Insensitizing controls with vanishing components for the Boussinesq system

LXXXIII Encuentro Anual - Sociedad de Matemática de Chile
Sesión Problemas Inversos y de Control de EDP

Nicolás Carreño Godoy
Universidad Técnica Federico Santa María

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Outline

Insensitizing controls

- Ω bounded connected regular open subset of $\mathbb{R}^N$ ($N = 2$ or 3)
- $T > 0$
- $\omega \subset \Omega$ (control set), $Q := \Omega \times (0,T)$, $\Sigma := \partial\Omega \times (0,T)$

We consider the Boussinesq system:

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = f + v\mathbb{1}_\omega + (0,0,\theta), \quad \nabla\cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta\theta + y\cdot\nabla\theta = f_0 + v_0\mathbb{1}_\omega & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0 + \tau\widehat{y}_0, \quad \theta(0) = \theta^0 + \tau\widehat{\theta}_0 & \text{in } \Omega. \end{cases}$$

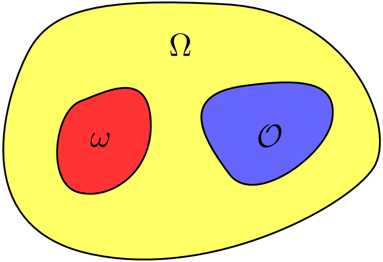
where τ is a small constant and $\|\widehat{y}^0\|_{L^2(\Omega)^3} = \|\widehat{\theta}^0\|_{L^2(\Omega)} = 1$. Unknown.

Insensitizing control problem: To find controls v and v_0 in $L^2(\omega \times (0,T))$ such that the functional (Sentinel)

$$J_\tau(y,\theta) := \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} \left(|y|^2 + |\theta|^2\right) \mathrm{d}x\, \mathrm{d}t, \ \mathcal{O} \subset \Omega \ \text{(Observation set)}$$

is not affected by the uncertainty of the initial data, that is,

$$\left.\frac{\partial J_\tau(y,\theta)}{\partial \tau}\right|_{\tau=0} = 0 \quad \forall\, (\widehat{y}_0, \widehat{\theta}_0) \in L^2(\Omega)^4 \text{ s.t. } \|\widehat{y}_0\|_{L^2(\Omega)^3} = \|\widehat{\theta}_0\|_{L^2(\Omega)} = 1.$$



A cascade system

The previous condition is equivalent to the following null controllability problem: To find controls v and v_0 such that $z(0) = 0$ and $q(0) = 0$, where

$$\left\{\begin{array}{rcll} w_t - \Delta w + (w \cdot \nabla) w + \nabla p_0 & = & f + v \mathbb{1}_\omega + (0,0,r), \quad \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + (z \cdot \nabla^t) w - (w \cdot \nabla) z + \nabla p_1 & = & w \mathbb{1}_\mathcal{O}, \quad \nabla \cdot z = 0 & \text{in } Q, \\ r_t - \Delta r + (w \cdot \nabla) r & = & f_0 + v_0 \mathbb{1}_\omega & \text{in } Q, \\ -q_t - \Delta q - (w \cdot \nabla) q & = & z_3 + r \mathbb{1}_\mathcal{O} & \text{in } Q, \end{array}\right.$$

with boundary and initial conditions:

$$\left\{\begin{array}{ll} w = z = 0, \quad r = q = 0 & \text{on } \Sigma, \\ w(0) = y^0, \quad z(T) = 0, \quad r(0) = \theta^0, \quad q(T) = 0 & \text{in } \Omega. \end{array}\right.$$

We are interested in controls of the form

1. $v = (v_1, 0, 0)$, $v_0 \neq 0$
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Null controllability results

Assume:

- $y^0 = 0$, $\theta^0 = 0$
- $\mathcal{O} \cap \omega \neq \emptyset$
- $\|e^{K/t^{10}} f\|_{L^2(Q)^3} < +\infty$, $\|e^{K/t^{10}} f_0\|_{L^2(Q)} < +\infty$, some $K > 0$

Theorem (Guerrero, Gueye, C.)

