Local null controllability of the Navier-Stokes system with a reduced number of scalar controls

Journée Jeunes Contrôleurs

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Outline

Let Ω bounded connected regular open subset of $\mathbf{R}^N$ ($N = 2$ or 3).
$\omega \subset \Omega$, $T > 0$, $Q := \Omega \times (0, T)$, $\Sigma := \partial\Omega \times (0, T)$.

$$V = \{y \in H_0^1(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega\}$$

$$H = \{y \in L^2(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega,\ y \cdot n = 0 \text{ on } \partial\Omega\}$$

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = v\mathbb{1}_\omega,\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{NS}$$

where v stands for the control which acts over the set ω.

Goal: Local null controllability of system (NS) with $N-1$ scalar controls, that is, if $\|y^0\|$ is small, we can find a control $v \in L^2(\omega \times (0,T))^N$, with $v_i \equiv 0$ ($i \in \{1, \dots, N\}$), s.t. the corresponding solution to (NS) satisfies

$$y(T) = 0.$$

Previous results

[Fernández-Cara, Guerrero, Imanuvilov, Puel, 2006]: Local exact controllability to the trajectories with $N-1$ scalar controls when $\overline{\omega} \cap \partial\Omega \neq \phi$:

$$\begin{cases} \bar{y}_t - \Delta \bar{y} + (\bar{y} \cdot \nabla)\bar{y} + \nabla \bar{p} = 0, \ \nabla \cdot \bar{y} = 0 & \text{in } Q, \\ \bar{y} = 0 & \text{on } \Sigma, \\ \bar{y}(0) = \bar{y}^0 & \text{in } \Omega, \end{cases}$$

If $\|y^0 - \bar{y}^0\|$ is small, we can find a control $v \in L^2(\omega \times (0,T))^N$, with $v_i \equiv 0$ $(i \in \{1, \dots, N\})$, s.t. the corresponding solution to (NS) satisfies

$$y(T) = \bar{y}(T).$$

Novelty here: We remove this geometric assumption.

In this direction: Null controllability of the Stokes system [Coron, Guerrero, 2009]

Local null controllability of (NS)

Theorem 1

Let $i \in \{1, \dots, N\}$. Then, for every $T > 0$ and $\omega \subset \Omega$, there exists $\delta > 0$ such that, for every $y^0 \in V$ satisfying

$$\|y^0\|_V \leq \delta,$$

we can find a control $v \in L^2(\omega \times (0, T))$, with $v_i \equiv 0$, and a corresponding solution (y, p) to (NS) such that

$$y(T) = 0,$$

i.e., the nonlinear system (NS) is locally null controllable by means of $N - 1$ scalar controls for an arbitrary control domain.

General strategy

Linearized system around 0:

$$\begin{cases} y_t - \Delta y + \nabla p = f + (v_1, v_2, 0)\mathbb{1}_\omega,\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \qquad \text{(L)}$$

where f is taken to decrease exponentially to zero in $t = T$.

Null controllability for (L) + Inverse mapping theorem
$\Rightarrow$ Local null controllability of (NS)

Need a suitable Carleman inequality for the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = g,\ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T & \text{in } \Omega, \end{cases} \qquad \text{(A)}$$

where $g \in L^2(Q)^3$ and $\varphi^T \in H$.

Weight functions

Let ω_0 be a nonempty open subset of $\mathbf{R}^3$ such that $\overline{\omega_0} \subset \omega$ and $\lambda > 1$

$$\alpha(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\ell^8(t)},\ \xi(x,t) = \frac{e^{\lambda\eta(x)}}{\ell^8(t)},$$
$$\alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(x,t),\ \xi^*(t) = \min_{x\in\overline{\Omega}} \xi(x,t),$$
$$\widehat{\alpha}(t) = \min_{x\in\overline{\Omega}} \alpha(x,t),\ \widehat{\xi}(t) = \max_{x\in\overline{\Omega}} \xi(x,t).$$

where $\eta \in C^2(\overline{\Omega})$ and $\ell \in C^\infty([0,T])$ are s.t.

$$|\nabla\eta| > 0 \text{ in } \overline{\Omega}\setminus\omega_0,\ \eta > 0 \text{ in } \Omega \text{ and } \eta \equiv 0 \text{ on } \partial\Omega,$$

$$\ell(t) = t \quad \forall t \in [0, T/4],\ \ell(t) = T - t \quad \forall t \in [3T/4, T].$$

Existence of η : [Fursikov, Imanuvilov, 1996].

