Control of parabolic systems and application to a hierarchical control problem

Nicolás Carreño
Universidad Técnica Federico Santa María

Séminaire du Département Automatique
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Outline

Outline

PDE control system

A control system governed by a partial differential equation can be formulated

$$\begin{cases} y'(t) = f(t, y(t), v(t)), & t > 0 \\ y(0) = y_0, \end{cases}$$

- $y(t)$ is the state of the system.
- $v(t)$ is the control.
- Controllability problem: Given y_0 and $T > 0$, find $v(t)$ driving $y(t)$ to a target y_1 at time T, that is, $y(T) = y_1$.
- Controllability types
  - Exact.
  - Null: $y(T) = 0$.
  - Approximate: $y(T)$ close to y_1.
  - Local: y_0 close to y_1.

PDE control

Example: Heat equation

Consider a regular open $\Omega \subset \mathbb{R}^N$ and $\omega \subset \Omega$ (control domain)

$$\begin{cases} y_t - \Delta y = v\mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ y = 0 & x \in \partial\Omega, \\ y(0) = y_0 & x \in \Omega, \end{cases}$$

- $y = y(x,t)$: Temperature distribution.
- $v = v(x,t)$: Control supported in ω.

Question: Given $T > 0$ and $y_1 = y_1(x)$, is there v such that $y(T) = y_1$?

PDE control

Answer: In general, the answer is no due to the *regularizing effect*.

• It seems natural to consider the notion of control to the trajectories:
Consider a solution of

$$\begin{cases} \bar{y}_t - \Delta \bar{y} = 0 & (x,t) \in \Omega \times (0,T), \\ \bar{y} = 0 & x \in \partial\Omega, \\ \bar{y}(0) = \bar{y}_0 & x \in \Omega, \end{cases}$$

We look for a control v such that $y(T) = \bar{y}(T)$.

• By linearity (taking $\widetilde{y} := y - \bar{y}$), this is equivalent to the null controllability:

$$y(T) = 0.$$

Therefore, we concentrate in this case.

Duality Method: Hilbert Uniqueness Method (HUM)

Construction of the control:

- We multiply $y_t - \Delta y = v 1_\omega$ by φ solution to the (adjoint) equation

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & (x,t) \in \Omega \times (0,T), \\ \varphi = 0 & x \in \partial\Omega, \\ \varphi(T) = \varphi_T \in L^2(\Omega) & x \in \Omega, \end{cases}$$

and integrate in $\Omega \times (0,T)$:

$$\int_\Omega y(T)\varphi_T \,\mathrm{d}x = \iint_{\omega\times(0,T)} v\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega y_0\varphi(0)\,\mathrm{d}x, \quad \forall \varphi_T \in L^2(\Omega).$$

- v is a control such that $y(T) = 0$ if and only if

$$\iint_{\omega\times(0,T)} v\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega y_0\varphi(0)\,\mathrm{d}x = 0, \quad \forall \varphi_T \in L^2(\Omega).$$

Observability inequality

The previous condition can be seen as an optimality condition for

$$J(\varphi_T) = \frac{1}{2} \iint\limits_{\omega\times(0,T)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega y_0\varphi(0)\,\mathrm{d}x.$$

- J convex, continuous and coercive if there exists $C > 0$ such that

$$\int_\Omega |\varphi(0)|^2 \,\mathrm{d}x \leq C \iint\limits_{\omega\times(0,T)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

This is known as observability inequality.

- The control is given by

$$v := \widehat{\varphi},$$

where $\widehat{\varphi}$ is the solution of the adjoint equation associated to $\widehat{\varphi}_T$, minimum of J.

- Null controllability is equivalent to observability.

Carleman estimates

How to prove the observability inequality?

Powerful tool to prove observability: Carleman estimates

$$\iint\limits_{\Omega\times(0,T)} \rho|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C \iint\limits_{\Omega\times(0,T)} \rho|\varphi_t + \Delta\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + C \iint\limits_{\omega\times(0,T)} \rho|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t$$

- $\varphi(x,t) = 0$, $x \in \partial\Omega$.
- $\rho = \rho(x,t)$ is a positive function and continuous in $\overline{\Omega} \times (0,T)$ with critical points only in ω.
- To deduce the observability, we use dissipation properties as

$$\int_\Omega |\varphi(0)|^2 \,\mathrm{d}x \leq \int_\Omega |\varphi(t)|^2 \,\mathrm{d}x, \quad t \in (0,T).$$

