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Sur la contrôlabilité de quelques systèmes de type parabolique avec un nombre réduit de contrôles et d'une équation de KdV avec dispersion évanescente

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À mes parents,
ma petite sœur
et Valentina.

Ce que j'aime dans les mathématiques appliquées, c'est qu'elles ont pour ambition de donner du monde des systèmes une représentation qui permette de comprendre et d'agir. Et, de toutes les représentations, la représentation mathématique, lorsqu'elle est possible, est celle qui est la plus souple et la meilleure.
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Chapitre 1

Introduction Générale

1.1 Présentation

Dans cette thèse, on établit quelques résultats de contrôlabilité, dans une première partie, pour des systèmes de type Navier-Stokes avec un nombre réduit de contrôles distribués et, dans une deuxième partie, pour une équation de KdV linéaire avec dispersion évanescente et conditions au bord de type Colin-Ghidaglia.

De manière générale, on peut formuler un système de contrôle d'une équation aux dérivées partielles (ou EDP) comme

$$\begin{cases} y'(t) = f(t, y(t), v(t)) \quad t > 0, \\ y(0) = y^0, \end{cases} \tag{1.1}$$

où $t \mapsto y(t) \in \mathcal{X}$ décrit l'état du système et $t \mapsto v(t) \in \mathcal{U}$ est le contrôle à l'instant t. $\mathcal{X}$ et $\mathcal{U}$ sont des espaces de Banach appelés espace d'états et espace des contrôles admissibles, respectivement. Le problème de la contrôlabilité consiste, pour une condition initiale y^0 donnée et un temps final T, à mener la solution du système (1.1) vers une cible à l'instant T par l'action d'un contrôle v.

On rappelle les différentes notions de contrôlabilité.

Définition 1.1 (Contrôlabilité exacte)**.** *On dit que le système de contrôle* (1.1) *est exactement contrôlable au temps T si, pour tout y^0 et y^T dans $\mathcal{X}$, il existe un contrôle $v(t) \in \mathcal{U}$ tel que $y(T) = y^T$.*

Définition 1.2 (Contrôlabilité à zéro)**.** *On dit que le système de contrôle* (1.1) *est contrôlable à zéro au temps T si, pour tout y^0 dans $\mathcal{X}$, il existe un contrôle $v(t) \in \mathcal{U}$ tel que $y(T) = 0$.*

Définition 1.3 (Contrôlabilité approchée)**.** *On dit que le système de contrôle* (1.1) *est approximativement contrôlable au temps T si, pour tout y^0 et y^T dans $\mathcal{X}$, et tout $\varepsilon > 0$, il existe un contrôle $v(t) \in \mathcal{U}$ tel que $\|y^T - y(T)\|_{\mathcal{X}} \leq \varepsilon$.*

Ces définitions sont de type *globale*, car elles ne restreignent pas la taille de la condition initiale. On parle de *contrôlabilité locale* si de plus y^0 est demandé d'être proche de la cible. Cette nouvelle notion est importante pour traiter des systèmes non linéaires.

La solvabilité de chaque problème de contrôlabilité dépend de la nature du système. Par exemple, en dimension finie, les propriétés de contrôlabilité sont bien comprises dans plusieurs cas linéaires et non linéaires. Notamment, pour un système linéaire autonome, une condition nécessaire et suffisante est le *critère de Kalman* (voir, par exemple, [20, chapitre 1]).

En dimension infinie, la situation est plus compliquée, car elle dépend des propriétés particulières de l'équation sur laquelle on travaille. Par exemple, il est bien connu que pour les équations paraboliques, comme l'équation de la chaleur, il n'y a pas d'espoir d'atteindre la contrôlabilité exacte dû à l'effet régularisant. Pour les équations hyperboliques, comme l'équation des ondes, la contrôlabilité n'est pas assurée si le temps final est petit à cause de la vitesse finie de propagation.

Quand le système de contrôle est non linéaire, on trouve en général des résultats locaux. En effet, la stratégie classique consiste à linéariser autour d'un état d'équilibre pour obtenir un résultat de contrôlabilité pour ce système. On revient au problème original par un argument de point fixe ou d'inversion locale.

Dans une grande partie de ce mémoire, on suit cette procédure et on se concentrera particulièrement sur la contrôlabilité à zéro des systèmes linéaires. Dans le paragraphe suivant, on présentera, de manière simple, un cadre général pour les équations linéaires et leur contrôlabilité à zéro qui sera la base pour les stratégies utilisées dans la suite.

1.1.1 Système linéaire abstrait et la méthode HUM

On considère ici le système linéaire autonome de contrôle suivant :

$$\begin{cases} y'(t) = Ay(t) + Bv(t), \\ y(0) = y^0. \end{cases} \tag{1.2}$$

Pour simplifier l'exposé, on suppose que les espaces d'état H et de contrôles admissibles U sont des espaces de Hilbert. De plus, $A : D(A) \subset H \to H$ est un opérateur non borné, $B \in \mathcal{L}(U; H)$ et $y^0 \in H$. Pour une présentation complète et plus générale, nous renvoyons à [20, Section 2.3] et [77, Chapitre 11].

Le but de cette partie est de présenter une méthode pour construire un contrôle u tel que la solution y de (1.2) satisfasse $y(T) = 0$. C'est la méthode de dualité ou Hilbert Uniqueness Method (HUM), introduite par J.-L. Lions dans [61, 62].

On commence par introduire le problème rétrogade suivant, appelé aussi *le système adjoint* de (1.2),

$$\begin{cases} -\varphi'(t) = A^*\varphi(t), \\ \varphi(T) = \varphi^T, \end{cases} \tag{1.3}$$

où $\varphi^T \in H$ et A^* est l'opérateur adjoint de A.

En prenant le produit entre l'équation (1.2) et φ solution de (1.3), il n'est pas difficile de voir qu'un contrôle v tel que $y(T) = 0$ doit satisfaire, pour tout $\varphi^T \in H$,

$$\int_0^T (v(t), B^*\varphi(t))_U \, \mathrm{d}t + (y^0, \varphi(0))_H = 0. \tag{1.4}$$

On construit alors le contrôle de la manière suivante : soit une fonctionnelle $J : H \to \mathbb{R}$ définie par

$$J(\varphi^T) := \frac{1}{2} \int_0^T \|B^*\varphi(t)\|_U^2 \, \mathrm{d}t + (y^0, \varphi(0))_H, \tag{1.5}$$

où φ est la solution de (1.3) telle que $\varphi(T) = \varphi^T$ et on suppose que J admet un minimum que l'on appelle $\widehat{\varphi}^T$. On définit maintenant

$$\widehat{v}(t) := B^*\widehat{\varphi}(t), \tag{1.6}$$

où $\widehat{\varphi}$ es la solution de (1.3) associée à $\widehat{\varphi}^T$. Avec ce choix de contrôle, on voit directement que (1.4) correspond à la condition d'optimalité du premier ordre de J qui est donc satisfaite.

Comme on peut observer, pour obtenir la contrôlabilité à zéro de (1.2), il faut assurer l'existence d'un minimum pour la fonctionnelle J donnée par (1.5). Comme J est continue et convexe, il reste à savoir si elle est aussi coercive. Pour voir ce dernier point, on suppose qu'il existe une constante $C_0 > 0$ telle que pour tout $\varphi^T \in H$, la solution φ de (1.3) satisfait

$$\|\varphi(0)\|_H^2 \leq C_0 \int_0^T \|B^*\varphi(t)\|_U^2 \,\mathrm{d}t. \tag{1.7}$$

En vue de (1.7), J est coercive pour la norme

$$\|\varphi^T\|^2 := \int_0^T \|B^*\varphi(t)\|_U^2 \,\mathrm{d}t,$$

et possède un minimum $\widehat{\varphi}^T$ qui appartient à l'espace complété de H pour cette norme.

L'inégalité (1.7) est appelée *inégalité d'observabilité*. De plus, le contrôle donné par (1.6) est tel que

$$\|\widehat{v}\|_{L^2(0,T;U)} \leq \sqrt{C_0}\|y^0\|_H. \tag{1.8}$$

En effet, il suffit d'utiliser $\varphi^T = \widehat{\varphi}^T$ et (1.6) dans (1.4) et l'inégalité de Cauchy-Schwarz avec (1.7).

En outre, $\widehat{v}$ est celui de norme minimale : en effet, si $\bar{u}$ est un autre contrôle tel que $y(T) = 0$, il satisfait (1.4). En faisant la différence entre (1.4) satisfait par $\widehat{v}$ et $\bar{v}$ on trouve

$$\int_0^T (\widehat{v}(t), B^*\varphi(t))_U \,\mathrm{d}t = \int_0^T (\bar{v}(t), B^*\varphi(t))_U \,\mathrm{d}t \quad \forall \varphi^T \in H,$$

où φ est la solution de (1.3) associée à φ^T. On prend $\varphi^T = \widehat{\varphi}^T$, le minimum de la fonctionnelle J, dans cette dernière égalité. On a

$$\int_0^T \|\widehat{v}(t)\|_U^2 \,\mathrm{d}t = \int_0^T (\bar{v}(t), \widehat{v}(t))_U \,\mathrm{d}t \leq \left(\int_0^T \|\bar{v}(t)\|_U^2 \,\mathrm{d}t\right)^{1/2} \left(\int_0^T \|\widehat{v}(t)\|_U^2 \,\mathrm{d}t\right)^{1/2},$$

d'où on peut conclure.

La méthode HUM couvre de nombreux types d'équations, en particulier celles traitées dans cette thèse. Remarquons que la procédure précédente nous permet de résoudre la contrôlabilité à zéro en démontrant une inégalité (pour un problème adjoint), ce qui, en pratique, est plus abordable. Par contre, elle ne fournit pas une méthode pour la démontrer (voir le livre de J.-M. Coron [20] et le *survey* par E. Zuazua [80] pour une description détaillée des méthodes pour obtenir (1.7) pour quelques équations classiques).

On s'intéressera dans ce qui suit à un des plus communs et puissants outils pour montrer l'inégalité d'observabilité et qui s'applique à un grand spectre d'équations : les *inégalités de Carleman*. Dans le paragraphe suivant, on les présente dans le cadre de la contrôlabilité à zéro de l'équation de la chaleur, ce qui nous servira d'exemple modèle pour la suite.

1.1.2 Un exemple modèle : l'équation de la chaleur

On introduit d'abord quelques notations que l'on gardera pour les sections qui suivent. Soit $T > 0$, Ω un domaine régulier de $\mathbb{R}^N$ ($N \in \mathbb{N}$) et $\omega \subset \Omega$ que l'on appelle le *domaine de contrôle*. On notera $Q := \Omega \times (0,T)$ et $\Sigma := \partial\Omega \times (0,T)$. On considère l'équation de la chaleur

$$\begin{cases} y_t - \Delta y = v\,\mathbb{1}_\omega & \text{dans } Q, \\ y = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega, \end{cases} \tag{1.9}$$

où $y^0 \in L^2(\Omega)$ est la donnée initiale, $y = y(x,t)$ est la variable d'état qui décrit la distribution de température dans Ω à l'instant t et $v = v(x,t)$ dénote le contrôle qui est distribué dans le domaine ω.

Remarquons que, dû à l'effet régularisant de (1.9) (et en général des équations paraboliques), on ne peut pas espérer, quelque soit $y^0 \in L^2(\Omega)$, mener y vers une cible quelconque en temps T. C'est pour cela qu'il est convenable de considérer pour ce type d'équations la notion de *contrôlabilité aux trajectoires*, i.e., pour tout $y^0 \in L^2(\Omega)$ et toute solution $\bar{y}$ de (1.9) non contrôlée

$$\begin{cases} \bar{y}_t - \Delta \bar{y} = 0 & \text{dans } Q, \\ \bar{y} = 0 & \text{sur } \Sigma, \end{cases}$$

il existe un contrôle $v \in L^2(\omega \times (0,T))$ tel que $y(T) = \bar{y}(T)$. Néanmoins, par linéarité, cette notion de contrôlabilité est équivalente à la contrôlabilité à zéro.

D'après le Paragraphe 1.1.1 avec $H = L^2(\Omega)$, $U = L^2(\omega)$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $A : y \mapsto \Delta y$ et $B : v \mapsto v\mathbb{1}_\omega$, la contrôlabilité à zéro de (1.9) se réduit à montrer l'observabilité du système adjoint, i.e., l'existence d'une constante $C > 0$ telle que pour tout $\varphi^T \in L^2(\Omega)$, la solution φ de l'équation adjointe donnée par

$$\begin{cases} -\varphi_t - \Delta \varphi = 0 & \text{dans } Q, \\ \varphi = 0 & \text{sur } \Sigma, \\ \varphi(T) = \varphi^T & \text{dans } \Omega, \end{cases} \tag{1.10}$$

satisfasse

$$\int_\Omega |\varphi(0)|^2 \,\mathrm{d}x \leq C \iint_{\omega \times (0,T)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{1.11}$$

La contrôlabilité à zéro de (1.9) est due à G. Lebeau et L. Robbiano [60] et A. V. Fursikov et O. Yu. Imanuvilov [39] (voir [30] pour un résultat avec $N = 1$). Dans le premier, les auteurs prouvent (1.11) par une décomposition spectrale des solutions. Dans le deuxième, la preuve de (1.11) repose sur les *inégalités de Carleman globales* pour les solutions de (1.10). Nous allons développer à présent cette deuxième technique.