There exists $\delta > 0$ such that if $\|e^{K/t^{10}}(f, f_0)\|_{L^2(Q)^4} < \delta$, there exist a controls (v, v_0) in $L^2(\omega \times (0,T))$ of the form $v = (v_1, 0, 0)$, $v_0 \neq 0$ such that $z(0) = 0$ and $q(0) = 0$.

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Method of proof

- Linearization around zero
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Adjoint system and observability inequality

Dual variables: $\varphi \leftrightarrow w, \quad \psi \leftrightarrow z, \quad \phi \leftrightarrow r, \quad \sigma \leftrightarrow q$

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_\varphi = g^\varphi + \psi\,\mathbb{1}_\mathcal{O}, & \nabla\cdot\varphi = 0 & \text{in } Q, \\ \psi_t - \Delta\psi + \nabla\pi_\psi = g^\psi + (0,0,\sigma), & \nabla\cdot\psi = 0 & \text{in } Q, \\ -\phi_t - \Delta\phi = g^\phi + \varphi_3 + \sigma\,\mathbb{1}_\mathcal{O} & & \text{in } Q, \\ \sigma_t - \Delta\sigma = g^\sigma & & \text{in } Q, \end{cases}$$

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For general controls $v = (v_1, v_2, v_3)$ and v_0:

$$\iint_Q \rho_1(t)(|\varphi|^2+|\psi|^2+|\phi|^2+|\sigma|^2) \leq C \iint_Q \rho_2(t)(|g^\varphi|^2+|g^\psi|^2+|g^\phi|^2+|g^\sigma|^2)$$
$$+ C \iint_{\omega\times(0,T)} \rho_3(t)(|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 + |\phi|^2)$$

$\rho_i(t) \sim \exp(-C_i/t^{10}(T-t)^{10})$

Using energy estimate, we can change to $\rho_i(t) \sim \exp(-C_i/t^{10})$

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- Eliminate φ_3 using: $\quad \varphi_3 = -\phi_t - \Delta\phi - g^\phi - \sigma \quad$ in $\omega\cap\mathcal{O}$.
- At this point we have local terms of φ_1, ϕ and global terms of σ. Carleman for σ, but cannot have a local term like σ.

$$(\partial_1^2 + \partial_2^2)\sigma = -(\partial_t^2 - \Delta^2)\Delta\varphi_3 + F(g^\varphi, g^\psi, g^\sigma) \quad \text{in } \omega\cap\mathcal{O}$$

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$$(\partial_1^2+\partial_2^2)\sigma = -(\partial_t^2-\Delta^2)\Delta\varphi_3 + F(g^\varphi, g^\psi, g^\sigma) \quad \text{in } \omega\cap\mathcal{O}$$

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- Carleman with a local term like $(\partial_1^2+\partial_2^2)\sigma$.

Sketch of proof. Case: $v = (v_1, 0, 0)$ and v_0

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\begin{cases}
-\varphi_t - \Delta\varphi + \nabla\pi_\varphi = g^\varphi + \psi\,1\!\!1_{\mathcal{O}}, & \nabla\cdot\varphi = 0 \quad \text{in } Q,\\
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- At this point, it only remains to add to the left-hand side the weighted norm of ϕ.
- Cannot add a local term of ϕ. No way to eliminate with coupling. Instead, we use energy estimates with weights like $\rho(t) = \exp(-C/t^{10})$:

$$\begin{cases} -(\rho\phi)_t - \Delta(\rho\phi) = \rho g^\phi + \rho\varphi_3 + \rho\sigma \mathbb{1}_{\mathcal{O}} - \rho'(t)\phi \\ (\rho\phi)_{|\Sigma} = 0, \quad \boxed{(\rho\phi)(T) = 0} \end{cases}$$

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Perspectives

- Our method limits the quantity of vanishing components to two. Also, we need to have v_3 or v_0
- What about three vanishing components, e.g., $v = (0,0,0)$ and v_0? One possibility: use the Return method.
- On going work: Insensitize the functional

$$J_\tau(y,\theta) := \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} \left(|\nabla \times y|^2 + |\nabla \theta|^2 \right) \mathrm{d}x\, \mathrm{d}t, \; \mathcal{O} \subset \Omega.$$

Adjoint equation:

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Some references

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Available on: https://www.ljll.math.upmc.fr/∼ncarreno

Thank you for your attention!