Control of a system of two equations with one control

Consider the system with one scalar control

$$\begin{cases} y_t - \Delta y = z + v\mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ z_t - \Delta z = y & (x,t) \in \Omega \times (0,T), \\ y = z = 0 & x \in \partial\Omega, \\ y(0) = y^0, \quad z(0) = z^0 & x \in \Omega. \end{cases}$$

- We look for v such that $y(T) = z(T) = 0$.
- Observability inequality: There exists $C > 0$ such that

$$\int_\Omega \left(|\varphi(0)|^2 + |\psi(0)|^2\right) \mathrm{d}x \leq C \iint_{\omega\times(0,T)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t$$

where (φ, ψ) is the solution to the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = \psi & (x,t) \in \Omega \times (0,T), \\ -\psi_t - \Delta\psi = \varphi & (x,t) \in \Omega \times (0,T), \\ \varphi = \psi = 0 & x \in \partial\Omega, \\ \varphi(T) = \varphi_T, \quad \psi(T) = \psi_T & x \in \Omega. \end{cases}$$

Control of a system of two equations with one control

- The idea is to combine Carleman estimates for φ and ψ:

$$\iint_{\Omega\times(0,T)} \rho_1|\varphi|^2 \,dx\,dt \leq C \iint_{\Omega\times(0,T)} \rho_2|\psi|^2 \,dx\,dt + C \iint_{\omega\times(0,T)} \rho_1|\varphi|^2 \,dx\,dt$$

$$\iint_{\Omega\times(0,T)} \rho_1|\psi|^2 \,dx\,dt \leq C \iint_{\Omega\times(0,T)} \rho_2|\varphi|^2 \,dx\,dt + C \iint_{\omega\times(0,T)} \rho_1|\psi|^2 \,dx\,dt$$

We can choose the weights such that $\rho_2 \leq \dfrac{1}{4C}\rho_1$, but we need to estimate the local term of ψ. We use the equation ($\psi = -\varphi_t - \Delta\varphi$)

$$\iint_{\omega\times(0,T)} \rho_1|\psi|^2 \,dx\,dt = \iint_{\omega\times(0,T)} \rho_1\psi(-\varphi_t - \Delta\varphi) \,dx\,dt$$

$$\leq \frac{1}{8C} \iint_{\omega\times(0,T)} \rho_1|\psi|^2 \,dx\,dt + C \iint_{\omega\times(0,T)} \rho_1|\varphi|^2 \,dx\,dt.$$

Control of a system of two equations with one control

$$\begin{cases} y_t - \Delta y = z + v\mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ z_t - \Delta z = y & (x,t) \in \Omega \times (0,T), \\ y = z = 0 & x \in \partial\Omega, \\ y(0) = y^0, \quad z(0) = z^0 & x \in \Omega. \end{cases}$$

- Then, there is v such that $y(T) = z(T) = 0$.

Variant: Systems coupled by $\mathcal{O} \subset \Omega$.

$$\begin{cases} y_t - \Delta y = z + v\mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ z_t - \Delta z = y\mathbb{1}_{\mathcal{O}} & (x,t) \in \Omega \times (0,T). \end{cases}$$

- In this case, if $\mathcal{O} \cap \omega \neq \emptyset$, then there exists v such that $y(T) = z(T) = 0$.

Outline

Introduction
- Observability and Carleman estimates
- Systems

Application to a multi-objective control problem

Other results

Mono-objective vs Multi-objective

- Standard control problem:

$$\begin{cases} y_t + \mathcal{A}(y) = f\mathbb{1}_{\mathcal{O}} & \text{in } \Omega \times (0,T) \\ y(0) = y_0 & \text{in } \Omega \end{cases}$$

  - $f \longrightarrow y(T) = 0$ in Ω.

- Control problem with more agents:

$$\begin{cases} y_t + \mathcal{A}(y) = f\mathbb{1}_{\mathcal{O}} + v_1\mathbb{1}_{\mathcal{O}_1} + v_2\mathbb{1}_{\mathcal{O}_2} & \text{in } \Omega \times (0,T) \\ y(0) = y_0 & \text{in } \Omega \end{cases}$$

  - $f \longrightarrow y(T) = 0$ in Ω.
  - $v_1 \longrightarrow y \approx y_{1,d}$ in $\mathcal{O}_{1,d} \subset \Omega$.
  - $v_2 \longrightarrow y \approx y_{2,d}$ in $\mathcal{O}_{2,d} \subset \Omega$.