Pour ce faire, il faut d'abord introduire quelques fonctions poids. Soit $\omega_0 \Subset \omega$. Dans [39] il est montré qu'il existe une fonction $\eta \in \mathcal{C}^2(\overline{\Omega})$ telle que

$$\eta > 0 \text{ dans } \Omega, \quad \eta = 0 \text{ sur } \partial\Omega \quad \text{et} \quad |\nabla \eta| > 0 \text{ dans } \overline{\Omega \setminus \omega_0}.$$

Maintenant, on définit, pour $\lambda \geq 1$ et $k > 1$, les fonctions

$$\alpha(x,t) = \frac{e^{2\lambda k \|\eta\|_\infty} - e^{\lambda(k\|\eta\|_\infty + \eta(x))}}{t(T-t)}, \quad \xi(x,t) = \frac{e^{\lambda(k\|\eta\|_\infty + \eta(x))}}{T(T-t)}. \tag{1.12}$$

Le résultat qui suit a été prouvé dans [39]. Voir aussi [33, Lemme 1.3].

Théorème 1.4. *Il existe trois constantes positives λ_0, s_0 et C (qui ne dépendent que de Ω et ω) telles que pour tout $\lambda \geq \lambda_0$ et pour tout $s \geq s_0(T + T^2)$, on a*

$$s^{-1} \iint_Q e^{-2s\alpha} \xi^{-1} (|q_t|^2 + |\Delta q|^2)\, dx\, dt + s\lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla q|^2\, dx\, dt + s^3\lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |q|^2\, dx\, dt$$
$$\leq C \left(\iint_Q e^{-2s\alpha} |q_t + \Delta q|^2\, dx\, dt + s^3\lambda^4 \iint_{\omega \times (0,T)} e^{-2s\alpha} \xi^3 |q|^2\, dx\, dt \right) \tag{1.13}$$

pour tout $q \in \mathcal{C}^2(\overline{\Omega})$ tel que $q = 0$ sur Σ.

Nous allons déduire maintenant (1.11) à partir de (1.13). On commence par remarquer que (1.13) est valide pour les solutions de (1.10) avec $\varphi^T \in L^2(\Omega)$. Ceci est une conséquence de la densité des fonctions régulières avec trace nulle dans l'espace des solutions de (1.10), à savoir, $L^2(0,T;H_0^1(\Omega)) \cap \mathcal{C}^0([0,T];L^2(\Omega))$. On fixe $\lambda = \lambda_0$ et $s = s_0(T+T^2)$. On obtient en particulier

$$\iint_Q e^{-2s\alpha} t^{-3}(T-t)^{-3}|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \le C \iint_{\omega\times(0,T)} e^{-2s\alpha} t^{-3}(T-t)^{-3}|\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

On tronque en $(T/4, 3T/4)$ l'intégrale à gauche et on utilise

$$e^{-2s\alpha} t^{-3}(T-t)^{-3} \ge e^{-2C(1+1/T)} \frac{1}{T^6} \quad \text{dans } \Omega \times (T/4, 3T/4)$$

et

$$e^{-2s\alpha} t^{-3}(T-t)^{-3} \le e^{-C(1+1/T)} \frac{1}{T^6} \quad \text{dans } \Omega \times (0, T)$$

pour obtenir

$$\iint_{\Omega\times(T/4,3T/4)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \le Ce^{C(1+1/T)} \iint_{\omega\times(0,T)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{1.14}$$

Pour les solutions de l'équation (1.10) on a l'estimation d'énergie

$$\int_0^1 |\varphi(0)|^2 \,\mathrm{d}x \le \int_0^1 |\varphi(t)|^2 \,\mathrm{d}x, \quad \forall t \in [0,T],$$

et en intégrant dans $(T/4, 3T/4)$ on trouve

$$\int_0^1 |\varphi(0)|^2 \,\mathrm{d}x \le \frac{2}{T} \iint_{\Omega\times(T/4,3T/4)} |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Il suffit de combiner cette inégalité avec (1.14) pour obtenir l'inégalité d'observabilité (1.11).

Les inégalités de Carleman permettent d'obtenir le contrôle à zéro d'équations paraboliques plus générales. Par exemple, pour une équation avec termes d'ordre inférieur de la forme

$$\begin{cases} y_t - \Delta y + B(x,t)\cdot\nabla y + a(x,t)y = v\,\mathbb{1}_\omega & \text{dans } Q, \\ y = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega. \end{cases} \tag{1.15}$$

Ici, a et $B = (B_i)_{i=1}^N$ sont $N+1$ fonctions qui dépendent de x et de t qui sont supposées appartenir à $L^\infty(Q)$. Dans ce cas, il faut utiliser une inégalité de Carleman appropriée pour les solutions de l'équation de la chaleur avec un second membre dans un espace plus faible (voir [53]). En particulier, la contrôlabilité à zéro de (1.15) est utilisée pour traiter quelques équations paraboliques non linéaires à travers d'un argument de point fixe. Nous renvoyons à [29, 39, 31, 37, 27] pour quelques résultats concernants la contrôlabilité de l'équation de la chaleur semi-linéaire. Dans le cadre de l'équation des ondes, on cite le travail pionnier [79] par E. Zuazua où l'argument de point fixe a été introduit pour traiter un problème de contrôle non linéaire.

1.2 Résultats principaux et plan de la thèse

Dans cette section on présente les principaux résultats obtenus pendant la réalisation de cette thèse. Elle est divisée en deux parties. Dans la première, on s'intéresse à des résultats de contrôlabilité à zéro de quelques systèmes liés à la mécanique des fluides avec des contrôles à un nombre réduit de composantes. Notamment, dans le Chapitre 2 on établit la contrôlabilité locale à zéro pour le système de Navier-Stokes

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = v\mathbb{1}_\omega, \quad \nabla\cdot y = 0 & \text{dans } Q, \\ y = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega, \end{cases}$$

où le contrôle $v \in L^2(\omega\times(0,T))^N$ satisfait $v_i \equiv 0$ pour $0 < i \le N$ quelconque et $N = 2, 3$. La nouveauté principale est la perte des restrictions géométriques sur le domaine de contrôle imposées dans [35].

Dans le même esprit du problème précédent, dans le Chapitre 3 on démontre la contrôlabilité locale aux trajectoires pour le système de Boussinesq

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = v\mathbb{1}_\omega + \theta\, e_N, \quad \nabla\cdot y = 0 & \text{dans } Q, \\ \theta_t - \Delta\theta + y\cdot\nabla\theta = v_0\mathbb{1}_\omega & \text{dans } Q, \\ y = 0, \quad \theta = 0 & \text{sur } \Sigma, \\ y(0) = y^0, \quad \theta(0) = \theta^0 & \text{dans } \Omega, \end{cases}$$

où $v_0 \in L^2(\omega\times(0,T))$ et $v \in L^2(\omega\times(0,T))^N$ est tel que $v_N \equiv 0$ et $v_i \equiv 0$ pour $0 < i < N$. La trajectoire cible est de la forme $(0, \bar p, \bar\theta)$, i.e.,

$$\begin{cases} \nabla\bar p = \bar\theta\, e_N & \text{dans } Q, \\ \bar\theta_t - \Delta\bar\theta = 0 & \text{dans } Q, \\ \bar\theta = 0 & \text{sur } \Sigma, \\ \bar\theta(0) = \bar\theta^0 & \text{dans } \Omega. \end{cases} \tag{1.16}$$

Le Chapitre 4 concerne l'existence de contrôles insensibilisants avec des composantes nulles pour le système de Boussinesq. Il est bien connu que ce problème est équivalent à la contrôlabilité à zéro d'un système en cascade où la quantité de contrôles est inférieur au nombre d'équations. Plus précisément, pour le système

$$\begin{cases} w_t - \Delta w + (w\cdot\nabla)w + \nabla p_0 = f + v\mathbb{1}_\omega + r\, e_N, & \nabla\cdot w = 0 \quad \text{dans } Q, \\ -z_t - \Delta z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + q\nabla r + \nabla p_1 = w\mathbb{1}_{\mathcal{O}}, & \nabla\cdot z = 0 \quad \text{dans } Q, \\ r_t - \Delta r + w\cdot\nabla r = f_0 + v_0\mathbb{1}_\omega & \text{dans } Q, \\ -q_t - \Delta q - w\cdot\nabla q = z_N + r\,\mathbb{1}_{\mathcal{O}} & \text{dans } Q, \\ w = z = 0, \quad r = q = 0 & \text{sur } \Sigma, \\ w(0) = 0, \quad z(T) = 0, \quad r(0) = 0, \quad q(T) = 0 & \text{dans } \Omega, \end{cases}$$

on trouve des contrôles $v \in L^2(\omega\times(0,T))^N$, avec $v_N \equiv v_i \equiv 0$ pour $0 < i < N$ et $v_0 \in L^2(\omega\times(0,T))$ tels que $z(0) = 0$ et $q(0) = 0$.

Tous ces résultats reposent sur une méthode classique : la linéarisation autour de la trajectoire cible et la contrôlabilité à zéro de ce système linéaire. Ensuite, on revient au problème non linéaire par un argument d'inversion locale. Tous les détails sont présentés dans les paragraphes suivants.

Dans la deuxième partie du présent manuscrit, on s'intéresse à quelques propriétés associées à la contrôlabilité à zéro du système de Korteweg-de Vries (KdV) avec conditions au bord de type Colin-Ghidalia

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{dans } (0,T)\times(0,1), \\ y_{|x=0} = v(t), \quad y_{x|x=1} = 0, \quad y_{xx|x=1} = 0 & \text{dans } (0,T), \\ y_{|t=0} = y_0 & \text{dans } (0,1), \end{cases}$$

où $M \in \mathbb{R}$ et $\varepsilon > 0$. Premièrement, pour M et ε fixes, on montre qu'il existe $v \in L^2(0,T)$ tel que $y(T) = 0$ et, de plus, qu'il existe une constante positive C telle que

$$\|v\|_{L^2(0,T)} \leq C \exp\left(C\varepsilon^{-1/2}\right)\|y^0\|_{L^2(0,1)}.$$

En particulier, ce résultat améliore le coût du contrôle trouvé dans [50]. En effet, dans [50] il est démontré que le coût est majoré par $\exp\left(C\varepsilon^{-1}\right)$.

Ensuite, on montre que pour un temps de contrôle T suffisamment petit, le coût du contrôle à zéro explose comme $\exp(C\varepsilon^{-1/2})$ quand ε tend vers zéro, ce qui est naturel car l'équation de transport $y_t - My_x = 0$ dans $(0,T)\times(0,1)$ n'est pas contrôlable si $T < 1/|M|$. Ceci est l'objet du Chapitre 5.

Dans ce qui suit, on présente chaque chapitre de ce mémoire de façon détaillée.

1.2.1 Contrôlabilité locale à zéro du système de Navier-Stokes N-dimensionnel avec $N-1$ contrôles scalaires dans un domaine de contrôle arbitraire

On commence par introduire deux espaces classiques présents dans le contexte de fluides incompressibles, à savoir :

$$V = \{y \in H_0^1(\Omega)^N : \nabla\cdot y = 0 \text{ dans } \Omega\}$$

et

$$H = \{y \in L^2(\Omega)^N : \nabla\cdot y = 0 \text{ dans } \Omega,\ y\cdot n = 0 \text{ sur } \partial\Omega\}.$$

La contrôlabilité de l'équation de Navier-Stokes a été introduite par J.-L. Lions dans [63] et a été l'objet de nombreux travaux ces dernières années. Dans le cadre de la contrôlabilité approchée, les premiers résultats ont été obtenus par J.-M Coron dans [18] avec des conditions au bord de type Navier en utilisant la célèbre méthode du retour. Pour des conditions au bord de type Dirichlet homogènes, on mentionne [28] par C. Fabre et [67] par J.-L Lions et E. Zuazua.

