Stackelberg-Nash exact controllability for the Kuramoto-Sivashinsky equation

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Outline

Introduction

Stackelberg-Nash control strategy for the KS equation

Extensions, open problems

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Extensions, open problems

Control system. Mono-objective vs Multi-objective

- Standard control problem:

$$\begin{cases} y_t + \mathcal{A}(y) = f\mathbb{1}_{\mathcal{O}} & \text{in } \Omega \times (0,T) \\ y(0) = y_0 & \text{in } \Omega \end{cases}$$

  - $f \longrightarrow y(T) = 0$ in Ω.

- Control problem with more agents:

$$\begin{cases} y_t + \mathcal{A}(y) = f\mathbb{1}_{\mathcal{O}} + v_1\mathbb{1}_{\mathcal{O}_1} + v_2\mathbb{1}_{\mathcal{O}_2} & \text{in } \Omega \times (0,T) \\ y(0) = y_0 & \text{in } \Omega \end{cases}$$

  - $f \longrightarrow y(T) = 0$ in Ω.
  - $v_1 \longrightarrow y \approx y_{1,d}$ in $\mathcal{O}_{1,d} \subset \Omega$.
  - $v_2 \longrightarrow y \approx y_{2,d}$ in $\mathcal{O}_{2,d} \subset \Omega$.

Motivation: resort lake

$$\begin{cases} y_t + \mathcal{A}(y) = f\mathbb{1}_{\mathcal{O}} + v_1\mathbb{1}_{\mathcal{O}_1} + v_2\mathbb{1}_{\mathcal{O}_2} & \text{in } \Omega \times (0,T) \\ y(0) = y_0 & \text{in } \Omega \end{cases}$$

- Lake represented by $\Omega \subset \mathbb{R}^3$.
- $y = y(x,t)$: concentration of chemicals or of living organisms in the lake.
- Local agents P_1, P_2 that can decide their policy v_1, v_2 acting on $\mathcal{O}_1$, $\mathcal{O}_2$ (The followers).
- The manager of the resort decides the policy f acting on $\mathcal{O}$ (The leader).
- Goal of the manager: "Clean" the lake at time T ($y(T) = 0$).
- Goal of the agents: To be close to a target concentration $y_{i,d}$ in $\mathcal{O}_{i,d} \subset \Omega$ during the time interval $(0,T)$ ($y \approx y_{i,d}$ in $\mathcal{O}_{i,d}$).
- Other applications: Heat control, fluid mechanics, finances...
  Some authors: Lions, Díaz–Lions, Glowinski–Periaux–Ramos, Guillén-González–Marques-Lopes–Rojas-Medar, Araruna–Fernández-Cara–Santos.

Kuramoto-Sivashinsky equation

Let $Q := (0, L) \times (0, T)$. f the leader, v_1 and v_2 the followers.

$$\begin{cases} y_t + y_{xxxx} + \lambda y_{xx} + y y_x = f \mathbb{1}_{\mathcal{O}} + v_1 \mathbb{1}_{\mathcal{O}_1} + v_2 \mathbb{1}_{\mathcal{O}_2} & \text{in } Q \\ y_{|x=0} = y_{x|x=0} = 0 = y_{|x=L} = y_{x|x=L} & \text{in } (0,T) \\ y(0) = y_0 & \text{in } (0,L) \end{cases}$$

Consider the functionals $(i = 1, 2)$: $\alpha_i > 0, \mu_i > 0$

$$J_i(f; v_1, v_2) = \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d} \times (0,T)} |y - y_{i,d}|^2 \, dx \, dt + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0,T)} |v_i|^2 \, dx \, dt$$

- Task of the followers: $y \approx y_{i,d}$ in $\mathcal{O}_{i,d}$ with "little effort"

$$\min J_i(f; v_1, v_2), \quad i = 1, 2.$$

- Task of the leader: $y(T) = 0$ in Ω (or $y(T) = \bar{y}(T)$, $\bar{y}$ an uncontrolled trajectory).