Motivation: resort lake

$$\begin{cases} y_t + \mathcal{A}(y) = f\mathbb{1}_{\mathcal{O}} + v_1\mathbb{1}_{\mathcal{O}_1} + v_2\mathbb{1}_{\mathcal{O}_2} & \text{in } \Omega \times (0,T) \\ y(0) = y_0 & \text{in } \Omega \end{cases}$$

- Lake represented by $\Omega \subset \mathbb{R}^3$.
- $y = y(x,t)$: concentration of chemicals or of living organisms in the lake.
- Local agents P_1, P_2 that can decide their policy v_1, v_2 acting on $\mathcal{O}_1$, $\mathcal{O}_2$ (The followers).
- The manager of the resort decides the policy f acting on $\mathcal{O}$ (The leader).
- Goal of the manager: "Clean" the lake at time T ($y(T) = 0$).
- Goal of the agents: To be close to a target concentration $y_{i,d}$ in $\mathcal{O}_{i,d} \subset \Omega$ during the time interval $(0,T)$ ($y \approx y_{i,d}$ in $\mathcal{O}_{i,d}$).

A linear fourth-order equation

Let $Q := (0,L)\times(0,T)$, f the leader, v_1 and v_2 the followers.

$$\begin{cases} y_t + y_{xxxx} = f\mathbb{1}_{\mathcal{O}} + v_1\mathbb{1}_{\mathcal{O}_1} + v_2\mathbb{1}_{\mathcal{O}_2} & \text{in } Q \\ y_{|x=0} = y_{x|x=0} = 0 = y_{|x=L} = y_{x|x=L} & \text{in } (0,T) \\ y(0) = y_0 & \text{in } (0,L) \end{cases}$$

Consider the functionals ($i = 1,2$): $\alpha_i > 0, \mu_i > 0$

$$J_i(f; v_1, v_2) = \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\mu_i}{2} \iint_{\mathcal{O}_i\times(0,T)} |v_i|^2 \,\mathrm{d}x\,\mathrm{d}t$$

- Task of the followers: $y \approx y_{i,d}$ in $\mathcal{O}_{i,d}$ with "little effort"

$$\min J_i(f; v_1, v_2), \quad i = 1,2.$$

- Task of the leader: $y(T) = 0$ in Ω.

The Stackelberg-Nash strategy

$$\begin{cases} y_t + y_{xxxx} = f\mathbb{1}_{\mathcal{O}} + v_1\mathbb{1}_{\mathcal{O}_1} + v_2\mathbb{1}_{\mathcal{O}_2} & \text{in } Q \\ y_{|x=0} = y_{x|x=0} = 0 = y_{|x=L} = y_{x|x=L} & \text{in } (0,T) \\ y(0) = y_0 & \text{in } (0,L) \end{cases}$$

- **Step 1:** For fixed f, find a Nash equilibrium (v_1, v_2):

$$J_1(f; v_1, v_2) = \min_{\hat{v}_1} J_1(f; \hat{v}_1, v_2), \quad J_2(f; v_1, v_2) = \min_{\hat{v}_2} J_2(f; v_1, \hat{v}_2)$$

Of course, this equilibrium depends on f: $v_1 = v_1(f)$, $v_2 = v_2(f)$.

- **Step 2:** Find f such that $y(T) = 0$.

This is the *Stackelberg-Nash strategy*.

Optimality system

$$\begin{cases} y_t + y_{xxxx} = f\mathbb{1}_{\mathcal{O}} + v_1\mathbb{1}_{\mathcal{O}_1} + v_2\mathbb{1}_{\mathcal{O}_2} & \text{in } Q \\ y_{|x=0} = y_{x|x=0} = 0 = y_{|x=L} = y_{x|x=L} & \text{in } (0,T) \\ y(0) = y_0 & \text{in } (0,L) \end{cases}$$

- Nash equilibrium is equivalent to (due to the convexity of J_i)

$$\begin{cases} J_1'(f; v_1, v_2)(\hat{v}_1, 0) = 0 & \forall \hat{v}_1 \in L^2(\mathcal{O}_1 \times (0,T)) \\ J_2'(f; v_1, v_2)(0, \hat{v}_2) = 0 & \forall \hat{v}_2 \in L^2(\mathcal{O}_2 \times (0,T)) \end{cases}$$