On considère le système de Navier-Stokes avec un contrôle distribué :

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = v\mathbb{1}_\omega, \quad \nabla\cdot y = 0 & \text{dans } Q, \\ y = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega, \end{cases} \tag{1.17}$$

pour $y^0 \in H$ ou V. Le premier résultat concernant la contrôlabilité aux trajectoires de (1.17) est dû à O. Yu Imanuvilov [51]. Dans cette référence, pour une solution de

$$\begin{cases} \bar{y}_t - \Delta\bar{y} + (\bar{y}\cdot\nabla)\bar{y} + \nabla\bar{p} = 0, \quad \nabla\cdot\bar{y} = 0 & \text{dans } Q, \\ \bar{y} = 0 & \text{sur } \Sigma, \\ \bar{y}(0) = \bar{y}^0 & \text{dans } \Omega, \end{cases}$$

telle que $\bar{y} \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega)^N \cap V)$, l'auteur montre l'existence d'un $\delta > 0$ tel que si $\|y^0 - \bar{y}^0\|_V \leq \delta$, il existe un contrôle v tel que $y(T) = \bar{y}(T)$. Ce résultat a été amélioré dans [34] par rapport à la régularité de la trajectoire $\bar{y}$. Elle est supposée de satisfaire

$$\bar{y} \in L^\infty(Q)^N, \quad \bar{y} \in L^2(0,T;L^\sigma(\Omega)^N), \tag{1.18}$$

($\sigma > 1$ si $N = 2$ et $\sigma > 6/5$ si $N = 3$) et $y^0 \in L^{2N-2}(\Omega)^N \cap H$. La preuve repose sur une inégalité de Carleman pour le système adjoint du système linéarisé autour de $\bar{y}$. La partie la plus délicate est l'estimation du terme de pression.

Un problème intéressant est de se demander si l'on peut enlever quelques composantes du contrôle v en gardant la même propriété de contrôlabilité. Celle-ci est la problématique principale de cette partie de ce mémoire. Le premier résultat dans cette direction est montré dans [67] pour la contrôlabilité approchée du système de Stokes pour quelques géométries particulières. Dans [35], les auteurs démontrent la contrôlabilité locale aux trajectoires du type (1.18) avec un contrôle $v \in L^2(\omega \times (0,T))^N$ tel que $v_i \equiv 0$, $0 < i \leq N$, sous l'hypothèse géométrique sur le domaine de contrôle suivante :

$$\exists x^0 \in \partial\Omega, \quad \exists \varepsilon > 0 \quad \text{tel que} \quad B(x^0;\varepsilon) \cap \partial\Omega \subset \overline{\omega} \cap \partial\Omega \quad \text{et} \quad n_i(x^0) \neq 0, \tag{1.19}$$

où $n(x)$ dénote le vecteur normal extérieur à Ω au point $x \in \partial\Omega$. Comme dans [34], la preuve est aussi fondée sur une inégalité de Carleman, où l'i-ème composante de la variable adjointe n'apparaît pas dans le terme local (voir (1.23) ci-dessous). En fait, à partir de l'inégalité trouvée dans [34] et (1.19), en utilisant les conditions au bord et la condition de divergence nulle, l'i-ème composante peut être estimée en fonction des autres. Bien entendu, cette hypothèse restreint le choix de la composante à annuler. On s'intéresse à enlever la condition (1.19).

Le résultat principal que l'on obtient dans cette partie est le suivant :

Théorème 1.5. *Soit $i \in \{1,\dots,N\}$. Alors, pour tout $T > 0$ et $\omega \subset \Omega$, il existe $\delta > 0$ tel que, pour tout $y^0 \in V$ satisfaisant $\|y^0\|_V \leq \delta$, on peut trouver un contrôle $v \in L^2(\omega \times (0,T))^N$, avec $v_i \equiv 0$, et une solution associée (y,p) de* (1.17) *telle que $y(T) = 0$ dans Ω, i.e., le système non linéaire* (1.17) *est localement contrôlable à zéro à l'aide de $N-1$ contrôles scalaires pour un domaine de contrôle arbitraire.*

La preuve du Théorème 1.5 repose sur un résultat de contrôle à zéro pour le système linéaire avec un terme source

$$\begin{cases} y_t - \Delta y + \nabla p = f + v 1\!\!1_\omega, \quad \nabla\cdot y = 0 & \text{dans } Q, \\ y = 0 & \text{sur } \Sigma, \\ y(0) = y^0 & \text{dans } \Omega, \end{cases} \tag{1.20}$$

et un théorème d'inversion locale (voir Théorème 1.9 ci-dessous).

Dans [23], la contrôlabilité à zéro de (1.20) est démontrée avec $f \equiv 0$ et $v_i \equiv 0$. Pour ce faire, les auteurs montrent une inégalité d'observabilité pour le système adjoint

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = 0, \quad \nabla\cdot\varphi = 0 & \text{dans } Q, \\ \varphi = 0 & \text{sur } \Sigma, \\ \varphi(T) = \varphi^T & \text{dans } \Omega. \end{cases} \tag{1.21}$$

Concrètement, les auteurs montrent l'existence d'une constante $C > 0$ telle que, pour tout $\varphi^T \in H$, la solution de (1.21) safisfait ([23, inégalité (2)])

$$\int_\Omega |\varphi(0)|^2 \,\mathrm{d}x \leq C \sum_{j=1,j\neq i}^{N} \iint_{\omega\times(0,T)} |\varphi_j|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{1.22}$$

La preuve de (1.22) repose sur une inégalité de Carleman du type

$$\iint_Q \rho_1 |\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C \sum_{j=1,j\neq i}^{N} \iint_{\omega\times(0,T)} \rho_2 |\varphi_j|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{1.23}$$

où $\rho_1(x,t)$ et $\rho_2(x,t)$ sont des poids du type (1.12). Le point central dans la preuve de (1.23) est le fait que $\Delta\pi = 0$ (car $\nabla\cdot\varphi = 0$), ce qui permet d'en déduire des équations de quelques dérivées de φ_j $(j \neq i)$ qui ne dépendent pas de φ_i ni de π.

Un autre point important est l'utilisation d'une inégalité de Carleman pour l'équation de la chaleur avec des conditions au bord non homogènes ([32, Théorème 1]) satisfaite par $\nabla\Delta\varphi_j$ $(j \neq i)$:

$$-(\nabla\Delta\varphi_j)_t - \Delta(\nabla\Delta\varphi_j) = 0.$$

Les termes de bord sont estimés par des résultats de régularité pour le système de Stokes.

Cependant, une inégalité d'observabilité comme (1.22) ne suffit pas pour traiter le problème non linéaire (1.17). On aura besoin de déduire quelques proprietés de décroissance pour le terme $(y\cdot\nabla)y$. Pour ce faire, on démontre dans une première étape une inégalité de Carleman pour le système adjoint non homogène

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = g, \quad \nabla\cdot\varphi = 0 & \text{dans } Q, \\ \varphi = 0 & \text{sur } \Sigma, \\ \varphi(T) = \varphi^T & \text{dans } \Omega, \end{cases} \tag{1.24}$$

où $g \in L^2(\Omega)^N$. Plus précisément, on montre :

Proposition 1.6. *Il existe une constante λ_0, telle que pour tout $\lambda > \lambda_0$ il existe deux constantes $C(\lambda) > 0$ et $s_0(\lambda) > 0$ telles que pour tout $i \in \{1,\dots,N\}$, tout $g \in L^2(Q)^N$ et tout $\varphi^T \in H$, la solution de* (1.24) *satisfait*

$$s^4 \iint_Q e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2 dx\,dt \leq C\left(\iint_Q e^{-3s\alpha^*}|g|^2 dx\,dt + s^7 \sum_{j=1,j\neq i}^N \iint_{\omega\times(0,T)} e^{-2s\widehat{\alpha}-3s\alpha^*}(\widehat{\xi})^7|\varphi_j|^2 dx\,dt\right) \tag{1.25}$$

pour tout $s \geq s_0$.

Les fonctions poids dans (1.25) sont similaires à celles données par (1.12). En particulier, la puissance de $t(T-t)$ est 8 au lieu de 1. Pour la définition précise, voir (2.5).

Pour montrer (1.25), on suit la même stratégie que [23], mais il y a deux différences importantes. La première concerne le fait que la pression n'est plus une fonction harmonique. La deuxième, le fait que g n'appartienne qu'à $L^2(Q)^N$ fait que l'on ne puisse pas appliquer directement le même nombre de dérivées à l'équation satisfaite par φ_j $(j \neq i)$. Pour surmonter ces difficultés, on récrit $\rho\varphi$ ($\rho = \rho(t)$ étant une fonction poids qui s'annule en $t = T$) comme la somme de deux solutions de (1.24) ayant ρg et $-\rho'\varphi$ comme second membre, respectivement. Pour ces nouvelles variables, on utilise des résultats de régularité pour le système de Stokes et la méthode de [23], respectivement.

Une fois que (1.25) a été établi, on déduit l'inégalité d'observabilité suivante avec des poids qui ne s'annulent pas en $t = 0$ (voir (2.29) pour la définition précise) :

$$\iint_Q e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2 \mathrm{d}x\,\mathrm{d}t + \|\varphi(0)\|^2_{L^2(\Omega)^N} \leq C\left(\iint_Q e^{-3s\beta^*}|g|^2 \mathrm{d}x\,\mathrm{d}t + \sum_{j=1,j\neq i}^N \iint_{\omega\times(0,T)} e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7|\varphi_j|^2 \mathrm{d}x\,\mathrm{d}t\right). \tag{1.26}$$

On cherche des solutions de (1.20) dans l'espace

$$E_N^i = \{ (y,p,v) : \quad e^{3/2s\beta^*} y \in L^2(Q)^N, \quad e^{s\widehat{\beta}+3/2s\beta^*}\widehat{\gamma}^{-7/2}\, v\mathbb{1}_\omega \in L^2(Q)^N, \quad v_i \equiv 0,$$
$$e^{3/2s\beta^*}(\gamma^*)^{-9/8} y \in L^2(0,T;H^2(\Omega)^N) \cap L^\infty(0,T;V),$$
$$e^{5/2s\beta^*}(\gamma^*)^{-2}(y_t - \Delta y + \nabla p - v\mathbb{1}_\omega) \in L^2(Q)^N \}.$$

Remarque 1.7. *Remarquons que $(y,p,v) \in E_N^i$ résout* (1.20) *pour un second membre tel que $e^{5/2s\beta^*}(\gamma^*)^{-2} f \in L^2(Q)^N$ avec $y(T) = 0$ et $v_i \equiv 0$. De plus $(y \cdot \nabla)y$ appartient au même espace que f, ce qui nous permettra de résoudre le problème non linéaire.*

Plus précisément, on a :

Proposition 1.8. *Soit $i \in \{1,\dots,N\}$. On suppose que $y^0 \in V$ et $e^{5/2s\beta^*}(\gamma^*)^{-2} f \in L^2(Q)^N$. Alors, il existe un contrôle v tel que la solution associée (y,p) de* (1.20) *vérifie $(y,p,v) \in E_N^i$. En particulier, $v_i \equiv 0$ et $y(T) = 0$ dans Ω.*

La preuve suit les arguments de [51] et [34]. On considère le problème variationnel suivant :

$$a((\widehat{\chi},\widehat{\sigma}),(\chi,\sigma)) = \langle G,(\chi,\sigma)\rangle \quad \text{pour tout } (\chi,\sigma) \in P_0, \tag{1.27}$$

où

$$a((\widehat{\chi},\widehat{\sigma}),(\chi,\sigma)) = \iint_Q e^{-3s\beta^*}(-\widehat{\chi}_t - \Delta\widehat{\chi} + \nabla\widehat{\sigma}) \cdot (-\chi_t - \Delta\chi + \nabla\sigma)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \sum_{j=1, j\neq i}^{N} \iint_{\omega\times(0,T)} e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7\,\widehat{\chi}_j\,\chi_j\,\mathrm{d}x\,\mathrm{d}t,$$

$$\langle G,(\chi,\sigma)\rangle = \iint_Q f\cdot\chi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega y^0\cdot\chi(0)\,\mathrm{d}x.$$

et

$$P_0 = \{ (\chi,\sigma) \in \mathcal{C}^2(\overline{Q})^{N+1} : \ \nabla\cdot\chi = 0 \text{ dans } Q, \quad \chi = 0 \text{ sur } \Sigma \}$$

Grâce à (1.26), $a(\cdot,\cdot)$ définit un produit scalaire et G une forme linéaire et continue dans P_0. On trouve donc une solution $(\widehat{\chi},\widehat{\sigma})$ (unique) de (1.27) dans le complété de P_0 avec la norme définie $a(\cdot,\cdot)^{1/2}$. Finalement, on démontre que $(\widehat{y},\widehat{v})$, avec un certain $\widehat{p}$, définis comme

$$\begin{cases} \widehat{y} := e^{-3s\beta^*}(-\widehat{\chi}_t - \Delta\widehat{\chi} + \nabla\widehat{\sigma}), & \text{dans } Q, \\ \widehat{v}_j := -e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7\,\widehat{\chi}_j \quad (j\neq i), \quad \widehat{v}_i := 0 & \text{dans } \omega\times(0,T), \end{cases}$$

satisfont (1.20) et que $(\widehat{y},\widehat{p},\widehat{v})$ appartient à l'espace E_N^i.

