Insensitizing controls for the Boussinesq system with a reduced number of controls

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Outline

Introduction

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Some comments, perspectives

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Control system

A control system governed by a partial differential equation can be formulated as

$$\begin{cases} y'(t) = f(t, y(t), v(t)), & t > 0 \\ y(0) = y_0. \end{cases}$$

- $y(t) \in \mathcal{X}$ is the state of the system.
- $v(t) \in \mathcal{U}$ is the control.
- $\mathcal{X}, \mathcal{U}$ are the state and admissible controls spaces, respectively.
- Controllability problem: Given T and y_0, find $v(t)$ driving $y(t)$ to a target y_1 at time T, that is, $y(T) = y_1$.
- Controllability types:
  - Exact.
  - Null: $y(T) = 0$.
  - Approximate: $y(T)$ close to y_1.
  - Local: y_0 close to y_1.

Example model: Heat equation

Consider a regular open $\Omega \subset \mathbb{R}^N$ and $\omega \subset \Omega$ (control domain)

$$\begin{cases} y_t - \Delta y = v\mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ y = 0 & x \in \partial\Omega, \\ y(0) = y^0 & x \in \Omega, \end{cases}$$

$y_0 \in L^2(\Omega)$ and $\mathbb{1}_\omega(x)$ the characteristic function of ω.

- We look for $v \in L^2(\omega \times (0,T))$ such that $y(T) = 0$ and

$$\|v\|_{L^2(\omega\times(0,T))} \leq C\|y_0\|_{L^2(\Omega)}.$$

- By linearity, this is equivalent to the control to the trajectories: find $v \in L^2(\omega \times (0,T))$ such that $y(T) = \bar{y}(T)$, where $\bar{y}$ is solution of

$$\begin{cases} \bar{y}_t - \Delta \bar{y} = 0 & (x,t) \in \Omega \times (0,T), \\ \bar{y} = 0 & x \in \partial\Omega, \\ \bar{y}(0) = \bar{y}^0 & x \in \Omega. \end{cases}$$

Observability and Carleman estimates

Null controllability is equivalent to the observability inequality: There exists $C > 0$ such that

$$\int_\Omega |\varphi(0)|^2 \mathrm{d}x \leq C \iint_{\omega\times(0,T)} |\varphi|^2 \mathrm{d}x\, \mathrm{d}t$$

where φ is the solution of the adjoint equation

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & (x,t) \in \Omega\times(0,T), \\ \varphi = 0 & x \in \partial\Omega, \\ \varphi(T) = \varphi_T \in L^2(\Omega) & x \in \Omega. \end{cases}$$

- Carleman estimates: They have the form

$$\iint_{\Omega\times(0,T)} \rho|\varphi|^2 \mathrm{d}x\, \mathrm{d}t \leq C \iint_{\Omega\times(0,T)} \rho|\varphi_t + \Delta\varphi|^2 \mathrm{d}x\, \mathrm{d}t + C \iint_{\omega\times(0,T)} \rho|\varphi|^2 \mathrm{d}x\, \mathrm{d}t$$

  - $\rho = \rho(x,t)$ is a continuous and positive function.
  - To obtain observability, we combine it with the energy estimate

$$\int_\Omega |\varphi(0)|^2 \mathrm{d}x \leq \int_\Omega |\varphi(t)|^2 \mathrm{d}x, \quad t \in (0,T).$$

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Controllability of a system of two heat equations

Consider the system

$$\begin{cases} y_t - \Delta y = z + v\mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ z_t - \Delta z = y + u\mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ y = z = 0 & x \in \partial\Omega, \\ y(0) = y^0, \quad z(0) = z^0 & x \in \Omega. \end{cases}$$

- We look for $v, u \in L^2(\omega \times (0,T))$ such that $y(T) = z(T) = 0$.
- Observability inequality: There exists $C > 0$ such that

$$\int_\Omega \left(|\varphi(0)|^2 + |\psi(0)|^2\right) \mathrm{d}x \le C \iint_{\omega \times (0,T)} \left(|\varphi|^2 + |\psi|^2\right) \mathrm{d}x\, \mathrm{d}t$$

where (φ, ψ) is the solution of the adjoint equation

$$\begin{cases} -\varphi_t - \Delta\varphi = \psi & (x,t) \in \Omega \times (0,T), \\ -\psi_t - \Delta\psi = \varphi & (x,t) \in \Omega \times (0,T), \\ \varphi = \psi = 0 & x \in \partial\Omega, \\ \varphi(T) = \varphi_T, \quad \psi(T) = \psi_T & x \in \Omega. \end{cases}$$