- Characterization of Nash equilibrium: Optimality system

$$\begin{cases} y_t + y_{xxxx} = f\mathbb{1}_{\mathcal{O}} - \dfrac{1}{\mu_1}\phi^1\mathbb{1}_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\phi^2\mathbb{1}_{\mathcal{O}_2} & \text{in } Q \\ -\phi^i_t + \phi^i_{xxxx} = \alpha_i(y - y_{i,d})\mathbb{1}_{\mathcal{O}_{i,d}} & \text{in } Q \\ y(0) = y_0, \quad \phi^i(T) = 0 & \text{in } (0,L) \end{cases}$$

- Followers: $v_1 = -\dfrac{1}{\mu_1}\phi^1$ and $v_2 = -\dfrac{1}{\mu_2}\phi^2$.
- Leader: $y(T) = 0$.

Result

Theorem[1]

Assume:

- $\mu_i >> 1$ (existence of Nash equilibrium).
- $\displaystyle\iint_{\mathcal{O}_{i,d}} \rho(t)^2 |y_{i,d}|^2 \, \mathrm{d}x \, \mathrm{d}t < +\infty$, with $\displaystyle\lim_{t\to T} \rho(t) = +\infty$.
- $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$, $i = 1, 2$ (No assumptions on $\mathcal{O}_i$).

There exist (a leader control) $f \in L^2(\mathcal{O} \times (0,T))$ and a Nash equilibrium for J_i (followers) $(v_1(f), v_2(f))$ such that $y(T) = 0$ in Ω.

- On the assumption $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$. Two cases:

1. $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$.
2. $\mathcal{O}_{1,d} \neq \mathcal{O}_{2,d}$: $\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}$ (different inside $\mathcal{O}$).

[1] C., Santos. Stackelberg-Nash exact controllability for the Kuramoto-Sivashinsky equation. *Submitted*, 2017. Available at http://ncarreno.mat.utfsm.cl

Adjoint system

$$\begin{cases} -\psi_t + \psi_{xxxx} = \alpha_1 \gamma^1 \mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2 \gamma^2 \mathbb{1}_{\mathcal{O}_{2,d}} & \text{in } Q \\ \gamma^i_t + \gamma^i_{xxxx} = -\dfrac{1}{\mu_i} \psi \mathbb{1}_{\mathcal{O}_i} & \text{in } Q \\ \psi(T) = \psi_T, \quad \gamma^i(0) = 0 & \text{in } (0, L) \end{cases}$$

Observability inequality:

$$\int_\Omega |\psi(0)|^2 \, dx + \sum_{i=1,2} \iint_Q \rho(t)^{-2} |\gamma^i|^2 \, dx \, dt \leq C \iint_{\mathcal{O} \times (0,T)} |\psi|^2 \, dx \, dt$$

Main tools:

- Carleman estimates for the fourth-order operator $\pm u_t + u_{xxxx}$.
- Energy estimates.

Observability inequality: general idea

$$\begin{cases} -\psi_t + \psi_{xxxx} = \alpha_1 \gamma^1 \mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2 \gamma^2 \mathbb{1}_{\mathcal{O}_{2,d}} & \text{in } Q \\ \gamma^i_t + \gamma^i_{xxxx} = -\dfrac{1}{\mu_i} \psi \mathbb{1}_{\mathcal{O}_i} & \text{in } Q \end{cases}$$

Classic approach: Fix $\omega \subset\subset \mathcal{O}_{i,d} \cap \mathcal{O}$.

- Carleman estimate for ψ, γ^1 and γ^2:

$$I(\psi) + I(\gamma^1) + I(\gamma^2) \leq C \iint_{\omega \times (0,T)} \rho(|\psi|^2 + |\gamma^1|^2 + |\gamma^2|^2) \, dx \, dt.$$

  Here, $I(\cdot)$ is the weighted energy, and ρ is the weight with critical points only in ω.
- Write γ^1 and γ^2 in terms of ψ using the coupling in $\mathcal{O}_{i,d} \cap \mathcal{O}$, $i = 1, 2$.
- Problem: we have a "loop"

$$I_\omega(\gamma^1) \lesssim I_\omega(\psi) + I_\omega(\gamma^2)$$

$$I_\omega(\gamma^2) \lesssim I_\omega(\psi) + I_\omega(\gamma^1)$$

Observability inequality. Case $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$

