Some control results for systems coupling second- and fourth-order parabolic equations

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Outline

Goal of this talk

Goal of this talk: To present some controllability results concerning systems coupling (one-dimensional) fourth- and second-order parabolic equations.
For instance:

$$\begin{cases} u_t + u_{xxxx} = 0 & \text{in } (0,T)\times(0,L), \\ u(0,t)=0, u(L,t)=0 & \text{in } (0,T), \\ u_x(0,t)=0, u_x(L,t)=0 & \text{in } (0,T), \\ u(x,0)=u_0(x) & \text{in } (0,L), \end{cases}$$

and

$$\begin{cases} v_t - v_{xx} = 0 & \text{in } (0,T)\times(0,L), \\ v(0,t)=0, v(L,t)=0 & \text{in } (0,T), \\ v(x,0)=v_0(x) & \text{in } (0,L). \end{cases}$$

Goal of this talk

- Many possibilities:
  - Different kinds of coupling.
  - Distributed controls (In which equation? both? just one?).
  - Boundary controls (Where in the boundary? everywhere or just some?)

- Here we will focus on two types of problem, which are treated with two methods:
  - One distributed control with first-order coupling (Carleman estimates).
  - One boundary control for a cascade system (Moments method).

Stabilized Kuramoto-Sivashinsky system

$$
\begin{cases}
u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} + u u_x = v_x, \\
v_t - \Gamma v_{xx} + c v_x = u_x, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x),
\end{cases}
$$

where $\gamma, a, \Gamma > 0$ and $c \in \mathbb{R}$ are fixed parameters.

- Proposed by Malomed, Feng, and Kawahara (2001).
- The first equation has unstable solitary pulse solutions.
- The second equation is added to allow the existence of stable solitary pulses.

Stabilized Kuramoto-Sivashinsky system in a bounded domain

Consider the fourth-second-order parabolic system:

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} + u u_x = v_x + f 1\!\!1_\omega & \text{in } (0,T)\times(0,L), \\ v_t - \Gamma v_{xx} + c v_x = u_x + h 1\!\!1_\omega & \text{in } (0,T)\times(0,L), \\ u(0,t) = u_x(0,t) = 0, \quad u(L,t) = u_x(L,t) = 0 & \text{in } (0,T), \\ v(0,t) = 0, \quad v(L,t) = 0 & \text{in } (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } (0,L), \end{cases}$$

where $\gamma, a, \Gamma > 0$ and $c \in \mathbb{R}$ are fixed parameters, and f and h are the controls acting on $\omega \subset (0,L)$.
Of course, the interesting case is when

- $h \equiv 0$; or
- $f \equiv 0$.

Case $h \equiv 0$

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} + u u_x = v_x + f \mathbb{1}_\omega & \text{in } (0,T)\times(0,L), \\ v_t - \Gamma v_{xx} + c v_x = u_x & \text{in } (0,T)\times(0,L). \end{cases}$$

Theorem (Cerpa, Mercado, Pazoto (2015))

Let $T>0$. Then, there exists $\delta>0$ such that for any initial conditions $u_0 \in H^{-2}(0,L)$ and $v_0 \in H^{-1}(0,L)$ verifying

$$\|u_0\|_{H^{-2}(0,L)} + \|v_0\|_{H^{-1}(0,L)} \leq \delta,$$

there exists a control $f \in L^2(\omega \times (0,L))$ such that the solution $(u,v) \in L^2((0,T)\times(0,L))^2 \cap C([0,T]; H^{-2}(0,L)\times H^{-1}(0,L))$ of the SKS system satisfies

$$u(\cdot,T) = 0 \text{ and } v(\cdot,T) = 0 \text{ in } (0,L).$$

Case $f \equiv 0$

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} + u u_x = v_x & \text{in } (0,T)\times(0,L), \\ v_t - \Gamma v_{xx} + c v_x = u_x + h 1\!\!1_\omega & \text{in } (0,T)\times(0,L). \end{cases}$$

Theorem (C., Cerpa (2016))

