On the cost of null controllability of some partial differential equations

Nicolás Carreño
Universidad Técnica Federico Santa María

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Outline

Introduction

I. On the uniform controllability of a linear KdV equation

II. On the uniform controllability of a fourth order equation

Open problems

Outline

Introduction

In this talk, we will discuss the cost of null controllability of two linear equations:

1. A KdV equation

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = v(t), \quad y_{x|x=L} = 0, \quad y_{xx|x=L} = 0 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,L). \end{cases}$$

2. A fourth order equation

$$\begin{cases} z_t + \varepsilon z_{xxxx} - M z_x = 0 & \text{in } (0,T)\times(0,L), \\ z_{|x=0} = v_1(t), \quad z_{|x=L} = 0 & \text{in } (0,T), \\ z_{xx|x=0} = v_2(t), \quad z_{xx|x=L} = 0 & \text{in } (0,T), \\ z_{|t=0} = z_0, & \text{in } (0,L). \end{cases}$$

We want to know how the size of the controls behaves with respect to $\varepsilon > 0$. In particular, as $\varepsilon \to 0^+$.

Introduction

The cost is measured as the best constant C^{ε}_{cost} such that

1. for all y_0, there is v^{ε} such that $y_{|t=T} = 0$, and
$$\|v^{\varepsilon}\| \leq C^{\varepsilon}_{cost}\|y_0\|.$$
2. for all z_0, there are v_1^{ε} and v_2^{ε} such that $z_{|t=T} = 0$, and
$$\|v_1^{\varepsilon}\| + \|v_2^{\varepsilon}\| \leq C^{\varepsilon}_{cost}\|z_0\|.$$

Question: What can be expected of C^{ε}_{cost} as $\varepsilon \to 0^+$?

On the control of the transport equation

To understand what to expect of C^{ε}_{cost} as $\varepsilon \to 0^{+}$, consider the transport equation ($\varepsilon = 0$)

$$\begin{aligned} &y_t - My_x = 0 && \text{in } (0,T)\times(0,L),\\ &y_{|t=0} = y_0 && \text{in } (0,L) \end{aligned}$$

with controls

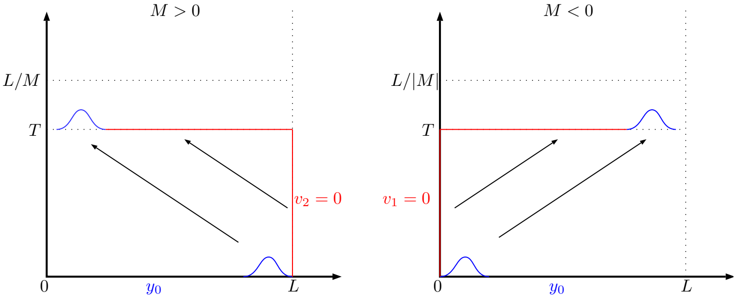
$$\begin{aligned} &y_{|x=0} = v_1(t) && \text{if } M<0,\\ &y_{|x=L} = v_2(t) && \text{if } M>0. \end{aligned}$$

- The transport equation is controllable if and only if $T \geq L/|M|$.

On the control of the transport equation

$$y_t - My_x = 0 \text{ in } (0,T)\times(0,L)$$

- $T < L/|M|$



- $C_{cost} = +\infty$
- Then, it is natural to expect that $\lim_{\varepsilon\to 0^+} C^{\varepsilon}_{cost} = +\infty$

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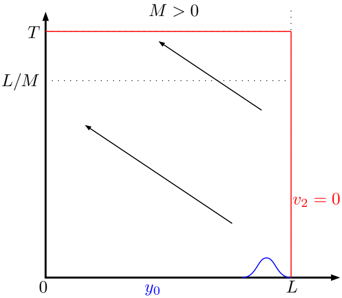

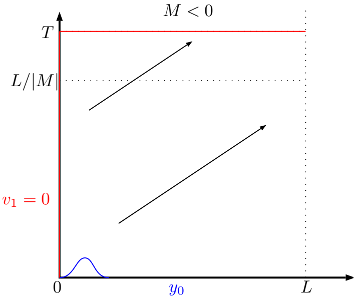

- $C_{cost} = 0$
- Then, it is natural to expect that $\lim_{\varepsilon\to 0^+} C^{\varepsilon}_{cost} = 0$

Some results

- For the heat equation:

$$\begin{cases} y_t - \varepsilon y_{xx} - M y_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = v(t), \quad y_{|x=L} = 0 & \text{in } (0,T), \end{cases}$$

Coron, Guerrero (2005) proved

1. $T < L/|M|$: $C^{\varepsilon}_{cost} \geq \exp(C\varepsilon^{-1})$ if $M \neq 0$.
2. $T \geq KL/|M|$: $C^{\varepsilon}_{cost} \leq \exp(-C\varepsilon^{-1})$ if $K > 0$ large (uniform contr.).