Kuramoto-Sivashinsky equation

$$\begin{cases} y_t + y_{xxxx} + \lambda y_{xx} + y y_x = f \mathbb{1}_{\mathcal{O}} & \text{in } Q \\ y_{|x=0} = y_{x|x=0} = 0 = y_{|x=L} = y_{x|x=L} & \text{in } (0,T) \\ y(0) = y_0 & \text{in } (0,L) \end{cases}$$

Several controllability results:

- distributed controls
- boundary controls
- systems
- other boundary conditions

Authors: Cerpa, Mercado, Pazoto, Guzmán, Gao, C..

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The Stackelberg-Nash strategy

$$\begin{cases} y_t + y_{xxxx} + \lambda y_{xx} + y y_x = f 1\!\!1_{\mathcal{O}} + v_1 1\!\!1_{\mathcal{O}_1} + v_2 1\!\!1_{\mathcal{O}_2} & \text{in } Q \\ y_{|x=0} = y_{x|x=0} = 0 = y_{|x=L} = y_{x|x=L} & \text{in } (0,T) \\ y(0) = y_0 & \text{in } (0,L) \end{cases}$$

- **Step 1:** For fixed f, find a Nash equilibrium (v_1, v_2):

$$J_1(f; v_1, v_2) = \min_{\hat{v}_1} J_1(f; \hat{v}_1, v_2), \quad J_2(f; v_1, v_2) = \min_{\hat{v}_2} J_2(f; v_1, \hat{v}_2)$$

Of course, this equilibrium depends on f: $v_1 = v_1(f)$, $v_2 = v_2(f)$.

- **Step 2:** Find f such that $y(T) = \bar{y}(T)$, where $\bar{y}$ solves

$$\begin{cases} \bar{y}_t + \bar{y}_{xxxx} + \lambda \bar{y}_{xx} + \bar{y}\bar{y}_x = 0 & \text{in } Q \\ \bar{y}_{|x=0} = \bar{y}_{x|x=0} = 0 = \bar{y}_{|x=L} = \bar{y}_{x|x=L} & \text{in } (0,T) \\ \bar{y}(0) = \bar{y}_0 & \text{in } (0,L) \end{cases}$$

This is the *Stackelberg-Nash strategy*.

Stackelberg-Nash strategy

Change of variables: $z := y - \bar{y}$

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} + z z_x + (\bar{y} z)_x = f 1\!\!1_{\mathcal{O}} + v_1 1\!\!1_{\mathcal{O}_1} + v_2 1\!\!1_{\mathcal{O}_2} & \text{in } Q \\ z_{|x=0} = z_{x|x=0} = 0 = z_{|x=L} = z_{x|x=L} & \text{in } (0,T) \\ z(0) = y_0 - \bar{y}_0 & \text{in } (0,L) \end{cases}$$

- $z_{i,d} := \bar{y} - y_{i,d}$

$$J_i(f; v_1, v_2) = \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d} \times (0,T)} |z - z_{i,d}|^2 \, \mathrm{d}x \, \mathrm{d}t + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0,T)} |v_i|^2 \, \mathrm{d}x \, \mathrm{d}t$$

- For the leader is a null controllability problem: $z(T) = 0$, and the followers intend to be a Nash equilibrium for J_i, $i = 1, 2$.

Linear equation

Let us consider a linearized version of the KS equation: $(\bar{y} \equiv 0)$

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = f 1\!\!1_{\mathcal{O}} + v_1 1\!\!1_{\mathcal{O}_1} + v_2 1\!\!1_{\mathcal{O}_2} & \text{in } Q \\ z_{|x=0} = z_{x|x=0} = 0 = z_{|x=L} = z_{x|x=L} & \text{in } (0,T) \\ z(0) = y_0 - \bar{y}_0 & \text{in } (0,L) \end{cases}$$

- Nash equilibrium is equivalent to (due to the convexity of J_i)

$$\begin{cases} J_1'(f; v_1, v_2)(\hat{v}_1, 0) = 0 & \forall \hat{v}_1 \in L^2(\mathcal{O}_1 \times (0,T)) \\ J_2'(f; v_1, v_2)(0, \hat{v}_2) = 0 & \forall \hat{v}_2 \in L^2(\mathcal{O}_2 \times (0,T)) \end{cases}$$