Carleman estimate

Proposition 2

There exists a constant λ_0, such that for any $\lambda > \lambda_0$ there exist two constants $C(\lambda) > 0$ and $s_0(\lambda) > 0$ such that for any $g \in L^2(Q)^3$ and any $\varphi^T \in H$, the solution of (A) satisfies

$$s^4 \iint\limits_Q e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2 \leq C\left(\iint\limits_Q e^{-3s\alpha^*}|g|^2 + s^7 \iint\limits_{\omega\times(0,T)} e^{-2s\widehat{\alpha}-3s\alpha^*}(\widehat{\xi})^7(|\varphi_1|^2+|\varphi_2|^2)\right)$$

for every $s \geq s_0$.

We actually need a Carleman inequality with weights not vanishing at $t = 0$.

Lemma 3

Let s and λ be like in Proposition 2. Then, there exists a constant $C > 0$ (depending on s and λ) such that every solution φ of (A) satisfies:

$$\iint\limits_Q e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2 + \|\varphi(0)\|^2_{L^2(\Omega)^3}$$

$$\leq C\left(\iint\limits_Q e^{-3s\beta^*}|g|^2 + \iint\limits_{\omega\times(0,T)} e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7(|\varphi_1|^2 + |\varphi_2|^2)\right).$$

where β and γ are defined as α and ξ with

$$\tilde{\ell}(t) = \begin{cases} \|\ell\|_\infty & 0 \leq t \leq T/2, \\ \ell(t) & T/2 < t \leq T. \end{cases}$$

This estimate is proved by the previous Carleman inequality and energy estimates for the Stokes system.

Null controllability for system (L)

We consider the variational problem

$$a((\widehat{\chi},\widehat{\sigma}),(\chi,\sigma)) = G(\chi,\sigma) \quad \forall(\chi,\sigma), \in P,$$

$$a((\widehat{\chi},\widehat{\sigma}),(\chi,\sigma)) = \iint\limits_Q e^{-3s\beta^*}\,(-\widehat{\chi}_t - \Delta\widehat{\chi} + \nabla\widehat{\sigma})\cdot(-\chi_t - \Delta\chi + \nabla\sigma)$$
$$+ \iint\limits_{\omega\times(0,T)} e^{-2s\widehat{\beta}-3s\beta^*}\,\widehat{\gamma}^7\,(\widehat{\chi}_1\,\chi_1 + \widehat{\chi}_2\,\chi_2)$$

$$G(\chi,\sigma) = \iint\limits_Q f\cdot\chi\,dx\,dt + \int\limits_\Omega y^0\cdot\chi(0)\,dx$$

Carleman + Lax Milgram Lemma $\Rightarrow \exists!\,(\widehat{\chi},\widehat{\sigma}) \in P$

$$\begin{cases} \widehat{y} = e^{-3s\beta^*}(-\widehat{\chi}_t - \Delta\widehat{\chi} + \nabla\widehat{\sigma}), & \text{in } Q,\\ (\widehat{v}_1,\widehat{v}_2) = -e^{-2s\widehat{\beta}-3s\beta^*}\,\widehat{\gamma}^7\,(\widehat{\chi}_1,\widehat{\chi}_2), \quad \widehat{v}_3\equiv 0 & \text{in } \omega\times(0,T). \end{cases}$$

Furthermore, $\widehat{y}$ (together with some pressure $\widehat{p}$) is the solution of (L) for $v = \widehat{v}$, and $(\widehat{y}, \widehat{p}, \widehat{v})$ belongs to the (Banach) space

$$E = \{ (y, p, v) : e^{3/2s\beta^*} y,\ e^{s\widehat{\beta}+3/2s\beta^*}\widehat{\gamma}^{-7/2}\, v\mathbb{1}_\omega \in L^2(Q)^3,\ v_3 \equiv 0,$$
$$e^{3/2s\beta^*}(\gamma^*)^{-9/8} y \in L^2(0, T; H^2(\Omega)^3) \cap L^\infty(0, T; V),$$
$$e^{5/2s\beta^*}(\gamma^*)^{-2}(y_t - \Delta y + \nabla p - v\mathbb{1}_\omega) \in L^2(Q)^3 \},$$

provided that $y^0 \in V$ and $e^{5/2s\beta^*}(\gamma^*)^{-2} f \in L^2(Q)^3$.

In particular, $\widehat{y}(T) = 0$.