Maintenant, pour obtenir le résultat de contrôlabilité pour le système non linéaire (1.17), on utilise le théorème suivant :

Théorème 1.9. *Soit $\mathcal{B}_1$ et $\mathcal{B}_2$ deux espaces de Banach et $\mathcal{A} : \mathcal{B}_1 \to \mathcal{B}_2$ vérifiant $\mathcal{A} \in \mathcal{C}^1(\mathcal{B}_1;\mathcal{B}_2)$. On suppose que $b_1 \in \mathcal{B}_1$, $\mathcal{A}(b_1) = b_2$ et que $\mathcal{A}'(b_1) : \mathcal{B}_1 \to \mathcal{B}_2$ est surjective. Alors, il existe $\delta > 0$ tel que pour tout $b' \in \mathcal{B}_2$ satisfaisant $\|b' - b_2\|_{\mathcal{B}_2} < \delta$, il existe une solution de l'équation*

$$\mathcal{A}(b) = b', \quad b \in \mathcal{B}_1.$$

Pour appliquer ce théorème dans notre contexte, on considère l'opérateur

$$\mathcal{A}(y,p,v) = (y_t - \Delta y + (y\cdot\nabla)y + \nabla p - v\mathbb{1}_\omega,\ y(0))$$

avec les espaces :

$$\mathcal{B}_1 = E^i_N$$

et

$$\mathcal{B}_2 = L^2(e^{5/2s\beta^*}(\gamma^*)^{-2}(0,T);L^2(\Omega)^N)\times V.$$

La Proposition 1.8 montre que $\mathcal{A}'(0,0,0)$ est surjective et de plus de classe $\mathcal{C}^1$ (voir la Section 2.4 pour les détails). Ceci montre le Théorème 1.5.

Tous les détails sont données dans le Chapitre 2 de ce mémoire.

1.2.2 Contrôlabilité locale du système de Boussinesq N-dimensionnel avec $N-1$ contrôles scalaires dans un domaine de contrôle arbitraire

Dans ce paragraphe on considère le système de Boussinesq suivant :

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = v\mathbb{1}_\omega + \theta\, e_N, \quad \nabla\cdot y = 0 & \text{dans } Q,\\ \theta_t - \Delta\theta + y\cdot\nabla\theta = v_0\mathbb{1}_\omega & \text{dans } Q,\\ y = 0, \quad \theta = 0 & \text{sur } \Sigma,\\ y(0) = y^0, \quad \theta(0) = \theta^0 & \text{dans } \Omega, \end{cases} \tag{1.28}$$

où

$$e_N = \begin{cases} (0,1) & \text{si } N = 2,\\ (0,0,1) & \text{si } N = 3, \end{cases}$$

représente la force gravitationnelle, $y = y(x,t)$ le champ de vitesse du fluide, $\theta = \theta(x,t)$ sa température et, v_0 et $v = (v_1,\dots,v_N)$ les contrôles distribués dans $\omega\subset\Omega$.

Les premiers résultats de contrôlabilité aux trajectoires du type

$$\begin{cases} \bar y_t - \Delta\bar y + (\bar y\cdot\nabla)\bar y + \nabla\bar p = \bar\theta\, e_N, \quad \nabla\cdot\bar y = 0 & \text{dans } Q,\\ \bar\theta_t - \Delta\bar\theta + \bar y\cdot\nabla\bar\theta = 0 & \text{dans } Q,\\ \bar y = 0, \quad \bar\theta = 0 & \text{sur } \Sigma,\\ \bar y(0) = \bar y^0, \quad \bar\theta(0) = \bar\theta^0 & \text{dans } \Omega, \end{cases}$$

ont été obtenus dans [40] et [38], avec des contrôles qui agissent sur toute la frontière de Ω et avec des contrôles distribués quand Ω est le tore, respectivement. Puis, dans [44] la contrôlabilité locale aux trajectoires qui vérifient (1.18) est démontrée.

Dans le même esprit que dans le paragraphe précédent pour le système de Navier-Stokes, on s'intéresse à l'obtention de contrôles ayant des composantes nulles sans restrictions géométriques sur le domaine de contrôle. Le premier résultat dans cette direction est montré dans [35] sous l'hypothèse (1.19) avec $v_N\equiv 0$ et $v_i\equiv 0$ $(0<i<N)$. Dans [36] un système de Boussinesq couplé avec une équation parabolique (qui est interprétée comme une concentration saline, par exemple) est considéré. En rajoutant un contrôle dans la nouvelle équation, les auteurs obtiennent des contrôles v avec $v_N\equiv v_{i_0}\equiv 0$ $(0<i_0<N)$ sans restriction géométrique sur ω. Lorsque $N=3$, il est possible de prendre $v\equiv 0$ si (1.19) est satisfait pour $i\neq i_0, 3$.

Dans ce paragraphe, nous considérons des trajectoires de la forme $(0,\bar p,\bar\theta)$, i.e.,

$$\begin{cases} \nabla\bar p = \bar\theta\, e_N & \text{dans } Q,\\ \bar\theta_t - \Delta\bar\theta = 0 & \text{dans } Q,\\ \bar\theta = 0 & \text{sur } \Sigma,\\ \bar\theta(0) = \bar\theta^0 & \text{dans } \Omega, \end{cases} \tag{1.29}$$

telles que

$$\bar{\theta} \in L^{\infty}(0,T;W^{3,\infty}(\Omega)) \text{ et } \nabla\bar{\theta}_t \in L^{\infty}(Q)^N. \tag{1.30}$$

Le résultat principal de cette partie est :

Théorème 1.10. *Soit $0 < i < N$ un entier et $(\bar{p},\bar{\theta})$ une solution de (1.29) satisfaisant (1.30). Alors, pour tout $T > 0$ et $\omega \subset \Omega$, il existe $\delta > 0$ tel que pour tout $(y^0,\theta^0) \in V \times H_0^1(\Omega)$ vérifiant $\|(y^0,\theta^0)-(0,\bar{\theta}^0)\|_{V\times H_0^1(\Omega)} \leq \delta$, on peut trouver des contrôles $v_0 \in L^2(\omega\times(0,T))$ et $v \in L^2(\omega\times(0,T))^N$, avec $v_i \equiv 0$ et $v_N \equiv 0$, tels que la solution associée (y,p,θ) de (1.28) satisfait $y(T) = 0$ et $\theta(T) = \bar{\theta}(T)$ dans Ω.*

La preuve du Théorème 1.10 est similaire à celle de Navier-Stokes adaptée à (1.28). On linéarise autour de $(0,\bar{p},\bar{\theta})$ avec des termes source

$$\begin{cases} y_t - \Delta y + \nabla p = f + v\mathbb{1}_\omega + \theta\, e_N, \quad \nabla\cdot y = 0 & \text{dans } Q, \\ \theta_t - \Delta\theta + y\cdot\nabla\bar{\theta} = f_0 + v_0\mathbb{1}_\omega & \text{dans } Q, \\ y = 0, \quad \theta = 0 & \text{sur } \Sigma, \\ y(0) = y^0, \quad \theta(0) = \theta^0 & \text{dans } \Omega. \end{cases} \tag{1.31}$$

Pour ce système, on cherche à démontrer la contrôlabilité à zéro. La preuve repose sur une inégalité de Carleman pour le système adjoint

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = g - \psi\nabla\bar{\theta}, \quad \nabla\cdot\varphi = 0 & \text{dans } Q, \\ -\psi_t - \Delta\psi = g_0 + \varphi_N & \text{dans } Q, \\ \varphi = 0, \quad \psi = 0 & \text{sur } \Sigma, \\ \varphi(T) = \varphi^T, \quad \psi(T) = \psi^T & \text{dans } \Omega, \end{cases} \tag{1.32}$$

où $g \in L^2(Q)^N$, $g_0 \in L^2(Q)$, $\varphi^T \in H$ et $\psi^T \in L^2(\Omega)$. On obtient :

Proposition 1.11. *Soit $N = 2$ ou 3, $\omega \subset \Omega$ et $\bar{\theta}$ satisfaisant (1.30). Il existe une constante $\lambda_0 > 0$, telle que pour tout $\lambda \geq \lambda_0$ il existe deux constantes $C(\lambda) > 0$ et $s_0(\lambda) > 0$ telles que pour tout $j \in \{1,\dots,N-1\}$, tout $g \in L^2(Q)^N$, tout $g_0 \in L^2(Q)$, tout $\varphi^T \in H$ et tout $\psi^T \in L^2(\Omega)$, la solution de (1.32) vérifie*

$$\begin{aligned} s^4 \iint_Q & e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2 dx\,dt + s^5 \iint_Q e^{-5s\alpha^*}(\xi^*)^5|\psi|^2 dx\,dt \\ &\leq C\left(\iint_Q e^{-3s\alpha^*}(|g|^2+|g_0|^2)dx\,dt + (N-2)s^7\iint_{\omega\times(0,T)} e^{-2s\widehat{\alpha}-3s\alpha^*}(\widehat{\xi})^7|\varphi_j|^2 dx\,dt \right. \\ &\qquad \left. + s^{12}\iint_{\omega\times(0,T)} e^{-4s\widehat{\alpha}-s\alpha^*}(\widehat{\xi})^{49/4}|\psi|^2 dx\,dt\right), \end{aligned} \tag{1.33}$$

pour tout $s \geq s_0$.

Les poids sont les mêmes que dans la Proposition 1.6. La preuve suit la même stratégie que pour montrer (1.25) dans la Proposition 1.6 (la méthode de [23]). On explique les principales différences. Dans une première partie, l'idée est d'utiliser une inégalité de Carleman pour l'équation de Stokes avec des termes locaux qui ne concernent que $N-1$ composantes de φ. Ensuite, l'idée est de combiner ceci avec une inégalité de Carleman pour l'équation de la chaleur (voir (1.13)) pour ψ, bien entendu avec un bon choix des poids. Néanmoins, si l'on regarde le terme $-\psi\nabla\bar{\theta}$ (qui appartient à $L^2(Q)^N$) comme un second membre, on ne pourra pas utiliser directement le résultat de la Proposition 1.6 car le poids qui multiplie le second membre est trop grand par rapport à celui de φ. Pour éviter ceci, on fait la décomposition décrite ci-dessus (Paragraphe 1.2.1) et on met le terme $-\rho\psi\nabla\theta$ dans l'équation de la solution la plus régulière (voir (3.22)-(3.23)-(3.24)). En faisant cela, on aura une inégalité comme

$$\iint_Q \rho_1 \left(|\varphi|^2 + |\psi|^2\right) \mathrm{d}x\,\mathrm{d}t \leq C \iint_Q \rho_2 \left(|g|^2 + |g_0|^2\right) \mathrm{d}x\,\mathrm{d}t \\ + C \iint_{\omega' \times (0,T)} \rho_3 \left((N-2)|\varphi_j|^2 + |\varphi_N|^2 + |\psi|^2\right) \mathrm{d}x\,\mathrm{d}t, \quad (1.34)$$

avec $\omega' \subset \omega$.

La deuxième partie consiste à éliminer le terme local en φ_N. On utilise simplement l'équation de ψ :

$$\varphi_N = -\psi_t - \Delta\psi - g_0,$$

et des intégrations par parties pour estimer φ_N par le terme de gauche de (1.34).