Controllability of a system of two heat equations

- The idea is to combine Carleman inequalities for φ and ψ:

$$\iint_{\Omega\times(0,T)} \rho_1 |\varphi|^2 \mathrm{d}x\,\mathrm{d}t \leq C \iint_{\Omega\times(0,T)} \rho_2 |\psi|^2 \mathrm{d}x\,\mathrm{d}t + C \iint_{\omega\times(0,T)} \rho_1 |\varphi|^2 \mathrm{d}x\,\mathrm{d}t$$

$$\iint_{\Omega\times(0,T)} \rho_1 |\psi|^2 \mathrm{d}x\,\mathrm{d}t \leq C \iint_{\Omega\times(0,T)} \rho_2 |\varphi|^2 \mathrm{d}x\,\mathrm{d}t + C \iint_{\omega\times(0,T)} \rho_1 |\psi|^2 \mathrm{d}x\,\mathrm{d}t$$

We can choose the weights to satisfy $\rho_2 \leq \dfrac{1}{4C}\rho_1$. Adding both estimates:

$$\iint_{\Omega\times(0,T)} \rho_1 \left(|\varphi|^2 + |\psi|^2\right) \mathrm{d}x\,\mathrm{d}t \leq C \iint_{\omega\times(0,T)} \rho_1 \left(|\varphi|^2 + |\psi|^2\right) \mathrm{d}x\,\mathrm{d}t.$$

Controllability of a system of two heat equations with one control

Consider the system with only one control:

$$\begin{cases} y_t - \Delta y = z + v\mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ z_t - \Delta z = y & (x,t) \in \Omega \times (0,T), \\ y = z = 0 & x \in \partial\Omega, \\ y(0) = y^0, \quad z(0) = z^0 & x \in \Omega. \end{cases}$$

- We look for $v \in L^2(\omega \times (0,T))$ such that $y(T) = z(T) = 0$.
- Observability inequality: There exists $C > 0$ such that

$$\int_\Omega \left(|\varphi(0)|^2 + |\psi(0)|^2\right) \mathrm{d}x \leq C \iint_{\omega \times (0,T)} |\varphi|^2 \mathrm{d}x\, \mathrm{d}t$$

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Controllability of a system of two heat equations with one control

- Again, the idea is to combine Carleman inequalities for φ and ψ:

$$\iint_{\Omega\times(0,T)} \rho_1 |\varphi|^2 \mathrm{d}x\,\mathrm{d}t \leq C \iint_{\Omega\times(0,T)} \rho_2 |\psi|^2 \mathrm{d}x\,\mathrm{d}t + C \iint_{\omega\times(0,T)} \rho_1 |\varphi|^2 \mathrm{d}x\,\mathrm{d}t$$

$$\iint_{\Omega\times(0,T)} \rho_1 |\psi|^2 \mathrm{d}x\,\mathrm{d}t \leq C \iint_{\Omega\times(0,T)} \rho_2 |\varphi|^2 \mathrm{d}x\,\mathrm{d}t + C \iint_{\omega\times(0,T)} \rho_1 |\psi|^2 \mathrm{d}x\,\mathrm{d}t$$

We can choose the weights to satisfy $\rho_2 \leq \frac{1}{4C}\rho_1$, but we need to estimate the <u>local term of ψ</u>. We use the equation ($\psi = -\varphi_t - \Delta\varphi$)

$$\iint_{\omega\times(0,T)} \rho_1 |\psi|^2 \mathrm{d}x\,\mathrm{d}t = \iint_{\omega\times(0,T)} \rho_1 \psi(-\varphi_t - \Delta\varphi) \mathrm{d}x\,\mathrm{d}t$$

$$\leq \frac{1}{8C} \iint_{\omega\times(0,T)} \rho_1 \left(|\psi_t|^2 + |\Delta\psi|^2\right) \mathrm{d}x\,\mathrm{d}t + C \iint_{\omega\times(0,T)} \rho_1 |\varphi|^2 \mathrm{d}x\,\mathrm{d}t.$$