Proof of Theorem 1

Theorem 4

Let B_1 and B_2 be two Banach spaces and let $\mathcal{A} : B_1 \to B_2$ satisfy $\mathcal{A} \in C^1(B_1; B_2)$. Assume that $b_1 \in B_1$, $\mathcal{A}(b_1) = b_2$ and that $\mathcal{A}'(b_1) : B_1 \to B_2$ is surjective. Then, there exists $\delta > 0$ such that, for every $b' \in B_2$ satisfying $\|b' - b_2\|_{B_2} < \delta$, there exists a solution of the equation

$$\mathcal{A}(b) = b', \quad b \in B_1.$$

$$B_1 = E, \qquad B_2 = L^2(e^{5/2s\beta^*}(\gamma^*)^{-2}(0,T); L^2(\Omega)^3) \times V$$

$$\mathcal{A}(y,p,v) = (\, y_t - \Delta y + (y \cdot \nabla)y + \nabla p - (v_1, v_2, 0)\mathbb{1}_\omega,\ y(0)\,) \in C^1(B_1; B_2)$$

$$\mathcal{A}'(0,0,0)(y,p,v) = (y_t - \Delta y + \nabla p - (v_1, v_2, 0)\mathbb{1}_\omega, y(0)\,)$$

surjective by NC of (L).

We apply Theorem 4 with $b_1 = (0,0,0)$, $b_2 = (0,0)$.

Proof of the Carleman inequality

Idea from [Coron, Guerrero, 2009]: $\Delta\pi = 0$. This is not the case here.
We consider $\rho(t) = e^{-3/2s\alpha^*}$ and the systems:

$$\begin{cases} -w_t - \Delta w + \nabla \pi_w = \rho g, \ \nabla \cdot w = 0 \text{ in } Q, \\ w = 0 \text{ on } \Sigma, \ w(T) = 0 \text{ in } \Omega, \end{cases}$$

$$\begin{cases} -z_t - \Delta z + \nabla \pi_z = -\rho' \varphi, \ \nabla \cdot z = 0 \text{ in } Q, \\ z = 0 \text{ on } \Sigma, \ z(T) = 0 \text{ in } \Omega. \end{cases}$$

$\Rightarrow (\rho\varphi, \rho\pi) = (w + z, \pi_w + \pi_z)$ and $\Delta\pi_z = 0$.
$\psi := \nabla \Delta z_1$:

$$-\psi_t - \Delta\psi = -\nabla(\Delta(\rho'\varphi_1))$$

Need Carleman estimate for parabolic equations with nonhomogeneous B.C.
For w, regularity estimate:

$$\|w\|^2_{L^2(0,T;H^2(\Omega)^2)} + \|w\|^2_{H^1(0,T;L^2(\Omega)^2)} \leq C\|\rho g\|^2_{L^2(Q)^2},$$

From [Imanuvilov, Puel, Yamamoto, 2009]:

$$\frac{1}{s}\iint\limits_{Q} e^{-2s\alpha}\frac{1}{\xi}|\nabla\psi|^2 + s\iint\limits_{Q} e^{-2s\alpha}\xi|\psi|^2$$
$$\leq C\left(s^{-\frac{1}{2}}\|e^{-s\alpha}\xi^{-\frac{1}{4}}\psi\|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)^2} + s^{-\frac{1}{2}}\|e^{-s\alpha}\xi^{-\frac{1}{8}}\psi\|^2_{L^2(\Sigma)^2}\right.$$
$$\left. + \iint\limits_{Q} e^{-2s\alpha}|\rho'|^2|\Delta\varphi_1|^2 + s\iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi|\psi|^2\right),$$

for every $\lambda \geq \widehat{\lambda_0}$ and $s \geq \widehat{s}$.

$$\|u\|_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} = \left(\|u\|^2_{H^{1/4}(0,T;L^2(\partial\Omega))} + \|u\|^2_{L^2(0,T;H^{1/2}(\partial\Omega))}\right)^{1/2}.$$

Other estimates:

$$s^3 \iint\limits_Q e^{-2s\alpha}\xi^3|\Delta z_1|^2 \le C\left(s\iint\limits_Q e^{-2s\alpha}\xi|\psi|^2 + s^3 \iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\Delta z_1|^2 \right),$$

$$s^6 \iint\limits_Q e^{-2s\alpha}\xi^6|z_1|^2 + s^4 \iint\limits_Q e^{-2s\alpha}\xi^4|\nabla z_1|^2$$
$$\le C\left(s^3\iint\limits_Q e^{-2s\alpha}\xi^3|\Delta z_1|^2 + s^6 \iint\limits_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^6|z_1|^2 \right),$$

for every $s \ge C$.

$$z_2|_\Sigma = 0,\ \nabla\cdot z = 0 \Rightarrow s^4 \iint\limits_Q e^{-2s\alpha^*}(\xi^*)^4|z_2|^2 \le C\, s^4 \iint\limits_Q e^{-2s\alpha^*}(\xi^*)^4|\partial_2 z_2|^2$$
$$\le C\, s^4 \iint\limits_Q e^{-2s\alpha}(\xi)^4|\nabla z_1|^2$$