- Solution 1: If $\mathcal{O}_{1,d} = \mathcal{O}_{2,d} = \mathcal{O}_d$, let $h := \alpha_1\gamma^1 + \alpha_2\gamma^2$.

$$\begin{cases} -\psi_t + \psi_{xxxx} = h\mathbb{1}_{\mathcal{O}_d} & \text{in } Q \\ h_t + h_{xxxx} = -\dfrac{1}{\mu_1}\psi\mathbb{1}_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\psi\mathbb{1}_{\mathcal{O}_2} & \text{in } Q \end{cases}$$

Same approach:

- $\omega \subset\subset \mathcal{O}_d \cap \mathcal{O}$

$$I(\psi) + I(h) \leq C\big(I_\omega(\psi) + I_\omega(h)\big).$$

- Using the equation: $I_\omega(h) \lesssim I_\omega(\psi)$.
- From energy estimates

$$\int_\Omega |\psi(0)|^2 \,\mathrm{d}x + I(\psi) + I(h) \leq C I_\omega(\psi)$$

Observability inequality. Case $\mathcal{O}_{1,d} \neq \mathcal{O}_{2,d}$

Suppose $\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}$.

$$\begin{cases} -\psi_t + \psi_{xxxx} = \alpha_1 \gamma^1 \mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2 \gamma^2 \mathbb{1}_{\mathcal{O}_{2,d}} & \text{in } Q \\ \gamma^i_t + \gamma^i_{xxxx} = -\dfrac{1}{\mu_i} \psi \mathbb{1}_{\mathcal{O}_i} & \text{in } Q \end{cases}$$

• Solution 2: A way around the "loop situation": Use two different weight functions (associated to ω_1 and ω_2).

- Carleman estimate for $\omega_1 \subset\subset \mathcal{O}_{1,d} \cap \mathcal{O}$, and $\omega_1 \cap \mathcal{O}_{2,d} \neq \emptyset$.
- Carleman estimate for $\omega_2 \subset\subset \mathcal{O}_{2,d} \cap \mathcal{O}$, and $\omega_2 \cap \mathcal{O}_{1,d} \neq \emptyset$.
- This way, they "do not see" each other:

$$I^1_{\omega_1}(\gamma^1) \lesssim I^1_{\omega_1}(\psi) \text{ and } I^2_{\omega_2}(\gamma^2) \lesssim I^2_{\omega_2}(\psi).$$

• This idea is due to S. Guerrero and M. C. Santos.
• **Important:** Weight functions should be equal outside $\mathcal{O}$, so we can compare the global terms coming from the Carleman estimates.

Outline

Null controllability of the Navier-Stokes system

- $\Omega \subset \mathbb{R}^3$. Fluid contained in Ω.

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = v\mathbb{1}_\omega, \quad \nabla\cdot y = 0 & (x,t)\in\Omega\times(0,T), \\ y = 0 & x\in\partial\Omega \\ y(0) = y^0 & x\in\Omega. \end{cases}$$

- $y = y(x,t)$: Velocity field.
- $v = (v_1, v_2, v_3)$ is the control.

- Controls of the form $v = (0, v_2, v_3)$.
- Local result: There exists $\delta > 0$ such that if $\|y^0\| \le \delta$, then there is a control of the form $v = (0, v_2, v_3)$ and an associated solution (y,p) such that

$$y(T) = 0.$$

Null controllability of the Boussinesq system

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = v\mathbb{1}_\omega + (0,0,\theta), \quad \nabla\cdot y = 0 & (x,t)\in\Omega\times(0,T),\\ \theta_t - \Delta\theta + y\cdot\nabla\theta = v_0\mathbb{1}_\omega & (x,t)\in\Omega\times(0,T),\\ y=0, \quad \theta = 0 & x\in\partial\Omega\\ y(0)=y^0, \quad \theta(0)=\theta^0 & x\in\Omega. \end{cases}$$

- $y = y(x,t)$: Velocity field.
- $\theta = \theta(x,t)$: Temperature of the fluid.

- Local result: There exists $\delta > 0$ such that if $\|(y^0,\theta^0)\| \leq \delta$, there are controls v_0 and $v = (v_1, 0, 0)$ and an associated solution (y,p,θ) such that

$$y(T) = 0 \text{ and } \theta(T) = 0.$$

Thank you

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