- Characterization of Nash equilibrium: Optimality system

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} = f 1\!\!1_{\mathcal{O}} - \dfrac{1}{\mu_1}\phi^1 1\!\!1_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\phi^2 1\!\!1_{\mathcal{O}_2} & \text{in } Q \\ -\phi^i_t + \phi^i_{xxxx} + \lambda \phi^i_{xx} = \alpha_i (z - z_{i,d}) 1\!\!1_{\mathcal{O}_{i,d}} & \text{in } Q \\ z(0) = y_0 - \bar{y}_0, \quad \phi^i(T) = 0 & \text{in } (0,L) \end{cases}$$

  - Followers: $v_1 = -\dfrac{1}{\mu_1}\phi^1$ and $v_2 = -\dfrac{1}{\mu_2}\phi^2$.
  - Leader: $z(T) = 0$.

Result for the linear system

Theorem

(Santos, C.) Assume:

- $\mu_i >> 1$ (existence of Nash equilibrium).
- $\iint_{\mathcal{O}_{i,d}} \rho(t)^2 |z_{i,d}|^2 \, dx \, dt < +\infty$, with $\lim_{t \to T} \rho(t) = +\infty$.
- $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$, $i = 1, 2$ (No assumptions on $\mathcal{O}_i$).

There exist (a leader control) $f \in L^2(\mathcal{O} \times (0,T))$ and a Nash equilibrium for J_i (followers) $(v_1(f), v_2(f))$ such that $z(T) = 0$ in Ω.

- On the assumption $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$. Two cases:

1. $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$.
2. $\mathcal{O}_{1,d} \neq \mathcal{O}_{2,d}$: $\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}$ (different inside $\mathcal{O}$).

Adjoint system

$$\begin{cases} -\psi_t + \psi_{xxxx} + \lambda\psi_{xx} = \alpha_1\gamma^1 \mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2\gamma^2 \mathbb{1}_{\mathcal{O}_{2,d}} & \text{in } Q \\ \gamma^i_t + \gamma^i_{xxxx} + \lambda\gamma^i_{xx} = -\dfrac{1}{\mu_i}\psi \mathbb{1}_{\mathcal{O}_i} & \text{in } Q \\ \psi(T) = \psi_T, \quad \gamma^i(0) = 0 & \text{in } (0,L) \end{cases}$$

Observability inequality:

$$\int_\Omega |\psi(0)|^2 \, \mathrm{d}x + \sum_{i=1,2} \iint_Q \rho(t)^{-2} |\gamma^i|^2 \, \mathrm{d}x \, \mathrm{d}t \leq C \iint_{\mathcal{O}\times(0,T)} |\psi|^2 \, \mathrm{d}x \, \mathrm{d}t$$

Main tools:

- Energy estimates.
- Carleman inequalities for the KS equation (Cerpa, Mercado, Pazoto, Guzmán, Gao, C.).

Observability inequality: general idea

$$\begin{cases} -\psi_t + \psi_{xxxx} + \lambda\psi_{xx} = \alpha_1\gamma^1 \mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2\gamma^2 \mathbb{1}_{\mathcal{O}_{2,d}} & \text{in } Q \\ \gamma^i_t + \gamma^i_{xxxx} + \lambda\gamma^i_{xx} = -\dfrac{1}{\mu_i}\psi \mathbb{1}_{\mathcal{O}_i} & \text{in } Q \end{cases}$$

Classic approach:

- Carleman estimate for ψ, γ^1 and γ^2:

  $$I(\psi)+I(\gamma^1)+I(\gamma^2) \le C \iint_{\omega\times(0,T)} \rho(|\psi|^2+|\gamma^1|^2+|\gamma^2|^2)\,\mathrm{d}x\,\mathrm{d}t,\ \omega \subset\subset \mathcal{O}_{i,d}\cap\mathcal{O}$$

  Here, $I(\cdot)$ is the weighted energy, and ρ is the weight with critical points only in ω.
- Write γ^1 and γ^2 in terms of ψ using the coupling in $\mathcal{O}_{i,d}\cap\mathcal{O}$, $i=1,2$.
- Problem: we have a "loop"

$$I_\omega(\gamma^1) \lesssim I_\omega(\psi) + I_\omega(\gamma^2)$$

$$I_\omega(\gamma^2) \lesssim I_\omega(\psi) + I_\omega(\gamma^1)$$

Observability inequality. Case $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$

