Control of parabolic systems and application to a hierarchical control problem

Nicolás Carreño
Universidad Técnica Federico Santa María

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Outline

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PDE control

Example: Heat equation

Consider a regular open $\Omega \subset \mathbb{R}^N$ and $\omega \subset \Omega$ (control domain)

$$\begin{cases} y_t - \Delta y = v \mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ y = 0 & x \in \partial\Omega, \\ y(0) = y_0 & x \in \Omega. \end{cases}$$

- $y = y(x,t)$: Temperature distribution.
- $v = v(x,t)$: Control supported in ω.

Question: Given $T > 0$ and $y_1 = y_1(x)$, is there v such that $y(T) = y_1$?

PDE control

Answer: In general, the answer is no due to the *regularizing effect*.

• It seems natural to consider the notion of control to the trajectories:
Consider a solution of

$$\begin{cases} \bar{y}_t - \Delta \bar{y} = 0 & (x,t) \in \Omega \times (0,T), \\ \bar{y} = 0 & x \in \partial\Omega, \\ \bar{y}(0) = \bar{y}_0 & x \in \Omega, \end{cases}$$

We look for a control v such that $y(T) = \bar{y}(T)$.

• By linearity (taking $\widetilde{y} := y - \bar{y}$), this is equivalent to the null controllability:

$$y(T) = 0.$$

Therefore, we concentrate in this case.

Duality Method: Hilbert Uniqueness Method (HUM)

Construction of the control:

- We multiply $y_t - \Delta y = v\mathbb{1}_\omega$ by φ solution to the (adjoint) equation

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & (x,t) \in \Omega \times (0,T), \\ \varphi = 0 & x \in \partial\Omega, \\ \varphi(T) = \varphi_T \in L^2(\Omega) & x \in \Omega, \end{cases}$$

and integrate in $\Omega \times (0,T)$:

$$\int_\Omega y(T)\varphi_T \,\mathrm{d}x = \iint_{\omega\times(0,T)} v\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega y_0\varphi(0)\,\mathrm{d}x, \quad \forall \varphi_T \in L^2(\Omega).$$

- v is a control such that $y(T) = 0$ if and only if

$$\iint_{\omega\times(0,T)} v\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega y_0\varphi(0)\,\mathrm{d}x = 0, \quad \forall \varphi_T \in L^2(\Omega).$$

Observability inequality

The previous condition can be seen as an optimality condition for

$$J(\varphi_T) = \frac{1}{2} \iint\limits_{\omega\times(0,T)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega y_0\varphi(0)\,\mathrm{d}x.$$

- J convex, continuous and coercive if there exists $C > 0$ such that

$$\int_\Omega |\varphi(0)|^2 \,\mathrm{d}x \leq C \iint\limits_{\omega\times(0,T)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

This is known as observability inequality.

- The control is given by

$$v := \widehat{\varphi},$$

where $\widehat{\varphi}$ is the solution of the adjoint equation associated to $\widehat{\varphi}_T$, minimum of J.

- Null controllability is equivalent to observability.

Carleman estimates

How to prove the observability inequality?

Powerful tool to prove observability: Carleman estimates

$$\iint_{\Omega\times(0,T)} \rho|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C \iint_{\Omega\times(0,T)} \rho|\varphi_t + \Delta\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + C \iint_{\omega_0\times(0,T)} \rho|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t$$

- $\varphi(x,t) = 0$, $x \in \partial\Omega$.
- $\rho = \rho(x,t)$ is a positive function and continuous in $\overline{\Omega} \times (0,T)$ with critical points only in $\omega_0 \subset \omega$.
- To deduce the observability, we use dissipation properties as

$$\int_\Omega |\varphi(0)|^2 \,\mathrm{d}x \leq \int_\Omega |\varphi(t)|^2 \,\mathrm{d}x, \quad t \in (0,T).$$

Control of a system of two equations with one control

Consider the system with one scalar control

$$\begin{cases} y_t - \Delta y = z + v\mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ z_t - \Delta z = y\mathbb{1}_\mathcal{O} & (x,t) \in \Omega \times (0,T), \\ y = z = 0 & x \in \partial\Omega, \\ y(0) = y^0, \quad z(0) = z^0 & x \in \Omega. \end{cases}$$

- We look for v such that $y(T) = z(T) = 0$.
- Observability inequality: There exists $C > 0$ such that

$$\int_\Omega \left(|\varphi(0)|^2 + |\psi(0)|^2\right) \mathrm{d}x \leq C \iint_{\omega\times(0,T)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t$$

where (φ, ψ) is the solution to the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = \psi\mathbb{1}_\mathcal{O} & (x,t) \in \Omega \times (0,T), \\ -\psi_t - \Delta\psi = \varphi & (x,t) \in \Omega \times (0,T), \\ \varphi = \psi = 0 & x \in \partial\Omega, \\ \varphi(T) = \varphi_T, \quad \psi(T) = \psi_T & x \in \Omega. \end{cases}$$