Remarque 1.12. *Dans [36], il a été remarqué (Section 6.4 dans cette référence) que la méthode pour l'équation de Stokes présentée dans [23] peut servir pour montrer une inégalité du type* (1.33). *Cependant, les auteurs ne précisent pas la régularité nécessaire pour les seconds membres dans* (1.32). *Sans faire la décomposition précédente, il ne semble pas possible de conserver g et g_0 dans $L^2(Q)$.*

On passe à la contrôlabilité à zéro de (1.31). À partir de (1.33) on peut déduire une inégalité d'observabilité similaire à (1.26) :

$$\iint_Q e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2 dx\,dt + \iint_Q e^{-5s\beta^*}(\gamma^*)^5|\psi|^2 dx\,dt + \|\varphi(0)\|^2_{L^2(\Omega)^N} + \|\psi(0)\|^2_{L^2(\Omega)} \\ \leq C\Bigg(\iint_Q e^{-3s\beta^*}(|g|^2+|g_0|^2)dx\,dt + (N-2)\iint_{\omega\times(0,T)} e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7|\varphi_j|^2 dx\,dt \\ + \iint_{\omega\times(0,T)} e^{-4s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{49/4}|\psi|^2 dx\,dt\Bigg). \quad (1.35)$$

On définit l'espace

$$\begin{aligned} E_N^i = \{\,(y,p,v,\theta,v_0) :\ & e^{3/2s\beta^*}\, y \in L^2(Q)^N,\ e^{s\widehat{\beta}+3/2s\beta^*}\widehat{\gamma}^{-7/2}\, v\mathbb{1}_\omega \in L^2(Q)^N,\ v_i \equiv v_N \equiv 0, \\ & e^{3/2s\beta^*}\,\theta \in L^2(Q), \quad e^{2s\widehat{\beta}+1/2s\beta^*}\widehat{\gamma}^{-49/8}\, v_0\mathbb{1}_\omega \in L^2(Q), \\ & e^{3/2s\beta^*}(\gamma^*)^{-9/8} y \in L^2(0,T;H^2(\Omega)^N) \cap L^\infty(0,T;V), \\ & e^{3/2s\beta^*}(\gamma^*)^{-9/8}\theta \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H^1_0(\Omega)), \\ & e^{5/2s\beta^*}(\gamma^*)^{-2}(y_t - \Delta y + \nabla p - \theta e_N - v\mathbb{1}_\omega) \in L^2(Q)^N, \\ & e^{5/2s\beta^*}(\gamma^*)^{-5/2}(\theta_t - \Delta\theta + y\cdot\nabla\bar{\theta} - v_0\mathbb{1}_\omega) \in L^2(Q)\,\}. \end{aligned}$$

Notons que la Remarque 1.7 est pertinente aussi dans ce cadre. La contrôlabilité du système linéarisé est donnée par la proposition suivante :

Proposition 1.13. *Soit $N = 2$ ou 3, et $i \in \{1,\dots,N-1\}$. On suppose que $(\bar{p},\bar{\theta})$ satisfait* (1.29)-(1.30)*, $y^0 \in V$, $\theta_0 \in H^1_0(\Omega)$, $e^{5/2s\beta^*}(\gamma^*)^{-2} f \in L^2(Q)^N$ et $e^{5/2s\beta^*}(\gamma^*)^{-5/2} f_0 \in L^2(Q)$. Alors, on peut trouver des contrôles v et v_0 tels que la solution associée (y,p,θ) de* (1.31) *vérifie $(y,p,v,\theta,v_0) \in E_N^i$. En particulier, $v_i \equiv v_N \equiv 0$, $y(T) = 0$ et $\theta(T) = 0$ dans Ω.*

La preuve est totalement analogue à celle de la Proposition 1.8 avec les éléments suivants :

$$a((\widehat{\chi},\widehat{\sigma},\widehat{\kappa}),(\chi,\sigma,\kappa)) = \langle G,(\chi,\sigma,\kappa)\rangle \quad \text{pour tout } (\chi,\sigma)\in P_0, \tag{1.36}$$

avec

$$\begin{aligned}
a((\widehat{\chi},\widehat{\sigma},\widehat{\kappa}),(\chi,\sigma,\kappa)) = &\iint_Q e^{-3s\beta^*}\,(-\widehat{\chi}_t-\Delta\widehat{\chi}+\nabla\widehat{\sigma}+\widehat{\kappa}\nabla\bar{\theta})\cdot(-\chi_t-\Delta\chi+\nabla\sigma+\kappa\nabla\bar{\theta})\,\mathrm{d}x\,\mathrm{d}t\\
&+\iint_Q e^{-3s\beta^*}\,(-\widehat{\kappa}_t-\Delta\widehat{\kappa}-\widehat{\chi}_N)(-\kappa_t-\Delta\kappa-\chi_N)\,\mathrm{d}x\,\mathrm{d}t\\
&+(N-2)\iint_{\omega\times(0,T)} e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7\,\widehat{\chi}_j\,\chi_j\,\mathrm{d}x\,\mathrm{d}t+\iint_{\omega\times(0,T)} e^{-4s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{49/4}\,\widehat{\kappa}\,\kappa\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

$$\langle G,(\chi,\sigma,\kappa)\rangle = \iint_Q f\cdot\chi\,\mathrm{d}x\,\mathrm{d}t+\iint_Q f_0\,\kappa\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega y^0\cdot\chi(0)\,\mathrm{d}x+\int_\Omega \theta^0\,\kappa(0)\,\mathrm{d}x$$

et

$$P_0 = \{\,(\chi,\sigma,\kappa)\in C^2(\overline{Q})^{N+2} : \nabla\cdot\chi = 0 \text{ dans } Q,\quad \chi=0 \text{ sur } \Sigma,\quad \kappa = 0 \text{ sur } \Sigma\}.$$

Ici, $j\in\{1,\dots,N-1\}\setminus\{i\}$. On résout le problème variationnel (1.36) grâce à (1.35) et on montre que $(\widehat{y},\widehat{v},\widehat{\theta},\widehat{v}_0)$, avec une certaine pression $\widehat{p}$, définis comme

$$\begin{cases}
\widehat{y} = e^{-3s\beta^*}(-\widehat{\chi}_t-\Delta\widehat{\chi}+\nabla\widehat{\sigma}+\widehat{\kappa}\nabla\bar{\theta}), & \text{dans } Q,\\
\widehat{v}_j := -(N-2)e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7\,\widehat{\chi}_j,\quad \widehat{v}_i := 0,\quad \widehat{v}_N := 0 & \text{dans } \omega\times(0,T),\\
\widehat{\theta} := e^{-3s\beta^*}(-\widehat{\kappa}_t-\Delta\widehat{\kappa}-\widehat{\chi}_N), & \text{dans } Q,\\
\widehat{v}_0 := -e^{-4s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{49/4}\,\widehat{\kappa}, & \text{dans } \omega\times(0,T),
\end{cases}$$

est une solution de (1.31) telle que $(\widehat{y},\widehat{p},\widehat{v},\widehat{\theta},\widehat{v}_0)\in E_N^i$.

Finalement, on revient au système non linéaire (1.28) en utilisant le Théorème 1.9 adapté à ce cadre, à savoir (voir Section 3.4)

$$\mathcal{B}_1 = E_N^i,$$

$$\mathcal{B}_2 = L^2(e^{5/2s\beta^*}(\gamma^*)^{-2}(0,T);L^2(\Omega)^N)\times V\times L^2(e^{5/2s\beta^*}(\gamma^*)^{-5/2}(0,T);L^2(\Omega))\times H_0^1(\Omega)$$

et l'opérateur

$$\begin{aligned}
\mathcal{A}(\widetilde{y},\widetilde{p},v,\widetilde{\theta},v_0) = (&\widetilde{y}_t-\Delta\widetilde{y}+(\widetilde{y}\cdot\nabla)\widetilde{y}+\nabla\widetilde{p}-\widetilde{\theta}\,e_N-v\mathbb{1}_\omega,\ \widetilde{y}(0),\\
&\widetilde{\theta}_t-\Delta\widetilde{\theta}+\widetilde{y}\cdot\nabla\widetilde{\theta}+\widetilde{y}\cdot\nabla\bar{\theta}-v_0\mathbb{1}_\omega,\ \widetilde{\theta}(0)),
\end{aligned}$$

où

$$\widetilde{y} = y,\quad \widetilde{p} = p-\bar{p}\quad \text{et}\quad \widetilde{\theta} = \theta-\bar{\theta}.$$

En montrant que cet opérateur est de classe $\mathcal{C}^1$ et que $\mathcal{A}'(0,0,0,0,0)$ est surjective grâce à la Proposition 1.13, on conclut le résultat.

Cette partie fait l'objet du Chapitre 3.

1.2.3 Contrôles insensibilisants pour le système de Boussinesq avec deux composantes nulles

On s'intéresse maintenant à l'existence de contrôles insensibilisants pour le système de Boussinesq. Pour introduire ce problème, on considère le système suivant avec donnée initiale incomplète :

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = f + v\mathbb{1}_\omega + \theta\, e_N, \quad \nabla\cdot y = 0 & \text{dans } Q, \\ \theta_t - \Delta\theta + y\cdot\nabla\theta = f_0 + v_0\mathbb{1}_\omega & \text{dans } Q, \\ y = 0, \quad \theta = 0 & \text{sur } \Sigma, \\ y(0) = y^0 + \tau\widehat{y}_0, \quad \theta(0) = \theta^0 + \tau\widehat{\theta}_0 & \text{dans } \Omega, \end{cases} \tag{1.37}$$

où $v = (v_1, \dots, v_N)$ et v_0 sont les contrôles, $f \in L^2(Q)^N$ et $f_0 \in L^2(Q)$ sont deux forces externes et les conditions initiales $(y(0), \theta(0))$ sont partiellement inconnues dans le sens suivant :

- $y^0 \in H$ et $\theta^0 \in L^2(\Omega)$ sont connus,
- $\widehat{y}_0 \in H$ et $\widehat{\theta}_0 \in L^2(\Omega)$ sont inconnus avec $\|\widehat{y}^0\|_{L^2(\Omega)^N} = \|\widehat{\theta}_0\|_{L^2(\Omega)} = 1$, et
- τ est un nombre réel petit inconnu.

On observe la solution de (1.37) à travers d'une certaine fonctionnelle $J_\tau(y,\theta)$, appelée la *sentinelle*. Dans notre cas, on considère

$$J_\tau(y,\theta) := \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} \left(|y|^2 + |\theta|^2\right) \mathrm{d}x\,\mathrm{d}t, \tag{1.38}$$

où $\mathcal{O} \subset \Omega$ est appelé *l'observatoire*. On cherche des contrôles (v, v_0) qui insensibilisent J_τ par rapport à l'incertitude de la condition initiale, i.e.,

$$\left.\frac{\partial J_\tau(y,\theta)}{\partial\tau}\right|_{\tau=0} = 0 \quad \forall\, (\widehat{y}_0, \widehat{\theta}_0) \in L^2(\Omega)^{N+1} \text{ tels que } \|\widehat{y}_0\|_{L^2(\Omega)^N} = \|\widehat{\theta}_0\|_{L^2(\Omega)} = 1. \tag{1.39}$$

Ce type de problème a été proposé par J.-L. Lions dans [65]. Les premiers résultats concernent l'équation de la chaleur. Dans [5], O. Bodart et C. Fabre démontrent l'existence de contrôles ε-insensibilisants (i.e., $|\partial_\tau J_\tau(y)|_{\tau=0}| \leq \varepsilon$ au lieu de (1.39)) pour une équation de la chaleur semi-linéaire. Puis, pour la même équation, (1.39) est montré dans [25] par L. de Teresa. Pour un autre type de sentinelle, par exemple, le gradient de la solution, on cite le travail de S. Guerrero [46].

Pour le système de Stokes, on mentionne [70] pour des résultats de ε-insensibilisation et [45] par S. Guerrero pour l'existence de contrôles insensibilisants. Puis, M. Gueye dans [49] a traité le cas du système de Navier-Stokes. Récemment, dans [9] l'existence de contrôles insensibilisants ayant une composante nulle a été établi.

Dans ce paragraphe on cherche à montrer l'existence de contrôles insensibilisants (v_0, v) ayant deux composantes nulles, à savoir, $v_{i_0} \equiv 0$ pour $0 < i_0 < N$ et $v_N \equiv 0$,

Il est bien connu que la condition (1.39) pour une sentinelle comme (1.38) est équivalente à la contrôlabilité à zéro d'un système d'équations en cascade. Plus précisément, on a le résultat suivant qui est essentiellement démontré dans [64] et [5].