- Solution 1: If $\mathcal{O}_{1,d} = \mathcal{O}_{2,d} = \mathcal{O}_d$, let $h := \alpha_1\gamma^1 + \alpha_2\gamma^2$.

$$\begin{cases} -\psi_t + \psi_{xxxx} + \lambda\psi_{xx} = h\mathbb{1}_{\mathcal{O}_d} & \text{in } Q \\ h_t + h_{xxxx} + \lambda h_{xx} = -\dfrac{1}{\mu_1}\psi\mathbb{1}_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\psi\mathbb{1}_{\mathcal{O}_2} & \text{in } Q \end{cases}$$

Same approach:

- $\omega \subset\subset \mathcal{O}_d \cap \mathcal{O}$

$$I(\psi) + I(h) \leq C\big(I_\omega(\psi) + I_\omega(h)\big).$$

- Using the equation: $I_\omega(h) \lesssim I_\omega(\psi)$.
- From energy estimates ($\mu_i >> 1$, $\gamma^i(0) = 0$):

$$\int_\Omega |\psi(0)|^2 \,\mathrm{d}x + I(\psi) + I(h) \leq C I_\omega(\psi)$$

- Since $\gamma^i(0) = 0$, we have

$$\iint_Q \rho(t)^{-2}|\gamma^i|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C \iint_{\mathcal{O}_i\times(0,T)} \rho(t)^{-2}|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C I(\psi),$$

where $\rho(t)$ is a non-decreasing function.

Observability inequality. Case $\mathcal{O}_{1,d} \neq \mathcal{O}_{2,d}$

Suppose $\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{2,d} \cap \mathcal{O}$.

$$\begin{cases} -\psi_t + \psi_{xxxx} + \lambda\psi_{xx} = \alpha_1 \gamma^1 \mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2 \gamma^2 \mathbb{1}_{\mathcal{O}_{2,d}} & \text{in } Q \\ \gamma^i_t + \gamma^i_{xxxx} + \lambda\gamma^i_{xx} = -\dfrac{1}{\mu_i}\psi \mathbb{1}_{\mathcal{O}_i} & \text{in } Q \end{cases}$$

• Solution 2: A way around the "loop situation": Use two different weight functions (associated to ω_1 and ω_2).

- Carleman estimate for $\omega_1 \subset\subset \mathcal{O}_{1,d} \cap \mathcal{O}$, and $\omega_1 \cap \mathcal{O}_{2,d} \neq \emptyset$.
- Carleman estimate for $\omega_2 \subset\subset \mathcal{O}_{2,d} \cap \mathcal{O}$, and $\omega_2 \cap \mathcal{O}_{1,d} \neq \emptyset$.
- This way, they "do not see" each other:

$$I^1_{\omega_1}(\gamma^1) \lesssim I^1_{\omega_1}(\psi) \text{ and } I^2_{\omega_2}(\gamma^2) \lesssim I^2_{\omega_2}(\psi).$$

• This idea is due to S. Guerrero and M. C. Santos.

Observability inequality. Case $\mathcal{O}_{1,d} \neq \mathcal{O}_{2,d}$

$$\begin{cases} -\psi_t + \psi_{xxxx} + \lambda\psi_{xx} = \alpha_1\gamma^1 1\!\!1_{\mathcal{O}_{1,d}} + \alpha_2\gamma^2 1\!\!1_{\mathcal{O}_{2,d}} & \text{in } Q \\ \gamma_t^i + \gamma_{xxxx}^i + \lambda\gamma_{xx}^i = -\dfrac{1}{\mu_i}\psi 1\!\!1_{\mathcal{O}_i} & \text{in } Q \end{cases}$$

- We can obtain

$$I^1(\gamma^1) + I^2(\gamma^2) + I^1(\psi) \leq C\big(I^1_{\mathcal{O}}(\psi) + I^2_{\mathcal{O}}(\psi)\big).$$

  - Weight functions should be equal outside $\mathcal{O}$, so we can compare the global terms coming from the Carleman estimates.
  - Therefore, Carleman estimate for ψ should hide what happens inside $\mathcal{O}$: Carleman estimate for $(\theta\psi)$, where θ is a cutoff function such that

$$\theta(x) = 0,\ x \in \mathcal{O}' \text{ and } \theta(x) = 1,\ x \in \Omega \setminus \mathcal{O}.$$

    $\mathcal{O}'$ is such that $\omega_i \subset\subset \mathcal{O}'$.
  - New Carleman estimate is needed with a less regular right-hand side.