Let $T > 0$. Then, there exists $\delta > 0$ such that for any initial conditions $u_0 \in H^{-2}(0,L)$ and $v_0 \in H^{-1}(0,L)$ verifying

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$$u(\cdot,T) = 0 \text{ and } v(\cdot,T) = 0 \text{ in } (0,L).$$

Approach to controllability

Both cases follow the same strategy:

1. Linearization around the zero state.
2. Null-controllability result for the linear system:
   - Controllability-Observability duality.
   - Observability inequality for the adjoint system (Carleman estimates).
   - Solutions of the (controlled) linear system belong to a weighted space (decaying to zero at $t = T$).
3. Local inversion argument for the non-linear system.

Adjoint system and observability

For the solutions of the adjoint system

$$\begin{cases} -\varphi_t + \gamma\varphi_{xxxx} - \varphi_{xxx} + a\varphi_{xx} = -\psi_x & \text{in } (0,L)\times(0,T), \\ -\psi_t - \Gamma\psi_{xx} - c\psi_x = -\varphi_x & \text{in } (0,L)\times(0,T), \\ \varphi(0,t) = \varphi_x(0,t) = 0, \quad \varphi(L,t) = \varphi_x(L,t) = 0 & \text{in } (0,T), \\ \psi(0,t) = 0, \quad \psi(L,t) = 0 & \text{in } (0,T), \\ \varphi(x,T) = \varphi_T(x), \quad \psi(x,T) = \psi_T(x) & \text{in } (0,L), \end{cases}$$

we prove the observability inequality:

- Case $h \equiv 0$:

$$\int_0^L \rho_1 \left(|\varphi(x,0)|^2 + |\psi(x,0)|^2\right) \,\mathrm{d}x\,\mathrm{d}t \leq C \int_0^T\!\!\int_\omega |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

- Case $f \equiv 0$:

$$\int_0^L \rho_1 \left(|\varphi(x,0)|^2 + |\psi(x,0)|^2\right) \,\mathrm{d}x\,\mathrm{d}t \leq C \int_0^T\!\!\int_\omega |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Carleman estimates

Tool to prove observability: Carleman estimates

$$\int_0^T\int_0^L \rho_1|z|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C\int_0^T\int_0^L \rho_2|Az|^2 \,\mathrm{d}x\,\mathrm{d}t + C\int_0^T\int_\omega \rho_3|z|^2 \,\mathrm{d}x\,\mathrm{d}t$$

with $\rho_i = \rho_i(x,t)$ some positive weight functions. Here,

$$Az = z_t - z_{xx} \quad \text{or} \quad Az = z_t + z_{xxxx},$$

with (usually) homogeneous boundary conditions.
Widely use in the last 20 years for control, inverse problems, and just for fun (Right?). To name a few people:

- Second-order: Fursikov, Imanuvilov, Puel, Guerrero, Ervedoza, Osses, Araruna, Chaves-Silva, Santos, Souza, Montoya...
- Fourth-order: Cerpa, Guzmán, Mercado, Pazoto, Gao,...

Steps to prove observability:

1. Carleman for $\varphi_t + \varphi_{xxxx}$ (first equation).
2. Carleman for $\psi_t - \psi_{xx}$ (second equation).
   At this point we essentially have

$$\int_0^T\!\!\int_0^L \rho_1\left(|\varphi|^2 + |\psi|^2\right)\,\mathrm{d}x\,\mathrm{d}t \leq C\int_0^T\!\!\int_\omega \rho_2\left(|\varphi|^2 + |\psi|^2\right)\,\mathrm{d}x\,\mathrm{d}t$$

3. Eliminate one of the local terms using the equation:
   - $-\varphi_t + \gamma\varphi_{xxxx} - \varphi_{xxx} + a\varphi_{xx} = -\psi_x$; or
   - $-\psi_t - \Gamma\psi_{xx} - c\psi_x = -\varphi_x$.

Since the coupling is of first-order and ω does not touch the boundary, we need to use a Carleman estimate with non-homogeneous boundary conditions applied to $(\psi_x)_t - (\psi_x)_{xx}$ or $(\varphi_x)_t + (\varphi_x)_{xxxx}$.

