Control of parabolic systems and some applications to the control of fluids

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Recent trends in nonlinear PDEs

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Outline

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PDE control

Example: Heat equation

Consider a regular open $\Omega \subset \mathbb{R}^N$ and $\omega \subset \Omega$ (control domain)

$$\begin{cases} y_t - \Delta y = v \mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ y = 0 & x \in \partial\Omega, \\ y(0) = y_0 & x \in \Omega. \end{cases}$$

- $y = y(x,t)$: Temperature distribution.
- $v = v(x,t)$: Control supported in ω.

Question: Given $T > 0$ and $y_1 = y_1(x)$, is there v such that $y(T) = y_1$?

PDE control

Answer: In general, the answer is no due to the *regularizing effect*.

• It seems natural to consider the notion of control to the trajectories:
Consider a solution of

$$\begin{cases} \bar{y}_t - \Delta \bar{y} = 0 & (x,t) \in \Omega \times (0,T), \\ \bar{y} = 0 & x \in \partial\Omega, \\ \bar{y}(0) = \bar{y}_0 & x \in \Omega, \end{cases}$$

We look for a control v such that $y(T) = \bar{y}(T)$.

• By linearity (taking $\widetilde{y} := y - \bar{y}$), this is equivalent to the null-controllability:

$$y(T) = 0.$$

Therefore, we concentrate in this case.

Duality Method: Hilbert Uniqueness Method (HUM)

Construction of the control:

- We multiply $y_t - \Delta y = v\mathbb{1}_\omega$ by φ solution to the (adjoint) equation

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & (x,t) \in \Omega \times (0,T), \\ \varphi = 0 & x \in \partial\Omega, \\ \varphi(T) = \varphi_T \in L^2(\Omega) & x \in \Omega, \end{cases}$$

and integrate in $\Omega \times (0,T)$:

$$\int_\Omega y(T)\varphi_T \,\mathrm{d}x = \iint_{\omega\times(0,T)} v\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega y_0\varphi(0)\,\mathrm{d}x, \quad \forall \varphi_T \in L^2(\Omega).$$

- v is a control such that $y(T) = 0$ if and only if

$$\iint_{\omega\times(0,T)} v\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega y_0\varphi(0)\,\mathrm{d}x = 0, \quad \forall \varphi_T \in L^2(\Omega).$$

Observability inequality

The previous condition can be seen as an optimality condition for

$$J(\varphi_T) = \frac{1}{2} \iint_{\omega\times(0,T)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega y_0 \varphi(0)\,\mathrm{d}x.$$

- J convex, continuous and coercive if there exists $C > 0$ such that

$$\int_\Omega |\varphi(0)|^2 \,\mathrm{d}x \leq C \iint_{\omega\times(0,T)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

This is known as observability inequality.

- The control is given by

$$v := \widehat{\varphi},$$

where $\widehat{\varphi}$ is the solution of the adjoint equation associated to $\widehat{\varphi}_T$, minimum of J.

- Null-controllability is equivalent to observability.

Carleman estimates

How to prove the observability inequality?

Powerful tool to prove observability: Carleman estimates

$$\cdots + \iint_{\Omega\times(0,T)} \rho|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C \iint_{\Omega\times(0,T)} \rho|\varphi_t + \Delta\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + C \iint_{\omega_0\times(0,T)} \rho|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t$$

- $\varphi(x,t) = 0$, $x \in \partial\Omega$.
- $\rho = \rho(x,t)$ is a positive function and continuous in $\overline{\Omega} \times (0,T)$ with critical points only in $\omega_0 \subset \omega$.
- To deduce the observability, we use dissipation properties as

$$\int_\Omega |\varphi(0)|^2 \,\mathrm{d}x \leq \int_\Omega |\varphi(t)|^2 \,\mathrm{d}x, \quad t \in (0,T).$$

Control of a system of two equations with one control

Consider the system with one scalar control

$$\begin{cases} y_t - \Delta y = z + v\mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ z_t - \Delta z = y\mathbb{1}_{\mathcal{O}} & (x,t) \in \Omega \times (0,T), \\ y = z = 0 & x \in \partial\Omega, \\ y(0) = y^0, \quad z(0) = z^0 & x \in \Omega. \end{cases}$$

- We look for v such that $y(T) = z(T) = 0$.
- Observability inequality: There exists $C > 0$ such that

$$\int_\Omega \left(|\varphi(0)|^2 + |\psi(0)|^2\right) \mathrm{d}x \leq C \iint_{\omega \times (0,T)} |\varphi|^2 \, \mathrm{d}x \, \mathrm{d}t$$

where (φ, ψ) is the solution to the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = \psi\mathbb{1}_{\mathcal{O}} & (x,t) \in \Omega \times (0,T), \\ -\psi_t - \Delta\psi = \varphi & (x,t) \in \Omega \times (0,T), \\ \varphi = \psi = 0 & x \in \partial\Omega, \\ \varphi(T) = \varphi_T, \quad \psi(T) = \psi_T & x \in \Omega. \end{cases}$$