Non-linear equation

Back to the original equation...

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} + z z_x + (\bar{y} z)_x = f \mathbb{1}_{\mathcal{O}} + v_1 \mathbb{1}_{\mathcal{O}_1} + v_2 \mathbb{1}_{\mathcal{O}_2} & \text{in } Q \\ z_{|x=0} = z_{x|x=0} = 0 = z_{|x=L} = z_{x|x=L} & \text{in } (0,T) \\ z(0) = y_0 - \bar{y}_0 & \text{in } (0,L) \end{cases}$$

$$J_i(f; v_1, v_2) = \frac{\alpha_i}{2} \iint_{\mathcal{O}_{i,d} \times (0,T)} |z - z_{i,d}|^2 \, \mathrm{d}x \, \mathrm{d}t + \frac{\mu_i}{2} \iint_{\mathcal{O}_i \times (0,T)} |v_i|^2 \, \mathrm{d}x \, \mathrm{d}t$$

Here, we lose convexity of J_i. Let us weaken the definition of Nash equilibrium.

- (v_1, v_2) is a Nash quasi-equilibrium if

$$\begin{cases} J_1'(f; v_1, v_2)(\hat{v}_1, 0) = 0 & \forall \hat{v}_1 \in L^2(\mathcal{O}_1 \times (0,T)) \\ J_2'(f; v_1, v_2)(0, \hat{v}_2) = 0 & \forall \hat{v}_2 \in L^2(\mathcal{O}_2 \times (0,T)) \end{cases}$$

A local result

- Optimality system: $v_1 = -\frac{1}{\mu_1}\phi^1$ and $v_2 = -\frac{1}{\mu_2}\phi^2$ where

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} + z z_x + (\bar{y} z)_x = f 1\!\!1_{\mathcal{O}} - \frac{1}{\mu_1}\phi^1 1\!\!1_{\mathcal{O}_1} - \frac{1}{\mu_2}\phi^2 1\!\!1_{\mathcal{O}_2} & \text{in } Q \\ -\phi^i_t + \phi^i_{xxxx} + \lambda \phi^i_{xx} - (z+\bar{y})\phi^i_x = \alpha_i (z - z_{i,d}) 1\!\!1_{\mathcal{O}_{i,d}} & \text{in } Q \\ z(0) = y_0 - \bar{y}_0, \quad \phi^i(T) = 0, \quad z(T) = 0 & \text{in } (0,L) \end{cases}$$

Theorem

(Santos, C.) Assume $\mu_i >> 1$, $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$, $i = 1,2$ and $\lim_{t\to T} \rho(t) = +\infty$. There exists $\delta > 0$ such that if

$$\|y_0 - \bar{y}_0\|_{L^2} + \sum_{i=1,2} \|\rho(\bar{y} - y_{i,d})\|_{L^2} < \delta,$$

then, there exist (a leader control) $f \in L^2(\mathcal{O} \times (0,T))$ and a Nash quasi-equilibrium (followers) $(v_1(f), v_2(f))$ such that $z(T) = 0$ in Ω $(y(T) = \bar{y}(T))$.

A local result

- Null controllability of the linearized system

$$\begin{cases} z_t + z_{xxxx} + \lambda z_{xx} + (\bar{y}z)_x = F^0 + f\mathbb{1}_{\mathcal{O}} - \dfrac{1}{\mu_1}\phi^1\mathbb{1}_{\mathcal{O}_1} - \dfrac{1}{\mu_2}\phi^2\mathbb{1}_{\mathcal{O}_2} & \text{in } Q \\ -\phi^i_t + \phi^i_{xxxx} + \lambda\phi^i_{xx} - \bar{y}\phi^i_x = F^i + \alpha_i(z - z_{i,d})\mathbb{1}_{\mathcal{O}_{i,d}} & \text{in } Q \\ z(0) = y_0 - \bar{y}_0, \quad \phi^i(T) = 0 & \text{in } (0,L) \end{cases}$$

where F^0, F^1, F^2 belong to appropriate weighted spaces.