Controllability of a system of two heat equations with one control

Consider the system with only one control and couplings of first order:

$$\begin{cases} y_t - y_{xx} = z_x + v\mathbb{1}_\omega & (x,t) \in \Omega \times (0,T), \\ z_t - z_{xx} = y_x & (x,t) \in \Omega \times (0,T), \\ y = z = 0 & x \in \partial\Omega, \\ y(0) = y^0, \quad z(0) = z^0 & x \in \Omega. \end{cases}$$

- We look for $v \in L^2(\omega \times (0,T))$ such that $y(T) = z(T) = 0$.
- Observability inequality: There exists $C > 0$ such that

$$\int_\Omega \left(|\varphi(0)|^2 + |\psi(0)|^2\right) \mathrm{d}x \le C \iint_{\omega\times(0,T)} |\varphi|^2 \mathrm{d}x\, \mathrm{d}t$$

where (φ, ψ) is the solution of the adjoint equation

$$\begin{cases} -\varphi_t - \varphi_{xx} = -\psi_x & (x,t) \in \Omega \times (0,T), \\ -\psi_t - \psi_{xx} = -\varphi_x & (x,t) \in \Omega \times (0,T), \\ \varphi = \psi = 0 & x \in \partial\Omega, \\ \varphi(T) = \varphi_T, \quad \psi(T) = \psi_T & x \in \Omega. \end{cases}$$

Controllability of a system of two heat equations with one control

- How to use the equation now to eliminate the local term of ψ?

$$\iint_{\Omega\times(0,T)} \rho_1 |\psi|^2 \mathrm{d}x\,\mathrm{d}t \leq C \iint_{\Omega\times(0,T)} \rho_2 |\varphi_x|^2 \mathrm{d}x\,\mathrm{d}t + C \iint_{\omega\times(0,T)} \rho_1 |\psi|^2 \mathrm{d}x\,\mathrm{d}t$$

- We need Carleman estimates with non-homogenous boundary conditions.

$$\begin{cases} -(\psi_x)_t - (\psi_x)_{xx} = -\varphi_{xx} & (x,t) \in \Omega \times (0,T), \\ \psi_x \neq 0 & x \in \partial\Omega. \end{cases}$$

$$\iint_{\Omega\times(0,T)} \rho_1 |\psi_x|^2 \leq \mathrm{b.t.}(\psi) + C \iint_{\Omega\times(0,T)} \rho_2 |\varphi_{xx}|^2 + C \iint_{\omega\times(0,T)} \rho_1 |\psi_x|^2$$

  - b.t.(ψ) are estimated with regularity results.
  - Now we can use the equation $\psi_x = \varphi_t + \varphi_{xx}$.

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Insensitizing controls for the Boussinesq system

- Ω bounded connected regular open subset of $\mathbb{R}^3$, $T > 0$.
- $\omega \subset \Omega$ (control set), $Q := \Omega \times (0,T)$, $\Sigma := \partial\Omega \times (0,T)$

We consider the Boussinesq system with incomplete data:

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = f + v\mathbb{1}_\omega + (0,0,\theta), \quad \nabla\cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta\theta + y\cdot\nabla\theta = f_0 + v_0\mathbb{1}_\omega & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0 + \tau\widehat{y}_0, \quad \theta(0) = \theta^0 + \tau\widehat{\theta}_0 & \text{in } \Omega. \end{cases}$$

where τ is a small constant and $\|\widehat{y}^0\|_{L^2(\Omega)^3} = \|\widehat{\theta}^0\|_{L^2(\Omega)} = 1$. Unknown.

Insensitizing control problem: To find controls v and v_0 in $L^2(\omega\times(0,T))$ such that the functional (Sentinel)

$$J_\tau(y,\theta) := \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} \left(|y|^2 + |\theta|^2\right) \mathrm{d}x\,\mathrm{d}t, \ \mathcal{O} \subset \Omega \ (\text{Observation set})$$

is not affected by the uncertainty of the initial data, that is,

$$\left.\frac{\partial J_\tau(y,\theta)}{\partial\tau}\right|_{\tau=0} = 0 \quad \forall\,(\widehat{y}_0,\widehat{\theta}_0) \in L^2(\Omega)^4 \text{ s.t. } \|\widehat{y}_0\|_{L^2(\Omega)^3} = \|\widehat{\theta}_0\|_{L^2(\Omega)} = 1.$$