Control of a system of two equations with one control

- The idea is to combine Carleman estimates for φ and ψ:

$$\iint_{\Omega\times(0,T)} \rho_1|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C \iint_{\Omega\times(0,T)} \rho_2|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + C \iint_{\omega_0\times(0,T)} \rho_1|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t$$

$$\iint_{\Omega\times(0,T)} \rho_1|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C \iint_{\Omega\times(0,T)} \rho_2|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t + C \iint_{\omega_0\times(0,T)} \rho_1|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t$$

- Estimate the local term of ψ: $\psi = -\varphi_t - \Delta\varphi$ in $\mathcal{O}$.
- We assume $\omega \cap \mathcal{O} \neq \emptyset$ and choose $\omega_0 \subset \Omega \cap \mathcal{O}$.

$$\iint_{\omega_0\times(0,T)} \rho_1|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t = \iint_{\omega_0\times(0,T)} \rho_1\psi(-\varphi_t - \Delta\varphi) \,\mathrm{d}x\,\mathrm{d}t$$

$$\leq \frac{1}{2C} \iint_{\omega_0\times(0,T)} \rho_1|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + C \iint_{\omega_0\times(0,T)} \rho_1|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Outline

Some systems from fluid mechanics ($N = 2$ or 3)

- Navier-Stokes system (N scalar controls)

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = v 1_\omega, \quad \nabla \cdot y = 0 & (x,t) \in \Omega \times (0,T), \\ y = 0 & x \in \partial\Omega. \end{cases}$$

  - $y = y(x,t) \in \mathbb{R}^N$: Velocity field of the fluid.
  - $v \in \mathbb{R}^N$ is the control.

- Boussinesq system ($N+1$ scalar controls)

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = v 1_\omega + \theta\, e_N, \quad \nabla \cdot y = 0 & (x,t) \in \Omega \times (0,T), \\ \theta_t - \Delta \theta + y \cdot \nabla \theta = v_0 1_\omega & (x,t) \in \Omega \times (0,T), \\ y = 0, \quad \theta = 0 & x \in \partial\Omega. \end{cases}$$

  - $\theta = \theta(x,t) \in \mathbb{R}$: Temperature of the fluid.
  - $v_0 \in \mathbb{R}$: Control acting on the temperature.

Question: Is it possible to control these systems with less scalar controls?

Null controllability of the Navier-Stokes system

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = v\mathbb{1}_\omega, \quad \nabla\cdot y = 0 & (x,t)\in\Omega\times(0,T), \\ y = 0 & x\in\partial\Omega \\ y(0) = y^0 & x\in\Omega. \end{cases}$$

- $i_0 \in \{1,\dots,N\}$, $T>0$ y $\omega\subset\Omega$.
- C. Guerrero (2012)[1]: There is a $\delta>0$ such that if $\|y^0\|\le\delta$, then there is a control v, with $v_{i_0}\equiv 0$, and an associated solution (y,p) such that

$$y(T)=0.$$

Idea:

- Control of the linearized system around zero: $y_t - \Delta y + \nabla p = v\mathbb{1}_\omega$.
- Go back to the non-linear system using a local inversion argument.

[1] C., Guerrero. Local null controllability of the N-dimensional Navier-Stokes system with N-1 scalar controls in an arbitrary control domaine. *J. Math. Fluid Mech*, 2013.

Navier-Stokes system. Case $N = 2$.

- $y = (y_1, y_2)$, $v = (v_1, 0)$.

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = (v_1, 0)\mathbb{1}_\omega, \quad \nabla \cdot y = 0 & (x,t) \in \Omega \times (0,T), \\ y = 0 & x \in \partial\Omega. \end{cases}$$

- Linearization around zero:

$$\begin{cases} y_t - \Delta y + \nabla p = (v_1, 0)\mathbb{1}_\omega, \quad \nabla \cdot y = 0 & (x,t) \in \Omega \times (0,T), \\ y = 0 & x \in \partial\Omega. \end{cases}$$

- Observability inequality:

$$\int_\Omega \left(|\varphi_1(0)|^2 + |\varphi_2(0)|^2\right) \mathrm{d}x \leq C \iint_{\omega \times (0,T)} |\varphi_1|^2 \, \mathrm{d}x \, \mathrm{d}t$$

where $\varphi = (\varphi_1, \varphi_2)$ is the solution to the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = 0, \quad \nabla \cdot \varphi = 0 & (x,t) \in \Omega \times (0,T), \\ \varphi = 0 & x \in \partial\Omega. \end{cases}$$

Navier-Stokes system. Case $N = 2$.