Proposition 1.14. *Le problème d'insensibilisation* (1.38)-(1.39) *pour* (1.37) *est équivalent au problème de contrôle à zéro suivant : il existe des contrôles* (v, v_0) *tels que la solu-*

tion (w, z, r, q) de

$$\begin{cases} w_t - \Delta w + (w\cdot\nabla)w + \nabla p_0 = f + v\mathbb{1}_\omega + r\,e_N, & \nabla\cdot w = 0 \quad \textit{dans } Q,\\ -z_t - \Delta z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + q\nabla r + \nabla p_1 = w\mathbb{1}_{\mathcal{O}}, & \nabla\cdot z = 0 \quad \textit{dans } Q,\\ r_t - \Delta r + w\cdot\nabla r = f_0 + v_0\mathbb{1}_\omega & \textit{dans } Q,\\ -q_t - \Delta q - w\cdot\nabla q = z_N + r\,\mathbb{1}_{\mathcal{O}} & \textit{dans } Q,\\ w = z = 0, \quad r = q = 0 & \textit{sur } \Sigma,\\ w(0) = y^0, \quad z(T) = 0, \quad r(0) = \theta^0, \quad q(T) = 0 & \textit{dans } \Omega, \end{cases} \tag{1.40}$$

vérifie $z(0) = 0$ et $q(0) = 0$ dans Ω, où on a utilisé la notation

$$((z,\nabla^t)w)_i = \sum_{j=1}^{N} z_j\partial_i w_j \quad i = 1,\dots,N.$$

Démonstration. On développe les idées de la preuve. On commence par remarquer que pour tout $\widehat{y}_0 \in L^2(\Omega)^N$ et tout $\widehat{\theta}_0 \in L^2(\Omega)$ on a

$$\left.\frac{\partial J_\tau(y,\theta)}{\partial\tau}\right|_{\tau=0} = \iint_{\mathcal{O}\times(0,T)} (w\cdot y^\tau + r\,\theta^\tau)\,\mathrm{d}x\,\mathrm{d}t, \tag{1.41}$$

où $w := y_{|\tau=0}$, $r := \theta_{|\tau=0}$, $y^\tau := \dfrac{\partial y}{\partial\tau}_{|\tau=0}$ et $\theta^\tau := \dfrac{\partial\theta}{\partial\tau}_{|\tau=0}$. En fait, (y^τ, θ^τ) est la solution de

$$\begin{cases} y^\tau_t - \Delta y^\tau + (y^\tau\cdot\nabla)w + (w\cdot\nabla)y^\tau + \nabla p^\tau = \theta^\tau e_N, & \nabla\cdot y^\tau = 0 \quad \text{dans } Q,\\ \theta^\tau_t - \Delta\theta^\tau + (y^\tau\cdot\nabla)r + (w\cdot\nabla)\theta^\tau = 0 & \text{dans } Q,\\ y^\tau = 0, \quad \theta^\tau = 0 & \text{sur } \Sigma,\\ y^\tau(0) = \widehat{y}_0, \quad \theta^\tau(0) = \widehat{\theta}_0 & \text{dans } \Omega. \end{cases}$$

En utilisant (1.40)-(1.41), on trouve

$$\left.\frac{\partial J_\tau(y,\theta)}{\partial\tau}\right|_{\tau=0} = \int_\Omega (z(0)\cdot\widehat{y}_0 + q(0)\widehat{\theta}_0)\,\mathrm{d}x,$$

pour tout $(\widehat{y}^0, \widehat{\theta}_0) \in L^2(\Omega)^{N+1}$ tels que $\|\widehat{y}_0\|_{L^2(\Omega)^N} = \|\widehat{\theta}_0\|_{L^2(\Omega)^N} = 1$, d'où la conclusion. □

Remarque 1.15. *Bien entendu, la Proposition 1.14 s'applique au cas de contrôles ε-insensibilisants, la preuve étant la même.*

Grâce à la Proposition 1.14, on se concentre dans ce qui suit sur la contrôlabilité à zéro du système (1.40). Remarquons que le contrôle n'agit pas directement sur z et q, mais via le couplage avec w et r dans l'observatoire $\mathcal{O}$.

Dans [25], L. de Teresa a montré dans le cadre de l'équation de la chaleur qu'il existe des conditions initiales non triviales telles que la fonctionnelle J_τ ne peut pas satisfaire (1.39) si $\Omega\setminus\overline{\omega} \neq \emptyset$. Le fait que le système (1.40) combine des équations directes et des équations rétrogrades nous forcera à fixer $y^0 \equiv 0$ et $\theta^0 \equiv 0$.

En outre, on suppose aussi que $\omega\cap\mathcal{O} \neq \emptyset$. Cette hypothèse est vitale pour l'obtention de l'inégalité d'observabilité (voir l'ébauche de la preuve de (1.44) plus bas). On fait référence à [54] et [69] pour quelques résultats d'existence des contrôles ε-insensibilisants lorsque $\omega\cap\mathcal{O} = \emptyset$, et à [1, 2] par F. Alabau-Boussouira pour des résultats d'existence des contrôles insensibilisants sous la condition de contrôle géométrique [4].

Le résultat principal de cette partie est :

Théorème 1.16. *Soit* $0 < i_0 < N$ *et* $m \geq 10$ *un nombre réel. On suppose que* $\omega \cap \mathcal{O} \neq \emptyset$, $y^0 \equiv 0$ *et* $\theta^0 \equiv 0$. *Alors, il existe* $\delta > 0$ *et* $C > 0$, *qui ne dépendent que de* ω, Ω, $\mathcal{O}$ *et* T, *tels que pour tout* $f \in L^2(Q)^N$ *et tout* $f_0 \in L^2(Q)$ *satisfaisant* $\|e^{C/t^m}(f, f_0)\|_{L^2(Q)^{N+1}} < \delta$, *il existe des contrôles* $(v, v_0) \in L^2(Q)^{N+1}$ *avec* $v_{i_0} \equiv v_N \equiv 0$ *tels que la solution associée* (w, p_0, z, p_1, r, q) *de* (1.40) *vérifie* $z(0) = 0$ *et* $q(0) = 0$ *dans* Ω.

Remarque 1.17. *On pourrait considérer une sentinelle plus générale dans le Théorème 1.16, par exemple,*

$$J_\tau(y, \theta) := \frac{1}{2} \iint_{\mathcal{O}_1 \times (0,T)} |y|^2 \, dx \, dt + \frac{1}{2} \iint_{\mathcal{O}_2 \times (0,T)} |\theta|^2 \, dx \, dt,$$

avec $\mathcal{O}_1, \mathcal{O}_2 \subset \Omega$, *et ensembles de contrôle* ω_1 *(pour* v*) et* ω_2 *(pour* v_0*) tels que* $\omega_1 \cap \mathcal{O}_1 \neq \emptyset$ *et* $\omega_2 \cap \mathcal{O}_2 \neq \emptyset$. *Cependant, ceci ne nous permet pas d'obtenir* $v_N \equiv 0$, *sauf si* $\mathcal{O}_1 \cap \mathcal{O}_2 \cap \omega_1 \cap \omega_2 \neq \emptyset$. *Donc, on suppose dans la suite* $\mathcal{O}_1 = \mathcal{O}_2$ *et* $\omega_1 = \omega_2$. *On remercie L. de Teresa pour cette remarque.*

La preuve suit les mêmes idées que dans les paragraphes précédents. On montre d'abord la contrôlabilité à zéro du système linéaire :

$$\begin{cases} w_t - \Delta w + \nabla p_0 = f^w + v\, \mathbb{1}_\omega + r e_N, & \nabla \cdot w = 0 \quad \text{dans } Q, \\ -z_t - \Delta z + \nabla p_1 = f^z + w \mathbb{1}_{\mathcal{O}}, & \nabla \cdot z = 0 \quad \text{dans } Q, \\ r_t - \Delta r = f^r + v_0\, \mathbb{1}_\omega & \text{dans } Q, \\ -q_t - \Delta q = f^q + z_N + r \mathbb{1}_{\mathcal{O}} & \text{dans } Q, \\ w = z = 0, \quad r = q = 0 & \text{sur } \Sigma, \\ w(0) = 0, \quad z(T) = 0, \quad r(0) = 0, \quad q(T) = 0 & \text{dans } \Omega, \end{cases} \tag{1.42}$$

où f^w, f^z, f^r et f^q décroissent exponentiellement en $t = 0$. Ce résultat sera obtenu grâce à une inégalité de Carleman appropriée pour les solutions du système adjoint :

$$\begin{cases} -\varphi_t - \Delta \varphi + \nabla \pi_\varphi = g^\varphi + \psi\, \mathbb{1}_{\mathcal{O}}, & \nabla \cdot \varphi = 0 \quad \text{dans } Q, \\ \psi_t - \Delta \psi + \nabla \pi_\psi = g^\psi + \sigma e_N, & \nabla \cdot \psi = 0 \quad \text{dans } Q, \\ -\phi_t - \Delta \phi = g^\phi + \varphi_N + \sigma\, \mathbb{1}_{\mathcal{O}} & \text{dans } Q, \\ \sigma_t - \Delta \sigma = g^\sigma & \text{dans } Q, \\ \varphi = \psi = 0, \quad \phi = \sigma = 0 & \text{sur } \Sigma, \\ \varphi(T) = 0, \quad \psi(0) = \psi^0, \quad \phi(T) = 0, \quad \sigma(0) = \sigma^0 & \text{dans } \Omega. \end{cases} \tag{1.43}$$

Celle-ci est de la forme (voir Proposition 4.15)

$$\iint_Q \rho_1(t) \left(|\varphi|^2 + |\psi|^2 + |\phi|^2 + |\sigma|^2 \right) \mathrm{d}x \, \mathrm{d}t \leq C \left\| \rho_2(t)(g^\varphi, g^\psi, g^\phi, g^\sigma) \right\|_X^2 + C \iint_{\omega \times (0,T)} \rho_3(t) \Big((N-2) |\varphi_{j_0}|^2 + |\phi|^2 \Big) \mathrm{d}x \, \mathrm{d}t, \tag{1.44}$$

où $\rho_k(t)$ sont des fonctions poids de type exponentiel comme dans (1.33), $j_0 \in \{1, \ldots, N-1\} \setminus \{i_0\}$ et X est un espace de Banach approprié.

La première étape de la preuve de (1.44) consiste à montrer une inégalité comme

$$\iint_Q \rho_1 \Big(|\varphi|^2 + |\psi|^2 \Big) \mathrm{d}x \, \mathrm{d}t \leq C \left\| \rho_2 \, (g^\varphi, g^\psi, \sigma) \right\|_X^2 + C \iint_{\omega_0 \times (0,T)} \rho_3 \left((N-2) |\varphi_{j_0}|^2 + |\varphi_N|^2 \right) \mathrm{d}x \, \mathrm{d}t, \tag{1.45}$$

avec $\omega_0 \Subset \omega \cap \mathcal{O}$. Cette inégalité est essentiellement montrée dans [9] en combinant une inégalité de Carleman pour ψ ayant comme termes locaux $\Delta\psi_{j_0}$ et $\Delta\psi_N$ (Lemme 4.9) avec la Proposition 1.6 pour φ (avec $i = j_0$). Puis, pour estimer les termes locaux de ψ_{j_0} et ψ_N on utilise la formule

$$\Delta\psi_j = -(\Delta\varphi_j)_t - \Delta(\Delta\varphi_j) + \partial_j \nabla\cdot g^{\varphi} - \Delta g_j^{\varphi} \quad \text{dans } \omega' \times (0,T),\ j = j_0, N$$

à condition que $\omega' \Subset \omega_0$ et on obtient (1.45).

Ensuite, on utilise l'équation satisfaite par ϕ

$$\varphi_N = -\phi_t - \Delta\phi - g^{\phi} - \sigma \mathbb{1}_{\mathcal{O}},$$

ce qui nous permet d'éliminer φ_N. On obtient,

$$\iint_Q \rho_1 \Big(|\varphi|^2 + |\psi|^2 + |\phi|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \le C \left\|\rho_2 \left(g^{\varphi}, g^{\psi}, g^{\phi}, \sigma\right)\right\|_X^2 + (N-2)C \iint_{\omega_0\times(0,T)} \rho_3 |\varphi_{j_0}|^2\,\mathrm{d}x\,\mathrm{d}t + C \iint_{\omega\times(0,T)} \rho_4 |\phi|^2\,\mathrm{d}x\,\mathrm{d}t. \quad (1.46)$$

Quelques calculs élémentaires nous permettent de trouver la relation entre σ et ϕ suivante

$$\begin{aligned}(\partial_1^2 + (N-2)\partial_2^2)\sigma &= (\partial_t + \Delta)(\Delta\partial_t^2 - \Delta^3)\phi - (\Delta g_N^{\varphi})_t + \Delta^2 g_N^{\varphi} + \partial_N \nabla\cdot g_t^{\varphi} \\ &\quad - \Delta(\partial_N \nabla\cdot g^{\varphi}) - \Delta g_N^{\psi} + \partial_N \nabla\cdot g^{\psi} + (\Delta\partial_t^2 - \Delta^3) g^{\phi} + \Delta g_t^{\sigma} + \Delta^2 g^{\sigma}.\end{aligned} \quad (1.47)$$

On fait remarquer que cette identité a lieu dans $\omega \cap \mathcal{O}$ et qu'elle ne fait pas intervenir les termes de pression ni φ_N.

L'étape suivante consiste à appliquer une inégalité de Carleman à l'équation de la chaleur satisfaite par

$$\mathcal{D}\sigma := (\partial_1^2 + (N-2)\partial_2^2)\sigma. \quad (1.48)$$

Celle-ci nous donne un terme local à droite en $\mathcal{D}\sigma$. En tenant compte du fait que $\|\mathcal{D}\sigma\|_{L^2(\Omega)}$ définit une norme, cette inégalité nous donne aussi un terme global en σ à gauche.

En combinant ceci avec (1.46), on a

$$\begin{aligned}\iint_Q \rho_1 \Big(|\varphi|^2 + |\psi|^2 + |\phi|^2 + |\sigma|^2\Big)\,\mathrm{d}x\,\mathrm{d}t &\le C \left\|\rho_2 \left(g^{\varphi}, g^{\psi}, g^{\phi}, g^{\sigma}, \sigma\right)\right\|_X^2 \\ &\quad + (N-2)C \iint_{\omega_0\times(0,T)} \rho_3 |\varphi_{j_0}|^2\,\mathrm{d}x\,\mathrm{d}t + C \iint_{\omega\times(0,T)} \rho_4 |\phi|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + C \iint_{\omega_1\times(0,T)} \rho_5 \left|(\partial_1^2 + (N-2)\partial_2^2)\sigma\right|^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

où $\omega_1 \Subset \omega \cap \mathcal{O}$ est un ouvert.

