Some controllability results with a reduced number of controls via Carleman estimates

Journée Interne LJLL

Nicolás Carreño
(Thesis advisor: Sergio Guerrero)

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Outline

Framework:

- Ω bounded connected regular open subset of $\mathbf{R}^N$ ($N = 2$ or 3)
- $T > 0$
- $\omega \subset \Omega$ (control set), $Q := \Omega \times (0, T)$, $\Sigma := \partial\Omega \times (0, T)$

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = v 1\!\!1_\omega, \ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \qquad \text{(NS)}$$

where v stands for the control which acts over the set ω.

Controllability problem: Find a control $v \in L^2(\omega \times (0, T))^N$ such that

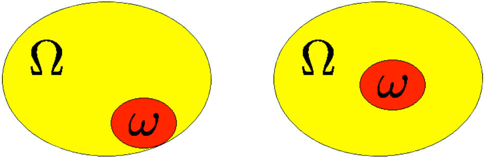
- $y(T) = 0$ (null controllability),
- $y(T) = \bar{y}(T)$, where $\bar{y}$ is a solution of the uncontrolled equation (controllability to trajectories).

If $\|y^0\|$ or $\|y^0 - \bar{y}(0)\|$ are required to be small: local controllability.

Our goal: To find a control $v \in L^2(\omega \times (0,T))^N$, with $v_i \equiv 0$ $(i \in \{1, \dots, N\})$.

Some previous results:

- [Fernández-Cara, Guerrero, Imanuvilov, Puel, 2006]: Local exact controllability to the trajectories of the N-S and Boussinesq system when $\overline{\omega} \cap \partial\Omega \neq \phi$
- [Coron, Guerrero, 2009] Null controllability of the Stokes system for a general $\omega \subset \Omega$.



We are interested in the general case for the nonlinear problem.

Carleman estimates and controllability

Consider the Stokes system and its adjoint:

$$\begin{cases} y_t - \Delta y + \nabla p = (v_1, v_2, 0)\mathbb{1}_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \qquad \begin{cases} -\varphi_t - \Delta \varphi + \nabla \pi = 0, & \text{in } Q, \\ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T & \text{in } \Omega. \end{cases}$$

Null controllability is equivalent to the Observability inequality

$$\int_\Omega |\varphi(0)|^2 \, dx \leq C \iint_{\omega\times(0,T)} (|\varphi_1|^2 + |\varphi_2|^2) dx\, dt.$$

Important tool: Carleman estimates

$$\iint_Q \rho_1(x,t)|\varphi|^2 dx\, dt \leq C \iint_{\omega\times(0,T)} \rho_2(x,t)(|\varphi_1|^2 + |\varphi_2|^2) dx\, dt$$

ρ_1, ρ_2 some positive weight functions.

Method introduced in [Coron, Guerrero, 2009]:

- $\Delta\pi = 0$,
- Look at the equations satisfied by $\nabla\Delta\varphi_1$ and $\nabla\Delta\varphi_2$, and apply Carleman estimates to them,
- Use $\nabla \cdot \varphi = 0$ to recover φ_3 in the LHS.

Remark: When applying the operator $\nabla\Delta$, we lose the boundary conditions. Special Carleman estimates are needed:

- [Fernández-Cara, González-Burgos, Guerrero, Puel, 2006]
- [Imanuvilov, Puel, Yamamoto, 2009]

The nonlinear problem

Linearized system around 0:

$$\begin{cases} y_t - \Delta y + \nabla p = f + (v_1, v_2, 0)\mathbb{1}_\omega, \ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \qquad \text{(L)}$$

where f is taken to decrease exponentially to zero in $t = T$.

Principle:

> Null controllability for (L) + Inverse mapping theorem
> $\Rightarrow$ Local null controllability of (NS)

Need a Carleman inequality for the nonhomogeneous adjoint system:

$$\begin{cases} -\varphi_t - \Delta \varphi + \nabla \pi = g, \ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T & \text{in } \Omega, \end{cases} \qquad \text{(A)}$$

where $g \in L^2(Q)^3$ and $\varphi^T \in L^2(\Omega)^3$.