Control of a system of two equations with one control

- The idea is to combine Carleman estimates for φ and ψ:

$$\iint_{\Omega\times(0,T)} \rho_1|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C \iint_{\Omega\times(0,T)} \rho_2|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + C \iint_{\omega_0\times(0,T)} \rho_1|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t$$

$$\iint_{\Omega\times(0,T)} \rho_1|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C \iint_{\Omega\times(0,T)} \rho_2|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + C \iint_{\omega_0\times(0,T)} \rho_1|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t$$

- Estimate the local term of ψ: $\psi = -\varphi_t - \Delta\varphi$ in $\mathcal{O}$.
- We assume $\omega \cap \mathcal{O} \neq \emptyset$ and choose $\omega_0 \subset \Omega \cap \mathcal{O}$.

$$\iint_{\omega_0\times(0,T)} \rho_1|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t = \iint_{\omega_0\times(0,T)} \rho_1\psi(-\varphi_t - \Delta\varphi) \,\mathrm{d}x\,\mathrm{d}t$$

$$\leq \frac{1}{2C} \iint_{\omega_0\times(0,T)} \rho_1|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + C \iint_{\omega_0\times(0,T)} \rho_1|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Outline

Mono-objective vs Multi-objective

- Standard control problem:

$$\begin{cases} y_t + \mathcal{A}(y) = f \mathbb{1}_{\mathcal{O}} & \text{in } \Omega \times (0,T) \\ y(0) = y_0 & \text{in } \Omega \end{cases}$$

  - $f \longrightarrow y(T) = 0$ in Ω.

- Control problem with more agents:

$$\begin{cases} y_t + \mathcal{A}(y) = f \mathbb{1}_{\mathcal{O}} + v_1 \mathbb{1}_{\mathcal{O}_1} + v_2 \mathbb{1}_{\mathcal{O}_2} & \text{in } \Omega \times (0,T) \\ y(0) = y_0 & \text{in } \Omega \end{cases}$$

  - $f \longrightarrow y(T) = 0$ in Ω.
  - $v_1 \longrightarrow y \approx y_{1,d}$ in $\mathcal{O}_{1,d} \subset \Omega$.
  - $v_2 \longrightarrow y \approx y_{2,d}$ in $\mathcal{O}_{2,d} \subset \Omega$.

Motivation: resort lake

$$\begin{cases} y_t + \mathcal{A}(y) = f \mathbb{1}_{\mathcal{O}} + v_1 \mathbb{1}_{\mathcal{O}_1} + v_2 \mathbb{1}_{\mathcal{O}_2} & \text{in } \Omega \times (0,T) \\ y(0) = y_0 & \text{in } \Omega \end{cases}$$

- Lake represented by $\Omega \subset \mathbb{R}^3$.
- $y = y(x,t)$: concentration of chemicals or of living organisms in the lake.
- Local agents P_1, P_2 that can decide their policy v_1, v_2 acting on $\mathcal{O}_1$, $\mathcal{O}_2$ (The followers).
- The manager of the resort decides the policy f acting on $\mathcal{O}$ (The leader).
- Goal of the manager: "Clean" the lake at time T ($y(T) = 0$).
- Goal of the agents: To be close to a target concentration $y_{i,d}$ in $\mathcal{O}_{i,d} \subset \Omega$ during the time interval $(0,T)$ ($y \approx y_{i,d}$ in $\mathcal{O}_{i,d}$).

A linear fourth-order equation

Let $Q := (0,L)\times(0,T)$, f the leader, v_1 and v_2 the followers.

$$\begin{cases} y_t + y_{xxxx} = f\mathbb{1}_{\mathcal{O}} + v_1\mathbb{1}_{\mathcal{O}_1} + v_2\mathbb{1}_{\mathcal{O}_2} & \text{in } Q \\ y_{|x=0} = y_{x|x=0} = 0 = y_{|x=L} = y_{x|x=L} & \text{in } (0,T) \\ y(0) = y_0 & \text{in } (0,L) \end{cases}$$

Consider the functionals $(i=1,2)$: $\alpha_i > 0, \mu_i > 0$

$$J_i(f; v_1, v_2) = \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\mu_i}{2} \iint_{\mathcal{O}_i\times(0,T)} |v_i|^2 \,\mathrm{d}x\,\mathrm{d}t$$

- Task of the followers: $y \approx y_{i,d}$ in $\mathcal{O}_{i,d}$ with "little effort"

$$\min J_i(f; v_1, v_2), \quad i = 1,2.$$

- Task of the leader: $y(T) = 0$ in Ω.