On élimine les termes locaux et globaux de σ à droite de cette inégalité avec (1.47) et les termes de gauche de σ, respectivement, et on obtient (1.44). Bien entendu, il faut bien choisir les poids pour les différentes inégalités de Carleman. Concrètement, dans ce paragraphe on utilise les poids considérés dans les deux paragraphes précédents mais avec $t^m(T-t)^m$ au lieu de $t^8(T-t)^8$ au dénominateur, où $m \ge 10$.

Pour traiter le système linéaire, on définit l'espace

$$\begin{aligned}
E_N^{i_0} = \{(w, p_0, z, p_1, r, q, v, v_0) : \quad & e^{13/4s\beta^*}(v, v_0)\mathbb{1}_\omega \in L^2(Q)^N, \quad v_{i_0} \equiv v_N \equiv 0 \\
& e^{13/4s\beta^*}(\gamma^*)^{-1-1/m} w \in L^2(0,T; H^2(\Omega)^N) \cap H^1(0,T; L^2(\Omega)^N), \\
& e^{13/4s\beta^*}(\gamma^*)^{-6-6/m} z \in L^2(0,T; H^2(\Omega)^N) \cap H^1(0,T; L^2(\Omega)^N), \quad z(T) = 0, \\
& e^{13/4s\beta^*}(\gamma^*)^{-1-1/m} r \in L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega)), \\
& e^{13/4s\beta^*}(\gamma^*)^{-15-15/m} q \in L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega)), \quad q(T) = 0, \\
& e^{25/4s\beta^*}\,(w_t - \Delta w + \nabla p_0 - v\mathbb{1}_\omega - r\, e_N, \quad -z_t - \Delta z + \nabla p_1 - w\mathbb{1}_{\mathcal{O}}) \in L^2(Q)^{2N}, \\
& e^{25/4s\beta^*}\,(r_t - \Delta r - v_0\,\mathbb{1}_\omega, \quad -q_t - \Delta q - z_N - r\mathbb{1}_{\mathcal{O}}) \in L^2(Q)^2\}.
\end{aligned}$$

La contrôlabilité à zéro de (1.42) est donnée par la Proposition suivante :

Proposition 1.18. *Soit* $i_0 \in \{1, \ldots, N-1\}$. *On suppose que*

$$e^{25/4s\beta^*}(f^w, f^z, f^r, f^q) \in L^2(Q)^{2N+2}. \tag{1.49}$$

Alors, il existe des contrôles $(v, v_0) \in L^2(Q)^{N+1}$ *tels que la solution associée* (w, p_0, z, p_1, r, q) *de* (1.42) *est telle que* $(w, p_0, z, p_1, r, q, v, v_0)$ *appartient à* $E_N^{i_0}$. *En particulier,* $v_{i_0} \equiv v_N \equiv 0$ *et* $(z(0), q(0)) = (0, 0)$ *dans* Ω.

La preuve de la Proposition 1.18 suit le même schéma que dans les paragraphes précédents à partir de (1.44). Cependant, la régularité supplémentaire demandée aux seconds membres de (1.43) nous oblige à adopter une stratégie différente. Les détails sont présentés dans la Section 4.4, Chapitre 4.

Concernant le problème non linéaire, à l'aide de la Proposition 1.18 on montre le Théorème 1.16 en appliquant le Théorème 1.9 avec l'opérateur

$$\begin{aligned}
\mathcal{A}(w, p_0, z, p_1, r, q, v, v_0) := (& w_t - \Delta w + (w \cdot \nabla) w + \nabla p_0 - v\mathbb{1}_\omega - r\, e_N, \\
& - z_t - \Delta z + (z \cdot \nabla^t) w - (w \cdot \nabla) z + q \nabla r + \nabla p_1 - w\mathbb{1}_{\mathcal{O}}, \\
& r_t - \Delta r + w \cdot \nabla r - v_0 \mathbb{1}_\omega, \quad -q_t - \Delta q - w \cdot \nabla q - z_N - r\,\mathbb{1}_{\mathcal{O}})
\end{aligned}$$

et les espaces

$$\mathcal{B}_1 := E_N^{i_0},$$
$$\mathcal{B}_2 := L^2(e^{25/4s\beta^*}(0,T); L^2(\Omega)^{2N+2}).$$

Cette partie est présentée dans le Chapitre 4.

1.2.4 Sur la contrôlabilité de l'équation de KdV avec conditions au bord de type Colin-Ghidaglia dans la limite de dispersion évanescente

L'équation de Korteweg-de Vries (KdV)

$$y_t + y_{xxx} + y y_x = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \tag{1.50}$$

a été introduite par D. J. Korteweg et G. de Vries dans [56]. Elle modélise la propagation d'une vague de petite amplitude se propageant à droite dans un canal uniforme peu profond, $y = y(t, x)$ étant l'amplitude de l'eau au point x au temps t. On fait référence au livre de Whitham [78] pour comprendre sa déduction et l'interprétation d'un point de vue physique.

Dans cette partie on s'intéresse aux propriétés de contrôle de la version linéaire de (1.50), posée sur un intervalle fini, suivante :

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{dans } (0,T) \times (0,1), \\ y_{|x=0} = v, \quad y_{x|x=1} = 0, \quad y_{xx|x=1} = 0 & \text{dans } (0,T), \\ y_{|t=0} = y_0 & \text{dans } (0,1). \end{cases} \tag{1.51}$$

Ici, $\varepsilon > 0$ est le coefficient de dispersion, $M \in \mathbb{R}$ le coefficient de transport, y_0 la condition initiale et $v = v(t)$ le contrôle.

Les conditions au bord de ce type ont été proposés par T. Colin et J.-M. Ghidaglia dans [14, 15] pour modéliser la propagation unidirectionnelle d'ondes où l'extrémité $x = 0$ est un générateur d'ondes et l'extrémité $x = 1$ est libre.

La plupart des résultats de contrôle pour l'équation de KdV posée sur un intervalle fini ont été obtenus pour les conditions au bord

$$y_{|x=0} = u_1, \quad y_{|x=1} = u_2, \quad y_{x|x=1} = u_3 \quad \text{dans } (0,T). \tag{1.52}$$

Quelques travaux classiques sont ceux de L. Rosier [72, 73]. On mentionne aussi [21] par J.-M. Coron et E. Crépeau, [10] par E. Cerpa et [12] par E. Cerpa et E. Crépeau. Dans ces travaux, les auteurs traitent le problème des longueurs critiques. Ces résultats ont été résumés dans le *survey* [11] réalisé par E. Cerpa. Pour une révision complète des résultats de contrôle pour l'équation de KdV on fait référence à [74] par L. Rosier et B.-Y. Zhang.

En ce qui concerne (1.51), le premier résultat de contrôlabilité est dû à J.-P. Guilleron [50], où la contrôlabilité à zero est établie (avec $\varepsilon = 1$ et $M = -1$). Puis, E. Cerpa, I. Rivas et B.-Y. Zhang dans [13] considèrent des contrôles dans toutes les conditions au bord, à savoir,

$$y_{|x=0} = v_1, \quad y_{x|x=1} = v_2, \quad y_{xx|x=1} = v_3 \quad \text{dans } (0,T), \tag{1.53}$$

et montrent des résultats de contrôlabilité exacte pour l'équation non linéaire avec toutes les combinaisons possibles de (v_1, v_2, v_3) sauf $(v_1, 0, 0)$.

Dans un premier temps, on améliore le résultat obtenu dans [50] par rapport au coût du contrôle. D'après le résultat principal de [50], on peut démontrer que pour tout $y_0 \in L^2(0,1)$, il existe un contrôle $v \in L^2(0,T)$ tel que la solution y de (1.51) satisfait $y_{|t=T} = 0$ et

$$\|v\|_{L^2(0,T)} \leq C \exp(C\varepsilon^{-1}) \|y_0\|_{L^2(0,1)}, \tag{1.54}$$

où $\varepsilon > 0$ et M sont fixes.

Dans cette partie, on établit

Théorème 1.19. *Soit $T > 0$, $M \in \mathbb{R}$ et $\varepsilon > 0$ trois nombres réels fixes. Alors, pour tout $y_0 \in L^2(0,1)$, il existe un contrôle $v \in L^2(0,T)$ tel que la solution associée de* (1.51) *satisfait $y_{|t=T} = 0$. De plus,*

$$\|v\|_{L^2(0,T)} \leq C_0 \exp\Big(C(\varepsilon^{-1/2}T^{-1/2} + M^{1/2}\varepsilon^{-1/2} + MT)\Big) \|y_0\|_{L^2(0,1)}, \tag{1.55}$$

si $M > 0$, et

$$\|v\|_{L^2(0,T)} \leq C_0 \exp\Big(C(\varepsilon^{-1/2}T^{-1/2} + |M|^{1/2}\varepsilon^{-1/2})\Big) \|y_0\|_{L^2(0,1)}, \tag{1.56}$$

si $M < 0$, où $C > 0$ est une constante indépendante de T, M et ε, et $C_0 > 0$ dépend polynomialement de ε^{-1}, T^{-1} et $|M|^{-1}$.

La preuve repose sur une inégalité d'observabilité pour l'équation adjointe de (1.51) (la méthode HUM, Paragraphe 1.1.1), à savoir :

$$\|\varphi_{|t=0}\|_{L^2(0,1)} \leq C_{obs}\|\varphi_{xx|x=0}\|_{L^2(0,T)},$$

où φ es la solution de

$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xxx} + M\varphi_x = 0 & \text{dans } (0,T)\times(0,1), \\ \varphi_{|x=0} = 0, \quad \varphi_{x|x=0} = 0, \quad (\varepsilon\varphi_{xx} - M\varphi)_{|x=1} = 0 & \text{dans } (0,T), \\ \varphi_{|t=T} = \varphi_T & \text{dans } (0,1), \end{cases} \tag{1.57}$$

avec $\varphi_T \in L^2(0,1)$. L'outil principal pour démontrer cette inégalité d'observabilité est une inégalité de Carleman. On considère la fonction

$$\alpha(t,x) = \frac{-x^2+4x+1}{t^m(T-t)^m}, \tag{1.58}$$

où $m \geq 1/2$. L'inégalité de Carleman est établie dans la proposition suivante.

Proposition 1.20. *Soit $\varepsilon, T > 0$, $M \in \mathbb{R}\backslash\{0\}$ et $m = 1/2$. Il existe une constante positive C indépendante de T, ε et M telle que, pour toute solution φ de (1.57), on a*

$$\iint_Q e^{-2s\alpha_{|x=1}}\Big(s^5\alpha^5_{|x=0}|\varphi|^2 + s^3\alpha^3_{|x=0}|\varphi_x|^2 + s\alpha_{|x=0}|\varphi_{xx}|^2\Big)\,dx\,dt$$
$$\leq C_0 \exp\big(C|M|^{1/2}\varepsilon^{-1/2}\big)s^5\int_0^T e^{-2s\alpha_{|x=0}}\alpha^5_{|x=0}|\varphi_{xx|x=0}|^2\,dt, \tag{1.59}$$

pour tout $s \geq C(T+\varepsilon^{-1/2}T^{1/2}+|M|^{1/2}\varepsilon^{-1/2}T)$, où C_0 dépend polynomialement de $|M|^{-1}$ et ε^{-1}.

Dans [50], une inégalité similaire à (1.59) est démontrée avec $m = 1$ (Proposition 3 dans cette référence). La difficulté principale est due à la condition au bord en $x = 1$ de (1.57). Ceci fait nécessaire de prendre $m = 1$ pour estimer les termes de bord en $x = 1$.

Ici, on réussit à prendre la puissance optimale $m = 1/2$ comme dans [42, 43], le point clé étant le changement de variable $\phi := \varepsilon\varphi_{xx} - M\varphi$. Cette nouvelle fonction satisfait l'équation de KdV (1.57) et les conditions au bord $\phi_{x|x=0} = \phi_{xx|x=0} = \phi_{|x=1} = 0$. Le prix à payer lorsque l'on utilise cette variable est un facteur $\exp\big(C|M|^{1/2}\varepsilon^{-1/2}\big)$ quand on récupère la variable originale φ, mais ceci ne change pas l'ordre de la constante d'observabilité par rapport à ε .