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A cascade system

The previous condition is equivalent to the following null controllability problem: To find a control v and v_0 such that $z(0)=0$ and $q(0)=0$, where

$$\begin{cases} w_t-\Delta w+(w\cdot\nabla)w+\nabla p_0 = f+v\,\mathbb{1}_\omega+(0,0,r), & \nabla\cdot w=0 \quad \text{in } Q,\\ -z_t-\Delta z+(z\cdot\nabla^t)w-(w\cdot\nabla)z+\nabla p_1 = w\mathbb{1}_{\mathcal{O}}, & \nabla\cdot z=0 \quad \text{in } Q,\\ r_t-\Delta r+(w\cdot\nabla)r = f_0+v_0\,\mathbb{1}_\omega & \text{in } Q,\\ -q_t-\Delta q-(w\cdot\nabla)q = z_3+r\mathbb{1}_{\mathcal{O}} & \text{in } Q,\end{cases}$$

with boundary and initial conditions:

$$\begin{cases} w=z=0, \quad r=q=0 & \text{on } \Sigma,\\ w(0)=y^0, \quad z(T)=0, \quad r(0)=\theta^0, \quad q(T)=0 & \text{in } \Omega.\end{cases}$$

We are interested in controls of the form

1. $v=(v_1,0,0)$, $v_0\neq 0$
2. $v=(v_1,0,v_3)$ and $v_0=0$.

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Null controllability result 1

$$\begin{cases} w_t - \Delta w + (w\cdot\nabla)w + \nabla p_0 = f + v\,\mathbb{1}_\omega + (0,0,r), & \nabla\cdot w = 0 & \text{in } Q,\\ -z_t - \Delta z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + \nabla p_1 = w\mathbb{1}_{\mathcal{O}}, & \nabla\cdot z = 0 & \text{in } Q,\\ r_t - \Delta r + (w\cdot\nabla)r = f_0 + v_0\,\mathbb{1}_\omega & & \text{in } Q,\\ -q_t - \Delta q - (w\cdot\nabla)q = z_3 + r\mathbb{1}_{\mathcal{O}} & & \text{in } Q,\\ w = z = 0, \quad r = q = 0 & & \text{on } \Sigma,\\ w(0) = y^0, \quad z(T) = 0, \quad r(0) = \theta^0, \quad q(T) = 0 & & \text{in } \Omega. \end{cases}$$

Theorem[1]**.** Let $y^0 = 0$, $\theta^0 = 0$ and $\mathcal{O}\cap\omega \neq \emptyset$. There exists $\delta > 0$ such that if $\|e^{K/t^{10}}(f,f_0)\|_{L^2(Q)^4} < \delta$, there exist controls (v, v_0) in $L^2(\omega\times(0,T))$ of the form $v = (v_1, 0, 0)$, $v_0 \neq 0$ such that $z(0) = 0$ and $q(0) = 0$.

[1] C., Guerrero, Gueye. Insensitizing controls with two vanishing components for the three-dimensional Boussinesq system. *ESAIM Control Optim. Calc. Var.*, 2015

Null controllability result 2

$$\begin{cases} w_t - \Delta w + (w\cdot\nabla)w + \nabla p_0 = f + v\,\mathbb{1}_\omega + (0,0,r), & \nabla\cdot w = 0 & \text{in } Q,\\ -z_t - \Delta z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + \nabla p_1 = w\mathbb{1}_{\mathcal{O}}, & \nabla\cdot z = 0 & \text{in } Q,\\ r_t - \Delta r + (w\cdot\nabla)r = f_0 & & \text{in } Q,\\ -q_t - \Delta q - (w\cdot\nabla)q = z_3 + r\mathbb{1}_{\mathcal{O}} & & \text{in } Q,\\ w = z = 0, \quad r = q = 0 & & \text{on } \Sigma,\\ w(0) = y^0, \quad z(T) = 0, \quad r(0) = \theta^0, \quad q(T) = 0 & & \text{in } \Omega. \end{cases}$$

Theorem[2]**.** Let $y^0 = 0$, $\theta^0 = 0$ and $\mathcal{O}\cap\omega \neq \emptyset$. There exists $\delta > 0$ such that if $\|e^{K/t^{10}}(f,f_0)\|_{L^2(Q)^4} < \delta$, there exist a control v in $L^2(\omega\times(0,T))$ of the form $v = (v_1, 0, v_3)$ such that $z(0) = 0$ and $q(0) = 0$.