The Stackelberg-Nash strategy

$$\begin{cases} y_t + y_{xxxx} = f 1\!\!1_{\mathcal{O}} + v_1 1\!\!1_{\mathcal{O}_1} + v_2 1\!\!1_{\mathcal{O}_2} & \text{in } Q \\ y_{|x=0} = y_{x|x=0} = 0 = y_{|x=L} = y_{x|x=L} & \text{in } (0,T) \\ y(0) = y_0 & \text{in } (0,L) \end{cases}$$

- **Step 1:** For fixed f, find a Nash equilibrium (v_1, v_2):

$$J_1(f; v_1, v_2) = \min_{\hat{v}_1} J_1(f; \hat{v}_1, v_2), \quad J_2(f; v_1, v_2) = \min_{\hat{v}_2} J_2(f; v_1, \hat{v}_2)$$

Of course, this equilibrium depends on f: $v_1 = v_1(f)$, $v_2 = v_2(f)$.

- **Step 2:** Find f such that $y(T) = 0$.

This is the *Stackelberg-Nash strategy*.

Optimality system

$$\begin{cases} y_t + y_{xxxx} = f\mathbb{1}_{\mathcal{O}} + v_1\mathbb{1}_{\mathcal{O}_1} + v_2\mathbb{1}_{\mathcal{O}_2} & \text{in } Q \\ y_{|x=0} = y_{x|x=0} = 0 = y_{|x=L} = y_{x|x=L} & \text{in } (0,T) \\ y(0) = y_0 & \text{in } (0,L) \end{cases}$$

- Nash equilibrium is equivalent to (due to the convexity of J_i)

$$\begin{cases} J_1'(f; v_1, v_2)(\hat{v}_1, 0) = 0 & \forall \hat{v}_1 \in L^2(\mathcal{O}_1 \times (0,T)) \\ J_2'(f; v_1, v_2)(0, \hat{v}_2) = 0 & \forall \hat{v}_2 \in L^2(\mathcal{O}_2 \times (0,T)) \end{cases}$$

- Characterization of Nash equilibrium: Optimality system

$$\begin{cases} y_t + y_{xxxx} = f\mathbb{1}_{\mathcal{O}} - \frac{1}{\mu_1}\phi^1\mathbb{1}_{\mathcal{O}_1} - \frac{1}{\mu_2}\phi^2\mathbb{1}_{\mathcal{O}_2} & \text{in } Q \\ -\phi^1_t + \phi^1_{xxxx} = \alpha_1(y - y_{1,d})\mathbb{1}_{\mathcal{O}_{1,d}} & \text{in } Q \\ -\phi^2_t + \phi^2_{xxxx} = \alpha_2(y - y_{2,d})\mathbb{1}_{\mathcal{O}_{2,d}} & \text{in } Q \\ y(0) = y_0, \quad \phi^i(T) = 0 & \text{in } (0,L) \end{cases}$$

  - Followers: $v_1 = -\frac{1}{\mu_1}\phi^1$ and $v_2 = -\frac{1}{\mu_2}\phi^2$.
  - Leader: $y(T) = 0$.

Result

Theorem[1]

Assume:

- $$\iint_{\mathcal{O}_{i,d}\times(0,T)} \rho(t)^2|y_{i,d}|^2\,\mathrm{d}x\,\mathrm{d}t < +\infty, \text{ with } \lim_{t\to T}\rho(t) = +\infty.$$
- $\mathcal{O}_{i,d}\cap\mathcal{O}\neq\emptyset$, $i=1,2$ (No assumptions on $\mathcal{O}_i$).

There exist (a leader control) $f\in L^2(\mathcal{O}\times(0,T))$ and a Nash equilibrium for J_i (followers) $(v_1(f), v_2(f))$ such that $y(T)=0$ in Ω.