- For the classic KdV equation:

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = v(t), \quad y_{|x=L} = 0, \quad y_{x|x=L} = 0 & \text{in } (0,T), \end{cases}$$

Glass, Guerrero (2009) proved

1. $T < L/|M|$: $C^{\varepsilon}_{cost} \geq \exp(C\varepsilon^{-1/2})$ if $M \neq 0$.
2. $T \geq KL/M$: $C^{\varepsilon}_{cost} \leq \exp(-C\varepsilon^{-1/2})$ if $M > 0$, $K > 0$ large (u.c.).

Outline

A KdV equation in a bounded interval

- $T > 0$, $M \in \mathbb{R} \setminus \{0\}$ (transport coefficient), $\varepsilon > 0$ (dispersion coefficient).

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } (0,T) \times (0,L), \\ y_{|x=0} = v(t), \quad y_{x|x=L} = 0, \quad y_{xx|x=L} = 0 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,L). \end{cases}$$

- Controllability results by Guilleron (2014) and Cerpa, Rivas, Zhang (2013).
- Null controllability: For all $y_0 \in L^2(0,L)$, there is $v \in L^2(0,T)$ such that $y_{|t=T} = 0$, and there is $C > 0$ such that

$$\|v\|_{L^2(0,T)} \leq C \|y_0\|_{L^2(0,L)}.$$

- In particular, $C^{\varepsilon}_{cost} \leq C$.

Duality

C^{ε}_{cost} is the best constant such that the observability inequality

$$\|\varphi_{|t=0}\|_{L^2(0,L)} \leq \varepsilon C^{\varepsilon}_{cost} \|\varphi_{xx|x=0}\|_{L^2(0,T)},$$

holds for every φ solution of the adjoint equation

$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xxx} + M\varphi_x = 0 & \text{in } (0,T)\times(0,L), \\ \varphi_{|x=0} = 0, \quad \varphi_{x|x=0} = 0, \quad (\varepsilon\varphi_{xx} - M\varphi)_{|x=L} = 0 & \text{in } (0,T), \\ \varphi_{|t=T} = \varphi_T & \text{in } (0,L). \end{cases}$$

It is natural to focus in the analisys of the observability inequality.

• In general, for $T > 0$ and $\varepsilon > 0$, it can be proved (by Carleman inequalities) that

$$C^{\varepsilon}_{cost} \leq \exp\left(CT^{-1}\varepsilon^{-1}\right).$$

Is it possible to obtain uniform controllability with respect to $\varepsilon \to 0^+$?

- $C^{\varepsilon}_{cost} \leq C_\varepsilon \exp(-C(T,M)\varepsilon^{-1/2})$, T large enough?
- A possible strategy is to combine an observability inequality:

$$\|\varphi_{|t=T/2}\|_{L^2(0,L)} \leq C_\varepsilon \exp\left(CT^{-1/2}\varepsilon^{-1/2}\right)\|\varphi_{xx|x=0}\|_{L^2(0,T)}$$

with an exponential dissipation ($M>0$ (negative transport), T large):

$$\|\varphi_{|t=0}\|_{L^2(0,L)} \leq C_\varepsilon \exp\left(-CT\varepsilon^{-1/2}\right)\|\varphi_{|t=T/2}\|_{L^2(0,L)}.$$

- Observability OK. Improvement from $\exp\left(C\varepsilon^{-1}\right)^{\text{Guilleron(2014)}}$ to $\exp\left(C\varepsilon^{-1/2}\right)$.
- Dissipation is not possible.

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- Dissipation is not possible.

Explosion result for arbitrary T

Theorem[1]. Let $T, L, M > 0$. Then, there is $\varepsilon_0 > 0$ such that

$$C_{cost}^{\varepsilon} \geq C_\varepsilon \exp(CLM^{1/2}\varepsilon^{-1/2}), \quad \forall \varepsilon \in (0, \varepsilon_0)$$

where C_ε depends polynomially on ε^{-1} and ε.

- There exists $y_0 \in L^2(0, L)$ such that every control $v \in L^2(0, T)$ such that $y_{|t=T} = 0$ satisfy

$$\|v\|_{L^2(0,T)} \geq C_\varepsilon \exp\left(CLM^{1/2}\varepsilon^{-1/2}\right)\|y_0\|_{L^2(0,L)}, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

[1] C., Guerrero. On the non-uniform null controllability of a linear KdV equation. *Asymptot. Anal.*, 2015.

Outline

A linear fourth order equation

- $T > 0$, $M \in \mathbb{R} \setminus \{0\}$ (transport coefficient), $\varepsilon > 0$ (diffusion coeficient).

$$\begin{cases} z_t + \varepsilon z_{xxxx} - M z_x = 0 & \text{in } (0,T) \times (0,L), \\ z_{|x=0} = v_1(t), \qquad z_{|x=L} = 0 & \text{in } (0,T), \\ z_{xx|x=0} = v_2(t), \quad z_{xx|x=L} = 0 & \text{in } (0,T), \\ z_{|t=0} = z_0, & \text{in } (0,L). \end{cases}$$

- Controllability results by Cerpa, Mercado, Guzmán.
- Null controllability: For all $z_0 \in L^2(0,L)$, there are $v_1 \in L^2(0,T)$ and $v_2 \in L^2(0,T)$ such that $z_{|t=T} = 0$, and there is $C > 0$ such that

$$\|v_1\|_{L^2(0,T)} + \|v_2\|_{L^2(0,T)} \leq C \|y_0\|_{L^2(0,L)}.$$

- In particular, $C^{\varepsilon}_{cost} \leq C$.