- Adjoint system

$$\begin{cases} -\psi_t + \psi_{xxxx} + \lambda\psi_{xx} - \bar{y}\psi_x = G^0 + \alpha_1\gamma^1\mathbb{1}_{\mathcal{O}_{1,d}} + \alpha_2\gamma^2\mathbb{1}_{\mathcal{O}_{2,d}} & \text{in } Q \\ \gamma^i_t + \gamma^i_{xxxx} + \lambda\gamma^i_{xx} + (\bar{y}\gamma^i)_x = G^i - \dfrac{1}{\mu_i}\psi\mathbb{1}_{\mathcal{O}_i} & \text{in } Q \\ \psi(T) = \psi_T, \quad \gamma^i(0) = 0 & \text{in } (0,L) \end{cases}$$

where $G^0, G^1, G^2 \in L^2(Q)$.

A local result

- If we assume $\bar{y}, \bar{y}_x \in L^\infty$, we can prove the observability inequality

$$\int_\Omega |\psi(0)|^2 \,\mathrm{d}x + \iint_Q \rho(t)^{-2}|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + \sum_{i=1,2} \iint_Q \rho(t)^{-2}|\gamma^i|^2 \,\mathrm{d}x\,\mathrm{d}t$$

$$\leq C \iint_{\mathcal{O}\times(0,T)} \rho(t)^{-2}|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t + C \iint_Q \rho(t)^{-2}\left(|G^0|^2+|G^1|^2+|G^2|^2\right) \mathrm{d}x\,\mathrm{d}t.$$

- We conclude using a local inversion argument.

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Extensions, open problems

Extension, open problems

- Distributed leader, boundary followers

$$\begin{cases} y_t + y_{xxxx} + \lambda y_{xx} + y y_x = f\mathbb{1}_{\mathcal{O}} & \text{in } Q \\ y_{|x=0} = v_1, \quad y_{|x=L} = v_2 & \text{in } (0,T) \\ y_{x|x=0} = v_3, \quad y_{x|x=L} = v_4 & \text{in } (0,T) \\ y(0) = y_0 & \text{in } (0,L) \end{cases}$$

- Boundary leader, distributed followers

$$\begin{cases} y_t + y_{xxxx} + \lambda y_{xx} + y y_x = v_1\mathbb{1}_{\mathcal{O}_1} + v_2\mathbb{1}_{\mathcal{O}_2} & \text{in } Q \\ y_{|x=0} = f, \quad y_{|x=L} = 0 & \text{in } (0,T) \\ y_{x|x=0} = 0, \quad y_{x|x=L} = 0 & \text{in } (0,T) \\ y(0) = y_0 & \text{in } (0,L) \end{cases}$$

Lots of options: Location of the leader, quantity of followers...

- Regularity of the target trajectory. Does the results hold for $\bar{y}$ satisfiying only $\bar{y} \in L^\infty$?

Some additional sharp Carleman estimates are required...

Extension, open problems

- A variation of the problem:

Insensitizing controls following a Stackelberg-Nash strategy.

$$\begin{cases} y_t - \Delta y = f \mathbb{1}_{\mathcal{O}} + v_1 \mathbb{1}_{\mathcal{O}_1} + v_2 \mathbb{1}_{\mathcal{O}_2} & \text{in } \Omega \times (0,T) \\ y = 0 & \text{in } \partial\Omega \times (0,T) \\ y(0) = y_0 + \tau \hat{y}_0 & \text{in } \Omega \end{cases}$$

where $\tau \hat{y}_0$ is an unknown perturbation of y_0.

- Given f, find a Nash equilibrium (v_1, v_2).
- Find f insensitizing the functional

$$J(f; y) = \frac{1}{2} \iint_{\mathcal{O}' \times (0,T)} |y|^2 \, \mathrm{d}x \, \mathrm{d}t, \quad \mathcal{O}' \subset \Omega$$

with respect to $\tau \hat{y}_0$.

This is equivalent to the null controllability of a system of 6 equations! (we are still trying...)

¡Gracias!