Combining these estimates (and after some calculations) with regularity estimates for w:

$$s^6 \iint\limits_Q e^{-2s\alpha}\xi^6|z_1|^2 + s^4 \iint\limits_Q e^{-2s\alpha^*}(\xi^*)^4|z_2|^2 + s^3 \iint\limits_Q e^{-2s\alpha}\xi^3|\Delta z_1|^2$$

$$+ \frac{1}{s}\iint\limits_Q e^{-2s\alpha}\frac{1}{\xi}|\nabla\psi|^2 + s\iint\limits_Q e^{-2s\alpha}\xi|\psi|^2$$

$$\leq C\Big(s^{-\frac{1}{2}}\|e^{-s\alpha^*}(\xi^*)^{-\frac{1}{4}}\psi\|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)^2} + s^{-\frac{1}{2}}\|e^{-s\alpha^*}(\xi^*)^{-\frac{1}{8}}\psi\|^2_{L^2(\Sigma)^2}$$

$$+ \|\rho g\|^2_{L^2(Q)^2} + s^7 \iint\limits_{\omega\times(0,T)} e^{-2s\alpha}\xi^7|z_1|^2$$

for every $s \geq C$.

$$\|u\|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)^2} \leq C(\|u\|^2_{L^2(0,T;H^1(\Omega)^2)} + \|u\|^2_{H^1(0,T;H^{-1}(\Omega)^2)}).$$

Estimate of boundary terms

We prove that $e^{-s\alpha^*(\xi^*)^{-1/4}}z$ is sufficiently regular. We do it in two steps:

$$(\widetilde{z},\widetilde{\pi}_z) := se^{-s\alpha^*}(\xi^*)^{7/8}(z,\pi_z) :$$

$$\begin{cases} -\widetilde{z}_t - \Delta\widetilde{z} + \nabla\widetilde{\pi}_z = -se^{-s\alpha^*}(\xi^*)^{7/8}\rho'\varphi - (se^{-s\alpha^*}(\xi^*)^{7/8})_t z, \ \nabla\cdot\widetilde{z} = 0 \text{ in } Q, \\ \widetilde{z} = 0 \text{ on } \Sigma, \ \widetilde{z}(T) = 0 \text{ in } \Omega. \end{cases}$$

Since $|\alpha_t^*| \leq C(\xi^*)^{9/8}$, $|\rho'| \leq Cs\rho(\xi^*)^{9/8}$

$\Rightarrow \widetilde{z} \in L^2(0,T;H^2(\Omega)^2) \cap H^1(0,T;L^2(\Omega)^2)$ and

$$\|se^{-s\alpha^*}(\xi^*)^{7/8}z\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)}$$

is on the left-hand side of the Carleman estimate.

Next:

$$(\widehat{z}, \widehat{\pi}_z) := e^{-s\alpha^*}(\xi^*)^{-1/4}(z, \pi_z):$$

$$\begin{cases} -\widehat{z}_t - \Delta \widehat{z} + \nabla \widehat{\pi}_z = -e^{-s\alpha^*}(\xi^*)^{-1/4}\rho'\varphi - (e^{-s\alpha^*}(\xi^*)^{-1/4})_t z, \ \nabla\cdot\widehat{z} = 0, \\ \widehat{z} = 0 \text{ on } \Sigma, \ \widehat{z}(T) = 0 \text{ in } \Omega. \end{cases}$$

By the previous step, this right-hand side belongs to $L^2(0,T;H^2(\Omega)^2) \cap H^1(0,T;L^2(\Omega)^2)$ and thus $\widehat{z} \in L^2(0,T;H^4(\Omega)^2) \cap H^1(0,T;H^2(\Omega)^2)$, and

$$\|e^{-s\alpha^*}(\xi^*)^{-1/4} z\|^2_{L^2(0,T;H^4(\Omega)^2)\cap H^1(0,T;H^2(\Omega)^2)}$$

is on the left-hand side of the Carleman estimate. In particular, $e^{-s\alpha^*}(\xi^*)^{-1/4}\psi \in L^2(0,T;H^1(\Omega)^2) \cap H^1(0,T;H^{-1}(\Omega)^2)$ ($\psi = \nabla\Delta z_1$) and

$$\|e^{-s\alpha^*}(\xi^*)^{-1/4}\psi\|^2_{L^2(0,T;H^1(\Omega)^2)} \text{ and } \|e^{-s\alpha^*}(\xi^*)^{-1/4}\psi\|^2_{H^1(0,T;H^{-1}(\Omega)^2)}$$

are on the left-hand side of the Carleman estimate (and absorb the boundary term for s large enough).

Thank you for your attention