Duality

Adjoint equation:

$$\begin{cases} -\varphi_t + \varepsilon\varphi_{xxxx} + M\varphi_x = 0 & \text{in } (0,T)\times(0,L), \\ \varphi_{|x=0} = 0, \qquad \varphi_{|x=L} = 0 & \text{in } (0,T), \\ \varphi_{xx|x=0} = 0, \quad \varphi_{xx|x=L} = 0 & \text{in } (0,T), \\ \varphi_{|t=T} = \varphi_T, & \text{in } (0,L). \end{cases}$$

- Observability:

$$\|\varphi_{|t=T/2}\|_{L^2(0,L)} \leq C_\varepsilon \exp\left(CT^{-1/3}\varepsilon^{-1/3}\right)\left(\|\varphi_{x|x=0}\|_{L^2(0,T)} + \|\varphi_{xxx|x=0}\|_{L^2(0,T)}\right)$$

- Exponencial dissipation(T large):

$$\|\varphi_{|t=0}\|_{L^2(0,L)} \leq C_\varepsilon \exp\left(-CT\varepsilon^{-1/3}\right)\|\varphi_{|t=T/2}\|_{L^2(0,L)}.$$

- Observability: Improvement from $\exp\left(C\varepsilon^{-1}\right)$ to $\exp\left(C\varepsilon^{-1/3}\right)$.

Uniform controllability uniform and explosion results[2]

Theorem (*Uniform controllability*). Let $L > 0$, $M \neq 0$ and $T \geq 40L/|M|$. Then, for all $z_0 \in L^2(0, L)$, there are $v_1, v_2 \in L^2(0, T)$ such that $z_{|t=T} = 0$ and

$$\|v_1\|_{L^2(0,T)} + \|v_2\|_{L^2(0,T)} \leq C_\varepsilon \exp\left(-C\varepsilon^{-1/3}\right)\|z_0\|_{L^2(0,L)}.$$

In particular: $\lim_{\varepsilon\to 0^+} C^\varepsilon_{cost} = 0$.

Theorem (*Explosion*). Let $L > 0$, $M \neq 0$ and $T < L/|M|$. Then, there is $z_0 \in L^2(0, L)$ and $\varepsilon_0 > 0$ such that every pair of controls $v_1, v_2 \in L^2(0, T)$ such that $z_{|t=T} = 0$ satisfy

$$\|v_1\|_{L^2(0,T)} + \|v_2\|_{L^2(0,T)} \geq C_\varepsilon \exp\left(C\varepsilon^{1/3}\right)\|z_0\|_{L^2(0,L)}, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

In particular: $\lim_{\varepsilon\to 0^+} C^\varepsilon_{cost} = +\infty$.

[2] C., Guzmán. On the cost of null controllability of a linear fourth order parabolic equation. *Submitted*.

Outline

Introduction

I. On the uniform controllability of a linear KdV equation

II. On the uniform controllability of a fourth order equation

Open problems

Open problems

- Two controls in KdV

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = v_0(t), \quad y_{x|x=L} = v_1(t), \quad y_{xx|x=L} = v_2(t) & \text{in } (0,T), \\ y_{|t=0} = y_0, \quad y_{|t=T} = 0 & \text{in } (0,L). \end{cases}$$

What happens in these cases?

1. $v_0(t) = 0$: Partially solved (restrictions on y_0).
2. $v_1(t) = 0$: Partially solved (restrictions on y_0)[3].
3. $v_2(t) = 0$: Open.

[3] C., Guerrero. Uniform null controllability of a linear KdV equation using two controls. *Preprint*.

Open problems

- $M > 0$ (negative transport) and $T > 0$ small.
  - For the heat equation

$$\begin{cases} y_t - \varepsilon y_{xx} - My_x = 0 & \text{in } (0,T)\times(0,L), \\ y_{|x=0} = v(t), \quad y_{|x=L} = 0 & \text{in } (0,T), \end{cases}$$

    it has been proved that $\lim_{\varepsilon\to 0^+} C^{\varepsilon}_{cost} = +\infty$, if $T < 2\sqrt{2}L/M$.
  - For the fourth order equation

$$\begin{cases} z_t + \varepsilon z_{xxxx} - Mz_x = 0 & \text{in } (0,T)\times(0,L), \\ z_{|x=0} = v_1(t), \quad z_{|x=L} = 0 & \text{in } (0,T), \\ z_{xx|x=0} = v_2(t), \quad z_{xx|x=L} = 0 & \text{in } (0,T). \end{cases}$$

    Is there $K \in (1,40)$ such that if $T < KL/M$, then $\lim_{\varepsilon\to 0^+} C^{\varepsilon}_{cost} = +\infty$?

Thank you