Local null controllability of (NS)

Proposition: Carleman inequality

There exists a constant $C > 0$ (depending on Ω, ω, T)

$$\iint_Q \rho_1(t)|\varphi|^2 \leq C \left(\iint_Q \rho_2(t)|g|^2 + \iint_{\omega\times(0,T)} \rho_3(t)(|\varphi_1|^2 + |\varphi_2|^2) \right)$$

for every φ solution of the adjoint system.

Theorem (joint work with S. Guerrero)

Let $i \in \{1, \ldots, N\}$. Then, for every $T > 0$ and $\omega \subset \Omega$, system (NS) is locally null controllable by a control $v \in L^2(\omega \times (0, T))$, with $v_i \equiv 0$.

Boussinesq system

We consider the Boussinesq system:

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = v 1\!\!1_\omega + \theta\, e_N,\ \nabla \cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta \theta + y \cdot \nabla \theta = v_0 1\!\!1_\omega & \text{in } Q, \\ y = 0,\ \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0,\ \theta(0) = \theta^0 & \text{in } \Omega, \end{cases} \tag{B}$$

Goal: To find controls $v \in L^2(\omega \times (0,T))^N$, with $v_i \equiv 0$ $(i < N)$, $v_N \equiv 0$, and $v_0 \in L^2(\omega \times (0,T))$ such that

$$y(T) = 0 \text{ and } \theta(T) = \bar\theta(T),$$

where

$$\begin{cases} \nabla \bar p = \bar\theta\, e_N & \text{in } Q, \\ \bar\theta_t - \Delta \bar\theta = 0 & \text{in } Q, \\ \bar\theta = 0 \text{ on } \Sigma, \bar\theta(0) = \bar\theta^0 & \text{in } \Omega, \end{cases}$$

i.e., local controllability to the trajectory $(0, \bar p, \bar\theta)$.

Linearized system around $(0, \bar{p}, \bar{\theta})$:

$$\begin{cases} y_t - \Delta y + \nabla p = f + (v_1, 0, 0)\mathbb{1}_\omega + \theta\, e_3, \nabla \cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta \theta + y \cdot \nabla \bar{\theta} = f_0 + v_0 \mathbb{1}_\omega & \text{in } Q, \\ y = 0,\ \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0,\ \theta(0) = \theta^0 & \text{in } \Omega, \end{cases} \quad \text{(LB)}$$

where f and f_0 will be taken to decrease exponentially to zero in T.
The (nonhomogeneous) adjoint system:

$$\begin{cases} -\varphi_t - \Delta \varphi + \nabla \pi = g - \psi \nabla \bar{\theta},\ \nabla \cdot \varphi = 0 & \text{in } Q, \\ -\psi_t - \Delta \psi = g_0 + \varphi_3 & \text{in } Q, \\ \varphi = 0,\ \psi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T,\ \psi(T) = \psi^T & \text{in } \Omega, \end{cases} \quad \text{(AB)}$$

where $g \in L^2(Q)^3$, $g_0 \in L^2(Q)$, $\varphi^T \in L^2(\Omega)^3$ and $\psi^T \in L^2(\Omega)$.

Proposition: Carleman inequality

There exists a constant $C > 0$ (depending on Ω, ω, T)

$$\iint_Q \rho_1(t)(|\varphi|^2 + |\psi|^2) \leq C\left(\iint_Q \rho_2(t)(|g|^2 + |g_0|^2) + \iint_{\omega\times(0,T)} \rho_3(t)(|\varphi_1|^2 + |\psi|^2)\right)$$

for every (φ, ψ) solution of the adjoint system.

