\Omega$, $\omega$, $T$)

$$\int_Q \rho_1(t)|\varphi|^2 \leq C\left(\int_Q \rho_2(t)|g|^2 + \int_{\omega\times(0,T)} \rho_3(t)(|\varphi_1|^2 + |\varphi_2|^2)\right)$$

- Inverse mapping theorem for the nonlinear system

$$\mathcal{A} = (y_t - \Delta y + (y\cdot\nabla)y + \nabla p - (v_1, v_2, 0)\mathbb{1}_\omega, y(0))$$

# What is different with the Stokes case?

- $g \neq 0$.
- $\Delta\pi \neq 0$.
- Applying $\nabla\Delta\cdot$ is more difficult now, since $g$ is not regular.
- We overcome this by separating $\varphi$ in $\varphi^g + \varphi^i$, with $\varphi^g$ not regular and $\varphi^i$ very regular.

# Null controllability of the linear system

We need weights that do not vanish at $t = 0$.
Let

$$\widetilde{\rho}_i(t) = \begin{cases} \rho_i(T/2) & 0 \leq t \leq T/2, \\ \rho_i(t) & T/2 < t \leq T. \end{cases}$$

$$\|\varphi(0)\|^2_{L^2(\Omega)^2} + \int_Q \widetilde{\rho}_1(t)|\varphi|^2 \leq C \left( \int_Q \widetilde{\rho}_2(t)|g|^2 + \int_{\omega\times(0,T)} \widetilde{\rho}_3(t)(|\varphi_1|^2 + |\varphi_2|^2) \right)$$

This is proved using classical energy estimates for Stokes and the previous Carleman inequality.

Recall the linear system:

$$\begin{cases} y_t - \Delta y + \nabla p = f + (v_1, v_2, 0)\mathbb{1}_\omega, \ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega. \end{cases}$$

If

$$\iint_Q \widetilde{\rho_1}^{-1} |f|^2 < +\infty,$$

then we can prove that there exists controls $v_1, v_2$ such that $\boxed{y(T)=0}$. Furthermore,

$$\boxed{\iint_Q \widetilde{\rho_2}^{-1} |y|^2 + \iint_Q \widetilde{\rho_3}^{-1} (|v_1|^2 + |v_2|^2)\mathbb{1}_\omega < +\infty,}$$

which gives that $y$ goes to zero at $T$ exponentially (so do the controls).

# Controllability of the NS system

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = (v_1, v_2, 0)\mathbb{1}_\omega, \ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega. \end{cases}$$

We consider the operator:

$$\mathcal{A}(y, p, v_1, v_2) = (\, y_t - \Delta y + (y \cdot \nabla) y + \nabla p - (v_1, v_2, 0)\mathbb{1}_\omega, \ y(0)\,)$$

$\mathcal{A}$ is $C^1$ between weighted spaces (where in particular $\boxed{y(T)=0}$).

$$\mathcal{A}'(0,0,0,0)(y, p, v_1, v_2) = (y_t - \Delta y + \nabla p - (v_1, v_2, 0)\mathbb{1}_\omega, y(0)\,)$$

is surjective by the null controllability of the linear system.
Inverse mapping theorem around $(0,0,0,0)$ gives the result for NS, i.e., there exists $\delta > 0$ such that if $\|y^0\| < \delta$, then there exists $(y, p, v_1, v_2)$ such that

$$\boxed{\mathcal{A}(y, p, v_1, v_2) = (0, y^0)}.$$

# Extension: Boussinesq system

Now we consider the Boussinesq system:

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = (v_1, 0, 0)\mathbb{1}_\omega + (0,0,\theta),\ \nabla\cdot y = 0 & \text{in } Q,\\ \theta_t - \Delta\theta + y\cdot\nabla\theta = v_0\mathbb{1}_\omega & \text{in } Q,\\ y = 0,\ \theta = 0 & \text{on } \Sigma,\\ y(0) = y^0,\ \theta(0) = \theta^0 & \text{in } \Omega. \end{cases}$$

**Theorem (C., 2013, MCRF)**

For every $T > 0$ and $\omega \subset \Omega$, the Boussinesq system is locally controllable to the trajectory $(0, \bar p, \bar\theta)$ by controls $v_0, v_1 \in L^2(\omega\times(0,T))$, where

$$\begin{cases} \nabla\bar p = \bar\theta\, e_3 & \text{in } Q,\\ \bar\theta_t - \Delta\bar\theta = 0 & \text{in } Q,\\ \bar\theta = 0 \text{ on } \Sigma,\ \bar\theta(0) = \bar\theta^0 & \text{in } \Omega. \end{cases}$$

- We can also choose $v = (0, v_2, 0)$.
- For $N = 2$, $v \equiv 0$: No control is needed in the fluid equation.

# Insensitizing controls for Navier-Stokes system

We consider the problem of insensitizing controls for the NS system:

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = f + v \mathbb{1}_\omega, \ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 + \tau \widehat{y}^0 & \text{in } \Omega, \end{cases}$$

where $\tau$ is a small constant and $\|\widehat{y}^0\|_{L^2(\Omega)^3} = 1$. Both are unknown.
**Insensitizing control problem:** To find a control $v \in L^2(\omega \times (0,T))^3$ such that the functional (Sentinel)

$$J(y) = \iint_{\mathcal{O} \times (0,T)} |y|^2 \, dx \, dt, \ \mathcal{O} \subset \Omega \ (\text{Observation set})$$

is not affected by the uncertainty of the initial data, that is,

$$\left. \frac{\partial J(y)}{\partial \tau} \right|_{\tau=0} = 0, \ \forall \widehat{y}^0 \in L^2(\Omega)^3 \text{ s.t. } \|\widehat{y}^0\|_{L^2(\Omega)^3} = 1.$$

# Some previous works

- Heat equation: [Bodart, Fabre, 1995], [de Teresa, 2000], [Bodart, González-Burgos, Pérez-García, 2002]
- Stokes: [Guerrero, 2007]
- Navier-Stokes: [Gueye, 2013]

We are interested in controls with one vanishing component.