[2] C.. Insensitizing controls for the Boussinesq system with no control on the temperature equation. *To appear in Adv. Differential Equations.*

Method of proof

- Linearization around zero
- Null controllability of the linearized system (Main part of the proof).
  Main tool: Carleman estimate for the adjoint system with source terms.
- Inverse mapping theorem for the nonlinear system

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Linearized system

The linearized system around zero with source terms:

$$\begin{cases} w_t - \Delta w + \nabla p_0 = f^w + v\,\mathbb{1}_\omega + (0,0,r), & \nabla\cdot w = 0 & \text{in } Q, \\ -z_t - \Delta z + \nabla p_1 = f^z + w\mathbb{1}_\mathcal{O}, & \nabla\cdot z = 0 & \text{in } Q, \\ r_t - \Delta r = f^r + v_0\,\mathbb{1}_\omega & & \text{in } Q, \\ -q_t - \Delta q = f^q + z_3 + r\mathbb{1}_\mathcal{O} & & \text{in } Q, \end{cases}$$

with

$$\begin{cases} w = z = 0, \quad r = q = 0 & \text{on } \Sigma, \\ w(0) = 0, \quad z(T) = 0, \quad r(0) = 0, \quad q(T) = 0 & \text{in } \Omega. \end{cases}$$

We want to prove $z(0) = 0$ and $q(0) = 0$ with controls of the form

$$v = (v_1, 0, 0), v_0 \neq 0 \quad \text{and} \quad v = (v_1, 0, v_3), v_0 = 0.$$

We prove an observability inequality for the adjoint system

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Adjoint system and observability inequality

Dual variables: $\varphi \leftrightarrow w, \quad \psi \leftrightarrow z, \quad \phi \leftrightarrow r, \quad \sigma \leftrightarrow q$

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_\varphi = g^\varphi + \psi\,\mathbb{1}_{\mathcal{O}}, & \nabla\cdot\varphi = 0 \quad \text{in } Q, \\ \psi_t - \Delta\psi + \nabla\pi_\psi = g^\psi + (0,0,\sigma), & \nabla\cdot\psi = 0 \quad \text{in } Q, \\ -\phi_t - \Delta\phi = g^\phi + \varphi_3 + \sigma\,\mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ \sigma_t - \Delta\sigma = g^\sigma & \text{in } Q, \end{cases}$$

with

$$\begin{cases} \varphi = \psi = 0, \quad \phi = \sigma = 0 & \text{on } \Sigma, \\ \varphi(T) = 0, \quad \psi(0) = \psi^0, \quad \phi(T) = 0, \quad \sigma(0) = \sigma^0 & \text{in } \Omega, \end{cases}$$

For general controls $v = (v_1, v_2, v_3)$ and v_0:

$$\iint_Q \rho_1(t)(|\varphi|^2+|\psi|^2+|\phi|^2+|\sigma|^2) \leq C\iint_Q \rho_2(t)(|g^\varphi|^2+|g^\psi|^2+|g^\phi|^2+|g^\sigma|^2)$$
$$+ C\iint_{\omega\times(0,T)} \rho_3(t)(|\varphi_1|^2+|\varphi_2|^2+|\varphi_3|^2+|\phi|^2)$$

$\rho_i(t) \sim \exp(-C_i/t^{10}(T-t)^{10})$

Using energy estimate, we can change to $\rho_i(t) \sim \exp(-C_i/t^{10})$

Adjoint system and observability inequality

Dual variables: $\varphi \leftrightarrow w, \quad \psi \leftrightarrow z, \quad \phi \leftrightarrow r, \quad \sigma \leftrightarrow q$

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_\varphi = g^\varphi + \psi\, 1\!\!1_{\mathcal{O}}, & \nabla\cdot\varphi = 0 & \text{in } Q, \\ \psi_t - \Delta\psi + \nabla\pi_\psi = g^\psi + (0,0,\sigma), & \nabla\cdot\psi = 0 & \text{in } Q, \\ -\phi_t - \Delta\phi = g^\phi + \varphi_3 + \sigma\, 1\!\!1_{\mathcal{O}} & & \text{in } Q, \\ \sigma_t - \Delta\sigma = g^\sigma & & \text{in } Q, \end{cases}$$

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$$\begin{cases} \varphi = \psi = 0, \quad \phi = \sigma = 0 & \text{on } \Sigma, \\ \varphi(T) = 0, \quad \psi(0) = \psi^0, \quad \phi(T) = 0, \quad \sigma(0) = \sigma^0 & \text{in } \Omega, \end{cases}$$