- Case $h \equiv 0$: Carleman for $\psi_t - \psi_{xx}$ (Guerrero et al., 2007).
- Case $f \equiv 0$: (New) Carleman for $\varphi_t - \varphi_{xxxx}$ (C., Cerpa, 2016).

Boundary controls

Result using Carleman estimates for the system:

$$\begin{cases} u_t + \gamma u_{xxxx} + u_{xxx} + a u_{xx} + u u_x = v_x + f \mathbb{1}_\omega & \text{in } (0,T)\times(0,L), \\ v_t - \Gamma v_{xx} + c v_x = u_x + h \mathbb{1}_\omega & \text{in } (0,T)\times(0,L), \\ u(0,t) = h_1(t), \quad u(L,t) = 0 & \text{in } (0,T), \\ u_x(0,t) = h_2(t), \quad u_x(L,t) = 0 & \text{in } (0,T), \\ v(0,t) = h_3(t), \quad v(L,t) = 0 & \text{in } (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } (0,L). \end{cases}$$

- Local null-controllability result from Cerpa, Mercado, Pazoto (2012).
- Carleman estimates does not allow to “remove” boundary controls.
- What is the alternative if we want to fix $h_1(t) = h_2(t) = 0$, for instance?

A friendly discussion

A cascade system with one control

Consider the system

$$\begin{cases} u_t + u_{xxxx} = v, & t > 0,\ x \in (0,\pi), \\ v_t - d v_{xx} = 0, & t > 0,\ x \in (0,\pi), \\ u(t,0) = u_{xx}(t,0) = 0, & t > 0, \\ u(t,L) = u_{xx}(t,\pi) = 0, & t > 0, \\ v(t,0) = h(t),\ v(t,\pi) = 0, & t > 0. \end{cases}$$

Goal: Study controllability properties in terms of the diffusion coefficient $d > 0$ using the moment method, introduced by Fattorini and Russell (1971).

Quick overview of the Moment Method

Consider the one-dimensional heat equation with a boundary control:

$$\begin{cases} u_t - u_{xx} = 0, & t \in (0,T),\ x \in (0,\pi), \\ u(t,0) = h(t),\ u(t,\pi) = 0 & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,\pi). \end{cases}$$

Using that the eigenfunctions $\{\sin(kx)\}_{k\geq 1}$ of $-\partial_{xx}$ is a basis of $L^2(0,\pi)$, writing $u_0(x) = \sum_{k\geq 1} a_k \sin(kx)$, the null-controllability is equivalent to the *moment problem*

$$k\int_0^T \tilde{h}(t)e^{-k^2 t}\,\mathrm{d}t = e^{-k^2 T} a_k\,\mathrm{d}x, \quad \forall k \geq 1.$$

Then, the problem is to find a biorthogonal family $\{q_k(t)\}_{k\geq 1}$ to $\{e^{-k^2 t}\}_{k\geq 1}$, and such that for any $\varepsilon > 0$:

$$\|q_k\|_{L^2(0,T)} \leq C(\varepsilon, T)e^{\varepsilon k^2}, \quad \forall k \geq 1.$$

Then: $h(t) := \tilde{h}(T-t) = \sum_{k\geq 1} b_k q_k(T-t) \in L^2(0,T)$, with $b_k = \dfrac{e^{-k^2 T} a_k}{k}$.

General result for the existence of biorthogonal families

Fattorini and Russell proved a general result on existence of a biorthogonal family to $\{e^{-\lambda_k t}\}_{k\geq 1}$ in $L^2(0,T)$ for a positive sequence $\Lambda = \{\lambda_k\}_{k\geq 1}$ such that satisfies:

- $\displaystyle\sum_{k\geq 1} \frac{1}{\lambda_k} < +\infty.$
- $|\lambda_k - \lambda_m| \geq \rho|k-m|, \quad \forall k,m \geq 1$ (Gap condition).