# A cascade Navier-Stokes system

The previous condition is equivalent to the following null controllability problem: To find a control $v = (v_1, v_2, 0)$ such that $z(0) = 0$, where

$$\begin{cases} w_t - \Delta w + (w \cdot \nabla) w + \nabla p^0 = f + (v_1, v_2, 0) 1\!\!1_\omega, \ \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + (z \cdot \nabla^t w) - (w \cdot \nabla) z + \nabla q = w 1\!\!1_{\mathcal{O}}, \ \nabla \cdot z = 0 & \text{in } Q, \\ w = z = 0 & \text{on } \Sigma, \\ w(0) = y^0, \ z(T) = 0 & \text{in } \Omega. \end{cases}$$

**Theorem (Gueye, C., 2013, to appear in JMPA)**

Assume $y^0 = 0$ and $\mathcal{O} \cap \omega \neq \emptyset$. There exists $\delta > 0$ such that if $\|e^{K/t^{10}} f\|_{L^2(Q)^3} < \delta$, there exist a controls $v_1, v_2 \in L^2(\omega \times (0, T))$ such that $z(0) = 0$.

- We can also choose: $v = (v_1, 0, v_3)$ or $v = (0, v_2, v_3)$.

# Same strategy as before

Null controllability of the linearized system around 0:

$$\begin{cases} w_t - \Delta w + \nabla p^0 = f^0 + (v_1, v_2, 0)\mathbb{1}_\omega,\ \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + \nabla q = f^1 + w\mathbb{1}_\mathcal{O},\ \nabla \cdot z = 0 & \text{in } Q, \\ w = z = 0 & \text{on } \Sigma, \\ w(0) = 0,\ z(T) = 0 & \text{in } \Omega, \end{cases}$$

where $f^0$ and $f^1$ decrease exponentially to zero at $t = 0$.

- The control acts on $z$ through $w$.

As before, we want to show an estimate of the form

$$\iint_Q \rho_1(t)(|\varphi|^2 + |\psi|^2) \leq C \left( \iint_Q \rho_2(t)(|g^0|^2 + |g^1|^2 + |\nabla g^1|^2) + \iint_{\omega\times(0,T)} \rho_3(t)(|\varphi_1|^2 + |\varphi_2|^2) \right)$$

where $(\varphi, \psi) = (\varphi_1, \varphi_2, \varphi_3, \psi)$ is the solution of the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = g^0 + \psi \mathbb{1}_{\mathcal{O}}, & \text{in } Q, \\ \psi_t - \Delta\psi + \nabla\kappa = g^1, & \text{in } Q, \\ \nabla\cdot\varphi = \nabla\cdot\psi = 0, \text{ in } Q, \ \varphi = \psi = 0 \text{ on } \Sigma & \\ \varphi(T) = 0, \ \psi(0) = \psi^0 & \text{in } \Omega. \end{cases}$$

- $g^1 \in L^2(0, T; H_0^1(\Omega)^3)$ with $\nabla \cdot g^1 = 0$.

# Idea of proof

- Carleman for $\psi_1$ and $\psi_2$.
- Carleman for $\varphi_1$ and $\varphi_2$.
- Estimate local terms on $\psi_1$ and $\psi_2$ using

$$\Delta\psi_i = -\Delta\varphi_{i,t} - \Delta^2\varphi_i + \partial_i\nabla\cdot g^0 - \Delta g_i^0, \text{ in } \omega\cap\mathcal{O},\ i = 1, 2.$$

# Perspectives, open problems

- What about controllability to trajectories for Navier-Stokes? Adjoint equation:

$$-\varphi_t - \Delta\varphi + \bar{y} \cdot D\varphi + \nabla\pi = g.$$

  Problem: The components of $\varphi$ are mixed.

- What about two vanishing components, e.g., $v = (v_1, 0, 0)$?. [Coron, Lissy, 2013]: Return method.
- Boundary controllability with one vanishing component (taking the trace of an extended controlled solution does not work).

# Some references


N. C. and S. Guerrero, Local null controllability of the $N$-dimensional Navier-Stokes system with $N-1$ scalar controls in an arbitrary control domain, *J. Math. Fluid Mech.*, **15** (2013), no. 1, 139–153.

N. C., Local controllability of the $N$-dimensional Boussinesq system with $N-1$ scalar controls in an arbitrary control domain, *Math. Control Relat. Fields*, **2** (2012), no. 4, 361–382.

N. C. and M. Gueye, Insensitizing controls with one vanishing component for the Navier-Stokes system, to appear in *J. Math. Pures Appl.*.

N. C., S. Guerrero and M. Gueye, Insensitizing controls with two vanishing components for the three-dimensional Boussinesq system, submitted.


Available on: https://www.ljll.math.upmc.fr/∼ncarreno