For general controls $v = (v_1, v_2, v_3)$ and v_0:

$$\iint_Q \rho_1(t)(|\varphi|^2 + |\psi|^2 + |\phi|^2 + |\sigma|^2) \leq C \iint_Q \rho_2(t)(|g^\varphi|^2 + |g^\psi|^2 + |g^\phi|^2 + |g^\sigma|^2)$$
$$+ C \iint_{\omega\times(0,T)} \rho_3(t)(|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 + |\phi|^2)$$

$\rho_i(t) \sim \exp(-C_i/t^{10}(T-t)^{10})$

Using energy estimate, we can change to $\rho_i(t) \sim \exp(-C_i/t^{10})$

Observability inequality

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_\varphi = g^\varphi + \psi\,\mathbb{1}_{\mathcal{O}}, & \nabla\cdot\varphi = 0 \quad \text{in } Q, \\ \psi_t - \Delta\psi + \nabla\pi_\psi = g^\psi + (0,0,\sigma), & \nabla\cdot\psi = 0 \quad \text{in } Q, \\ -\phi_t - \Delta\phi = g^\phi + \varphi_3 + \sigma\,\mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ \sigma_t - \Delta\sigma = g^\sigma & \text{in } Q. \end{cases}$$

- For controls $v = (v_1, 0, 0)$ and v_0: only local terms φ_1 and ϕ:

$$\ldots \leq \ldots + C \iint_{\omega\times(0,T)} \rho_3(t)(|\varphi_1|^2 + |\phi|^2)$$

- For controls $v = (v_1, 0, v_3)$ and $v_0 = 0$: only local terms φ_1 and φ_3:

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- Carleman for φ_1 and φ_3.
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- $\Delta\psi_i = -\Delta\varphi_{i,t} - \Delta^2\varphi_i + \partial_i\nabla\cdot g^\varphi - \Delta g_i^\varphi$ in $\omega\cap\mathcal{O}$, $i = 1, 3$.
- Eliminate φ_3 using: $\varphi_3 = -\phi_t - \Delta\phi - g^\phi - \sigma$ in $\omega\cap\mathcal{O}$.
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$$(\partial_1^2 + \partial_2^2)\sigma = -(\partial_t^2 - \Delta^2)\Delta\varphi_3 + F(g^\varphi, g^\psi, g^\sigma) \quad \text{in } \omega\cap\mathcal{O}$$

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$$(\partial_1^2 + \partial_2^2)\sigma = -(\partial_t^2 - \Delta^2)\Delta\varphi_3 + F(g^\varphi, g^\psi, g^\sigma) \quad \text{in } \omega \cap \mathcal{O}$$

- At this point, it only remains to add to the left-hand side the weighted norm of ϕ.
- Cannot add a local term of ϕ. No way to eliminate with coupling. Instead, we use energy estimates with weights like $\rho(t) = \exp(-C/t^{10})$:

$$\begin{cases} -(\rho\phi)_t - \Delta(\rho\phi) = \rho g^\phi + \rho\varphi_3 + \rho\sigma \mathbb{1}_{\mathcal{O}} - \rho'(t)\phi \\ (\rho\phi)_{|\Sigma} = 0, \quad \boxed{(\rho\phi)(T) = 0} \end{cases}$$

$$\|\rho\phi\|^2_{L^2} \le C(\|\rho g^\phi\|^2_{L^2} + \|\rho\varphi_3\|^2_{L^2} + \|\rho\sigma\|^2_{L^2}) - \iint_Q \rho'\rho|\phi|^2$$

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Sketch of proof. Case: $v = (v_1, 0, v_3)$ and $v_0 = 0$

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$$(\partial_1^2 + \partial_2^2)\sigma = -(\partial_t^2 - \Delta^2)\Delta\varphi_3 + F(g^\varphi, g^\psi, g^\sigma) \quad \text{in } \omega \cap \mathcal{O}$$

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Outline

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Insensitizing controls for the Boussinesq system

Some comments, perspectives

Perspectives

- Our method limits the quantity of vanishing components to two. Also, we need to have v_3 or v_0
- What about three vanishing components, e.g., $v = (0,0,0)$ and v_0? One possibility: use the Return method.
- Alternative: Insensitize the functional

$$J_\tau(y,\theta) := \frac{1}{2} \iint_{\mathcal{O}\times(0,T)} \left(|\nabla \times y|^2 + |\nabla\theta|^2 \right) \mathrm{d}x\,\mathrm{d}t, \; \mathcal{O} \subset \Omega.$$

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Thank you for your attention