- No coupling between φ_1 and φ_2? Yes: $\partial_1 \varphi_1 + \partial_2 \varphi_2 = 0$.
- Carleman estimate for φ_1. Since $\Delta \pi = 0$, we have

$$-(\Delta \varphi_1)_t - \Delta(\Delta \varphi_1) = 0,$$

  but no boundary condition for $\Delta \varphi_1$.
- Carleman estimate with non-homogenous boundary conditions:

$$\cdots + \iint_{\Omega \times (0,T)} \rho_1 |\Delta \varphi_1|^2 \, dx \, dt \leq C \iint_{\omega_0 \times (0,T)} \rho_1 |\Delta \varphi_1|^2 \, dx \, dt + b.t.$$

- Recover φ_2 from $\varphi|_{\partial\Omega} = 0$ and $\partial_1 \varphi_1 + \partial_2 \varphi_2 = 0$.

Boussinesq system

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = v\mathbb{1}_\omega + \theta\, e_N, \quad \nabla\cdot y = 0 & (x,t)\in\Omega\times(0,T), \\ \theta_t - \Delta\theta + y\cdot\nabla\theta = v_0\mathbb{1}_\omega & (x,t)\in\Omega\times(0,T), \\ y = 0, \quad \theta = 0 & x\in\partial\Omega \\ y(0) = y^0, \quad \theta(0) = \theta^0 & x\in\Omega. \end{cases}$$

- $i_0 \in \{1,\dots,N-1\}$, $T>0$ y $\omega\subset\Omega$.
- C. (2012)[2]: There is a $\delta>0$ such that if $\|(y^0,\theta^0)\|\le\delta$, there are controls v_0 and v, with $v_{i_0}\equiv v_N\equiv 0$, and an associated solution (y,p,θ) such that

$$y(T)=0 \text{ y } \theta(T)=0.$$

[2] C. Local controllability of the N-dimensional Boussinesq system with N-1 scalar controls in an arbitrary control domain. Math. Control Relat. Fields, 2012.

Boussinesq system. Case $N = 2$.

- $y = (y_1, y_2)$, $\theta \in \mathbb{R}$, $v_0 \in \mathbb{R}$.

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = (0, \theta), & \nabla \cdot y = 0 \quad (x,t) \in \Omega \times (0,T), \\ \theta_t - \Delta \theta + y \cdot \nabla \theta = v_0 \mathbb{1}_\omega & (x,t) \in \Omega \times (0,T). \end{cases}$$

- Linearization around zero:

$$\begin{cases} y_t - \Delta y + \nabla p = (0, \theta), & \nabla \cdot y = 0 \quad (x,t) \in \Omega \times (0,T), \\ \theta_t - \Delta \theta = v_0 \mathbb{1}_\omega & (x,t) \in \Omega \times (0,T). \end{cases}$$

- Observability inequality:

$$\int_\Omega \left(|\varphi_1(0)|^2 + |\varphi_2(0)|^2 + |\theta(0)|^2 \right) \mathrm{d}x \leq C \iint_{\omega \times (0,T)} |\theta|^2 \, \mathrm{d}x \, \mathrm{d}t$$

where $(\varphi, \theta) = (\varphi_1, \varphi_2, \theta)$ is the solution to the adjoint system

$$\begin{cases} -\varphi_t - \Delta \varphi + \nabla \pi = 0, & \nabla \cdot \varphi = 0 \quad (x,t) \in \Omega \times (0,T), \\ -\theta_t - \Delta \theta = \varphi_2 & (x,t) \in \Omega \times (0,T). \end{cases}$$

Boussinesq system. Case $N = 2$.

- We combine the previous ideas.
- Carleman for φ_2 and θ:

$$\cdots + \iint_{\Omega\times(0,T)} \rho_1 |\Delta\varphi_2|^2 \,dx\,dt \leq C \iint_{\omega_0\times(0,T)} \rho_1 |\Delta\varphi_2|^2 \,dx\,dt + b.t.$$

$$\cdots + \iint_{\Omega\times(0,T)} \rho_1 |\theta|^2 \,dx\,dt \leq C \iint_{\omega_0\times(0,T)} \rho_1 |\theta|^2 \,dx\,dt$$

- Recover φ_1 from $\varphi|_{\partial\Omega} = 0$ and $\partial_1\varphi_1 + \partial_2\varphi_2 = 0$.
- Use the equation $\varphi_2 = -\theta_t - \Delta\theta$.

Some comments

This method has its limitations:

- Seems difficult to consider a control like $v = (v_1, 0, 0)$.
  [Coron, Lissy, 2014]: Return method.
- Controllability to trajectories for Navier-Stokes:
  Adjoint equation:

$$-\varphi_t - \Delta\varphi + \bar{y} \cdot (\nabla\varphi + \nabla^t\varphi) + \nabla\pi = 0.$$

  Problem: The components of φ are mixed.
- Boundary controllability with one vanishing component (taking the trace of an extended controlled solution does not work).

Thank you