[1] C., Santos. Stackelberg-Nash exact controllability for the Kuramoto-Sivashinsky equation. *Submitted*, 2017. Available at http://ncarreno.mat.utfsm.cl

Adjoint system

$$\begin{cases} -\psi_t + \psi_{xxxx} = \alpha_1 \gamma^1 \mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2 \gamma^2 \mathbb{1}_{\mathcal{O}_{2,d}} & \text{in } Q \\ \gamma^1_t + \gamma^1_{xxxx} = -\dfrac{1}{\mu_1} \psi \mathbb{1}_{\mathcal{O}_1} & \text{in } Q \\ \gamma^2_t + \gamma^2_{xxxx} = -\dfrac{1}{\mu_2} \psi \mathbb{1}_{\mathcal{O}_2} & \text{in } Q \\ \psi(T) = \psi_T, \quad \gamma^1(0) = \gamma^2(0) = 0 & \text{in } (0, L) \end{cases}$$

Observability inequality:

$$\int_\Omega |\psi(0)|^2 \, \mathrm{d}x + \sum_{i=1,2} \iint_Q \rho(t)^{-2} |\gamma^i|^2 \, \mathrm{d}x \, \mathrm{d}t \leq C \iint_{\mathcal{O} \times (0,T)} |\psi|^2 \, \mathrm{d}x \, \mathrm{d}t$$

Main tools:

- Carleman estimates for the fourth-order operator $\pm u_t + u_{xxxx}$.
- Energy estimates.

Observability inequality: general idea

$$\begin{cases} -\psi_t + \psi_{xxxx} = \alpha_1 \gamma^1 1_{\mathcal{O}_{1,d}} + \alpha_2 \gamma^2 1_{\mathcal{O}_{2,d}} & \text{in } Q \\ \gamma^1_t + \gamma^1_{xxxx} = -\dfrac{1}{\mu_1} \psi 1_{\mathcal{O}_1} & \text{in } Q \\ \gamma^2_t + \gamma^2_{xxxx} = -\dfrac{1}{\mu_2} \psi 1_{\mathcal{O}_2} & \text{in } Q \end{cases}$$

Classic approach: Fix $\omega_0 \subset \mathcal{O}_{i,d} \cap \mathcal{O}$, $i = 1, 2$.

- Carleman estimate for ψ, γ^1 and γ^2:

$$I(\psi) + I(\gamma^1) + I(\gamma^2) \leq C \iint_{\omega_0 \times (0,T)} \rho(|\psi|^2 + |\gamma^1|^2 + |\gamma^2|^2) \, dx \, dt.$$

  Here, $I(\cdot)$ is the weighted energy and ρ is the weight function.
- Write γ^1 and γ^2 in terms of ψ using the coupling in $\mathcal{O}_{i,d} \cap \mathcal{O}$, $i = 1, 2$.
- Problem: we have a "loop"

$$I_{\omega_0}(\gamma^1) \lesssim I_{\omega_0}(\psi) + I_{\omega_0}(\gamma^2)$$

$$I_{\omega_0}(\gamma^2) \lesssim I_{\omega_0}(\psi) + I_{\omega_0}(\gamma^1)$$

Observability inequality: solutions to the "loop situation"

- Solution 1: If $\mathcal{O}_{1,d} = \mathcal{O}_{2,d} = \mathcal{O}_d$, let $h := \alpha_1\gamma^1 + \alpha_2\gamma^2$.

$$\begin{cases} -\psi_t + \psi_{xxxx} = h\mathbb{1}_{\mathcal{O}_d} & \text{in } Q \\ h_t + h_{xxxx} = -\dfrac{1}{\mu_1}\psi\mathbb{1}_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\psi\mathbb{1}_{\mathcal{O}_2} & \text{in } Q \end{cases}$$

Use Carleman with $\omega_0 \subset \mathcal{O}_d \cap \mathcal{O}$.

Observability inequality: solutions to the "loop situation"

$$\begin{cases} -\psi_t + \psi_{xxxx} = \alpha_1 \gamma^1 \mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2 \gamma^2 \mathbb{1}_{\mathcal{O}_{2,d}} & \text{in } Q \\ \gamma^1_t + \gamma^1_{xxxx} = -\dfrac{1}{\mu_1} \psi \mathbb{1}_{\mathcal{O}_1} & \text{in } Q \\ \gamma^2_t + \gamma^2_{xxxx} = -\dfrac{1}{\mu_2} \psi \mathbb{1}_{\mathcal{O}_2} & \text{in } Q \end{cases}$$

• Solution 2: Suppose $\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}$. Use two different weight functions (associated to ω_1 and ω_2).

- Carleman estimate for $\omega_1 \subset\subset \mathcal{O}_{1,d} \cap \mathcal{O}$, and $\omega_1 \cap \mathcal{O}_{2,d} \neq \emptyset$.
- Carleman estimate for $\omega_2 \subset\subset \mathcal{O}_{2,d} \cap \mathcal{O}$, and $\omega_2 \cap \mathcal{O}_{1,d} \neq \emptyset$.
- This way, γ^1 and γ^2 "do not see" each other.

$$I^1_{\omega_1}(\gamma^1) \lesssim I^1_{\omega_1}(\psi) \text{ and } I^2_{\omega_2}(\gamma^2) \lesssim I^2_{\omega_2}(\psi).$$

Thank you