Of course, $\Lambda = \{k^2\}_{k\geq 1}$ fulfills these properties and the previous control satisfies

$$\|h\|_{L^2(0,T)} \leq C(\varepsilon, T) \sum_{k\geq 1} \frac{|a_k|}{k} e^{-k^2(T-\varepsilon)}.$$

Extensions to systems

$$\begin{cases} y_t - (D\partial_{xx}^2 + A) = 0, & t \in (0,T),\, x \in (0,\pi), \\ u(t,0) = Bv(t),\, u(t,L) = 0 & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,\pi). \end{cases}$$

where $D = \text{diag}(d_1, \dots, d_n)$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$.

Some results:

- Fernández-Cara et al. (2010). $D = \text{Id}, n = 2, m = 1$.
- Ammar-Khodja et al. (2011). Generalization $D = \text{Id}, n \geq 2, m \geq 1$.

In these works, the eigenvalues of $D\partial_{xx}^2 + A$ satisfy the gap condition, which allows to have controllability for any $T > 0$.

Back to our system

$$\begin{cases} u_t + u_{xxxx} = v, & t > 0,\ x \in (0,\pi), \\ v_t - dv_{xx} = 0, & t > 0,\ x \in (0,\pi), \\ u(t,0) = u_{xx}(t,0) = 0, & t > 0, \\ u(t,L) = u_{xx}(t,\pi) = 0, & t > 0, \\ v(t,0) = h(t),\ v(t,\pi) = 0, & t > 0. \end{cases}$$

- The eigenvalues are given by $\Lambda = \{k^4, dk^2\}_{k\geq 1}$.
- Ideas from [Ammar-Khodja, Benabdallah, González-Burgos, de Teresa, 2014].
- A first result: If $\sqrt{d}$ is rational, the system is not approximate-controllable.
  • We can construct a solution of the adjoint system such that the unique continuation does not hold.

Condensation index and minimal time of controllability

- We assume that d is irrational. Therefore, the family $\Lambda = \{k^4, dk^2\}_{k\geq 1}$ has no repeated elements.
- In this case, there exists a biorthogonal family $\{q_k\}_{k\geq 1}$ to $\{e^{-\lambda_k t}\}_{k\geq 1}$.
- The gap condition may not be satisfied. However, it can be proved that

$$\|q_k\|_{L^2(0,T)} \leq C(\varepsilon, T) e^{(c(\Lambda)+\varepsilon)\lambda_k}$$

  where $c(\Lambda)$ is the *condensation index* of the sequence Λ. Roughly speaking, $c(\Lambda)$ is a measure of the way how λ_k approaches λ_m for $k \neq m$.
- Notice that $c(\Lambda)$ is the minimal time of null-controllability in the sense that:
  - The system is null-controllable if $T > c(\Lambda)$.
  - System is not null-controllable if $T < c(\Lambda)$.
- In particular, if Λ satisfies de gap condition: $c(\Lambda) = 0$ and the system is controllable at any time $T > 0$.

Characterization of the condensation index

- From the two branches of $\Lambda = \{k^4, dk^2\}_{k\geq 1}$, we have $c(\Lambda) = \max\{c_1, c_2\}$, where

$$c_1 := \limsup_{k\to\infty} \frac{-\ln|\sin(\pi\sqrt{k\sqrt{d}})|}{dk^2} \quad \text{and} \quad c_2 := \limsup_{k\to\infty} \frac{-\ln\left|\sin\left(\frac{\pi k^2}{\sqrt{d}}\right)\right|}{k^4}.$$

- With this characterization $c(\Lambda)$, we can prove that for any $T_0 \in [0, +\infty]$, there exists d irrational such that $T_0 = c(\Lambda)$.

Theorem (C., Cerpa, Mercado (submitted))

There are $d > 0$ irrational such that the system:

1. *is null-controllable in time T for any $T > 0$;*
2. *for a given $T_0 > 0$, is null-controllable in time T if $T > T_0$ and is not null-controllable if $T < T_0$; and*
3. *is not null-controllable.*

• The previous result depends on how well d is approximated by rational numbers (technical lemmas coming from number theory).

An awesome (?) team


thank
you