Theorem

Let $i < N$ be a positive integer and consider $(\bar{p}, \bar{\theta})$. Then, for every $T > 0$ and $\omega \subset \Omega$, system (B) is locally controllable to the trajectory $(0, \bar{p}, \bar{\theta})$ by controls $v_0 \in L^2(\omega \times (0, T))$ and $v \in L^2(\omega \times (0, T))^N$, with $v_i \equiv 0$ and $v_N \equiv 0$.

Remark: For $N = 2$, $v \equiv 0$.

Insensitizing controls for Stokes system

We consider the problem of insensitizing controls for the Stokes system:

$$\begin{cases} y_t - \Delta y + \nabla p = f + v\mathbb{1}_\omega,\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 + \tau \widehat{y}^0 & \text{in } \Omega, \end{cases} \qquad \text{(S)}$$

where τ is a small constant and $\|\widehat{y}^0\|_{L^2(\Omega)^N} = 1$. Both are unknown.
Goal: To find a control $v \in L^2(\omega \times (0,T))^N$, with $v_i \equiv 0$ ($i \in \{1, \dots, N\}$) such that the functional (Sentinel)

$$J(y) = \iint_{\mathcal{O}\times(0,T)} |y|^2\, dx\, dt,\ \mathcal{O} \subset \Omega \ (\text{Observation set})$$

is not affected by the uncertainty of the initial data, that is,

$$\left.\frac{\partial J(y)}{\partial \tau}\right|_{\tau=0} = 0,\ \forall \widehat{y}^0 \in L^2(\Omega)^N \text{ s.t. } \|\widehat{y}^0\|_{L^2(\Omega)^N} = 1.$$

A coupled Stokes system

The previous condition is equivalent to the following null controllability problem: To find a control $v = (v_1, v_2, 0)$ such that $z(0) = 0$, where

$$\begin{cases} w_t - \Delta w + \nabla p^0 = f + (v_1, v_2, 0)\mathbb{1}_\omega, \ \nabla \cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + \nabla q = w\mathbb{1}_{\mathcal{O}}, \ \nabla \cdot z = 0 & \text{in } Q, \\ w = z = 0 & \text{on } \Sigma, \\ w(0) = y^0, \ z(T) = 0 & \text{in } \Omega. \end{cases}$$

Equivalent to the observability inequality

$$\iint_Q e^{-K/t^m} |\varphi|^2 \, dx \, dt \le C \iint_{\omega \times (0,T)} (|\varphi_1|^2 + |\varphi_2|^2) \, dx \, dt,$$

where φ is the solution of the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = \psi\mathbb{1}_{\mathcal{O}}, \ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \psi_t - \Delta\psi + \nabla\kappa = 0, \ \nabla \cdot \psi = 0 & \text{in } Q, \\ \varphi = \psi = 0 & \text{on } \Sigma, \\ \varphi(T) = 0, \ \psi(0) = \psi^0 & \text{in } \Omega. \end{cases}$$

Proposition: Carleman inequality

There exists a constant $C > 0$ (depending on Ω, ω, $\mathcal{O}$, T)

$$\iint_Q \rho_3(t)(|\varphi|^2 + |\psi|^2) \leq C \iint_{\omega\times(0,T)} \rho_4(x,t)(|\varphi_1|^2 + |\varphi_2|^2)$$

for every (φ, ψ) solution of the adjoint system.

Theorem (joint work with M. Gueye)

Let $i \in \{1, \dots, N\}$. Assume $y^0 = 0$, $\|e^{K/t^{11}} f\|_{L^2(Q)^N} < \infty$ and $\mathcal{O} \cap \omega \neq \phi$. There exists a insensitizing control $v \in L^2(\omega \times (0,T))^N$, with $v_i \equiv 0$, for system (S).

Future work

- Local controllability to $\bar{y} \neq 0$ and $(\bar{y}, \bar{p}, \bar{\theta})$ for Navier-Stokes and Boussinesq systems.
- Insensitizing controls for Navier-Stokes system (on going).

Thank you for your attention