Pour introduire et motiver le deuxième résultat de cette partie, on considère l'équation de transport

$$y_t - My_x = 0 \quad \text{dans } (0,T)\times(0,1). \tag{1.60}$$

Il est bien connu que (1.60) est contrôlable si et seulement si $T \geq 1/|M|$ (voir, par exemple, [20, Theorem 2.6, page 29]), avec un contrôle $y_{|x=0} = v_1$ si $M < 0$ et avec un contrôle $y_{|x=1} = v_2$ si $M > 0$. De plus, le coût de contrôler à zéro est nul. En effet, on peut mener la solution de (1.60) à zéro si l'on prend $v_1 \equiv 0$ si $M < 0$, et $v_2 \equiv 0$ si $M > 0$. Donc, il est naturel de s'attendre à ce que le coût décroisse vers zéro lorsque $\varepsilon \to 0$ si $T \geq 1/|M|$, ou au moins pour T suffisamment grand. Par contre, si $T < 1/|M|$, on espère que le coût explose quand $\varepsilon \to 0$.

Ce problème a été traité dans [42] pour les conditions au bord (1.52) et dans [43] avec $u_2 = u_3 = 0$. On fait référence à [22] et [47] pour le cas de viscosité évanescente en une et plusieurs dimensions d'espace, respectivement. On mentionne aussi [41] par O. Glass et [68] par P. Lissy pour quelques résultats liés au temps minimal pour obtenir la décroissance exponentielle du coût du contrôle pour l'équation de transport-diffusion.

En ce qui concerne le résultat de décroissance du coût du contrôle, une stratégie possible consiste à combiner une inégalité de Carleman avec une dissipation exponentielle du type

$$\|\varphi_{|t=t_1}\|_{L^2(0,1)} \leq \exp\left(-CT\varepsilon^{-1/2}\right)\|\varphi_{|t=t_2}\|_{L^2(0,1)},\ 0 \leq t_1 \leq t_2 \leq T,\ t_2-t_1 > 1/|M|, \quad (1.61)$$

pour les solutions de (1.57), de telle sorte que pour T suffisamment grand cette dissipation compense la constante de (1.55). Remarquons que pour faire une telle comparaison, le choix d'une puissance optimale dans (1.58) ($m = 1/2$ dans ce cas) devient essentiel quand ε est petit. En outre, il a été remarqué dans [42] et [43] qu'une dissipation comme (1.61) est possible seulement si $M > 0$ dû à l'effet asymétrique du terme de dispersion. Malheureusement, la condition au bord sur $x = 1$ pose de gros problèmes pour obtenir (1.61). En fait, il semble aussi difficile d'obtenir une dissipation standard comme

$$\|\varphi_{|t=t_1}\|_{L^2(0,1)} \leq \|\varphi_{|t=t_2}\|_{L^2(0,1)}. \quad (1.62)$$

Néanmoins, on peut obtenir le résultat désiré quand T est petit par rapport à $1/|M|$.

Théorème 1.21. *Soit $M \neq 0$. Alors, il existe $T_0 < 1/|M|$ tel que pour tout $T \in (0, T_0)$ il existe des constantes $C > 0$ (indépendante de ε), $\varepsilon_0 > 0$ et des conditions initiales $y_0 \in L^2(0,1)$ telles que, si $v \in L^2(0,T)$ est un contrôle tel que la solution y de* (1.51) *satisfait $y_{|t=T} = 0$, alors, pour tout $\varepsilon \in (0, \varepsilon_0)$,*

$$\|v\|_{L^2(0,T)} \geq \exp\left(\frac{C}{\varepsilon^{1/2}}\right)\|y_0\|_{L^2(0,1)}.$$

De plus, si $M < 0$, on peut choisir $T_0 = 1/|M|$.

La preuve suit le schéma de [43, Theorem 1.4]. On montre

$$\|\widehat{\varphi}_{xx|x=0}\|_{L^2(0,T)} \leq C\exp\left[-\frac{C}{\varepsilon^{1/2}T^{1/2}}\right]\|\widehat{\varphi}_T\|_{L^2(0,1)} \quad (1.63)$$

et

$$\|\widehat{\varphi}_{|t=0}\|_{L^2(0,1)} \geq c > 0, \quad (1.64)$$

pour $\widehat{\varphi}$ une solution particulière de (1.57). Pour le cas $M < 0$, la preuve est identique à celle de [43] et (1.63)-(1.64) sont démontrés indépendamment. Par contre, quand $M > 0$, le manque d'une inégalité comme (1.62) nous force à prendre T suffisamment petit par rapport à M (mais indépendant de ε) pour obtenir (1.63). Puis, on obtient (1.64) comme conséquence de (1.63) si $\varepsilon \in (0, \varepsilon_0)$.

Les détails de cette partie sont présentés dans le Chapitre 5.

1.3 Commentaires

On finit ce chapitre introducteur en faisant une brève présentation de quelques travaux complémentaires concernant le contrôle insensibilisant des systèmes de Navier-Stokes et de Boussinesq, ainsi que quelques perspectives et problèmes qui demeurent ouverts.

1.3.1 Travaux complémentaires

Dans ce paragraphe, on présente le travail [9] ainsi qu'un travail en cours concernant l'existence de contrôles insensibilisants pour le système de Boussinesq.

Contrôles insensibilisants pour le système de Navier-Stokes ayant une composante nulle

Dans cette partie, on présente les résultats principaux obtenus dans [9] fait en collaboration avec M. Gueye. Dans ce travail, on montre l'existence de contrôles insensibilisants ayant une composante nulle pour le système de Navier-Stokes :

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = f + v\mathbb{1}_\omega, \quad \nabla\cdot y = 0 & \text{dans } Q,\\ y = 0 & \text{sur } \Sigma,\\ y(0) = y^0 + \tau\widehat{y}^0 & \text{dans } \Omega. \end{cases} \tag{1.65}$$

On garde la notation utilisée dans le Paragraphe 1.2.3 ci-dessus pour le système de Boussinesq. La sentinelle est donnée par :

$$J_\tau(y) := \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} |y|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{1.66}$$

et la condition d'insensibilisation par :

$$\left.\frac{\partial J_\tau(y)}{\partial\tau}\right|_{\tau=0} = 0 \quad \forall\,\widehat{y}^0\in L^2(\Omega)^N \quad \text{tel que} \quad \|\widehat{y}^0\|_{L^2(\Omega)^N} = 1. \tag{1.67}$$

En adaptant la Proposition 1.14 à (1.65)-(1.67), on sait que ce problème est équivalent à la contrôlabilité à zéro du système en cascade suivant :

$$\begin{cases} w_t - \Delta w + (w\cdot\nabla)w + \nabla p_0 = f + v\mathbb{1}_\omega, & \nabla\cdot w = 0 \quad \text{dans } Q,\\ -z_t - \Delta z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + \nabla p_1 = w\mathbb{1}_{\mathcal{O}}, & \nabla\cdot z = 0 \quad \text{dans } Q,\\ w = z = 0 & \text{sur } \Sigma,\\ w(0) = y^0, \quad z(T) = 0 & \text{dans } \Omega. \end{cases} \tag{1.68}$$

On énonce maintenant ce résultat de contrôlabilité ([9, Theorem 1.1]) :

Théorème 1.22. *Soit $i_0\in\{1,\dots,N\}$ et $m\geq 10$ un nombre réel. On suppose $\omega\cap\mathcal{O}\neq\emptyset$ et $y^0\equiv 0$. Alors, il existe des constantes $\delta>0$ et $C>0$, qui ne dépendent que de ω, Ω, $\mathcal{O}$ et T, telles que pour tout $f\in L^2(Q)^N$ satisfaisant $\|e^{C/t^m}f\|_{L^2(Q)^N}<\delta$, il existe un contrôle $v\in L^2(Q)^N$ avec $v_{i_0}\equiv 0$ tel que la solution associée (w,p_0,z,p_1) de (1.68) vérifie $z(0)=0$ dans Ω.*

Notons que les mêmes remarques faites dans le Paragraphe 1.2.3 sur la condition initiale et les ensembles de contrôle et d'observation sont pertinentes dans ce cas.

La preuve du Théorème 1.22 repose sur la contrôlabilité à zéro du système linéarisé :

$$\begin{cases} w_t - \Delta w + \nabla p_0 = f^w + v\mathbb{1}_\omega, & \nabla\cdot w = 0 \quad \text{dans } Q,\\ -z_t - \Delta z + \nabla p_1 = f^z + w\mathbb{1}_{\mathcal{O}}, & \nabla\cdot z = 0 \quad \text{dans } Q,\\ w = z = 0 & \text{sur } \Sigma,\\ w(0) = 0, \quad z(T) = 0 & \text{dans } \Omega, \end{cases} \tag{1.69}$$

où f^w et f^z décroissent exponentiellement en $t = 0$. Ce résultat est obtenu grâce à une inégalité de Carleman appropriée pour le système adjoint

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_\varphi = g^\varphi + \psi \mathbb{1}_{\mathcal{O}}, & \nabla\cdot\varphi = 0 \quad \text{dans } Q, \\ \psi_t - \Delta\psi + \nabla\pi_\psi = g^\psi, & \nabla\cdot\psi = 0 \quad \text{dans } Q, \\ \varphi = \psi = 0 & \text{sur } \Sigma, \\ \varphi(T) = 0, \quad \psi(0) = \psi^0 & \text{dans } \Omega \end{cases} \tag{1.70}$$

Cette inégalité de Carleman est donnée par la proposition suivante ([9, Proposition 3.1]) :

Proposition 1.23. *On suppose $\omega \cap \mathcal{O} \neq \emptyset$. Alors, il existe une constante λ_0, telle que pour tout $\lambda \geq \lambda_0$ il existe une constante $C > 0$ qui ne dépend que de λ, Ω, ω et ℓ telle que pour tout $i_0 \in \{1, \ldots, N\}$, tout $g^\varphi \in L^2(Q)^N$, tout $g^\psi \in L^2(0,T;V)$ et tout $\psi^0 \in H$, la solution (φ, ψ) de* (1.70) *satisfait*

$$\begin{aligned} s^4 \iint_Q e^{-7s\alpha^*}(\xi^*)^4|\varphi|^2\,dx\,dt + s^5 \iint_Q e^{-4s\alpha^*}(\xi^*)^5|\psi|^2\,dx\,dt \\ \leq C\left(s^9 \iint_Q e^{-3s\alpha - s\alpha^*}\xi^9|g^\varphi|^2\,dx\,dt + \iint_Q e^{-s\alpha^*}(|g^\psi|^2 + |\nabla g^\psi|^2)\,dx\,dt \right. \\ \left. + s^{13} \sum_{j=1, j\neq i_0}^{N} \iint_{\omega\times(0,T)} e^{-3s\alpha - s\alpha^*}\xi^{13}|\varphi_j|^2\,dx\,dt \right), \end{aligned} \tag{1.71}$$

pour tout $s \geq C$.

Une idée classique pour montrer (1.71) est de combiner une inégalité de Carleman pour φ et ψ du type (1.23) (avec i_0 au lieu de i), puis d'utiliser le couplage de l'équation satisfaite par φ pour éliminer les termes locaux de ψ_j $(j \neq i_0)$. Cependant, cette procédure fait apparaître le terme de pression π_φ. Par conséquent, appliquer directement la Proposition 1.6 ne semble pas être une bonne stratégie.

Cette difficulté a été déjà remarquée dans [45] et [49]. Dans ces travaux, les auteurs démontrent une inégalité de Carleman avec des termes locaux du type $\nabla \times \psi$ pour éviter le problème de la pression, mais dans notre cas, un tel terme local ferait apparaître la composante φ_{i_0} que l'on cherche à éviter.

Pour contourner ces deux difficultés, on montre d'abord une inégalité de Carleman avec des termes locaux dépendants de $\Delta\psi_j$ $(j \neq i_0)$ en suivant le schéma de la preuve de (1.25), mais en utilisant l'opérateur $\nabla\nabla\Delta$ au lieu de $\nabla\Delta$. Ce fait nous force à demander plus de régularité à g^ψ, comme indiqué dans la Proposition 1.23. Ensuite, on applique la Proposition 1.6 à l'équation de φ en regardant $g^\varphi + \psi\mathbb{1}_{\mathcal{O}}$ comme un second membre. Enfin, on élimine les termes locaux en ψ_j $(j \neq i_0)$ en utilisant

$$\Delta\psi_j = -(\Delta\varphi_j)_t - \Delta(\Delta\varphi_j) + \partial_j\nabla\cdot g^\varphi - \Delta g_j^\varphi \quad \text{dans } (\omega\cap\mathcal{O})\times(0,T),$$

pour obtenir (1.71).

Remarque 1.24. *La condition $\varphi(T) = 0$ ne joue aucun rôle dans la preuve de* (1.71)*, qui est toujours vraie si $\varphi(T) = \varphi^T \in H$ quelconque. Néanmoins, cette condition est essentielle pour obtenir une inégalité avec des poids qui ne soient pas dégénérés en $t = T$.*

Le Théorème 1.22 est une conséquence du Théorème 1.9 adapté à ce cadre. L'hypothèse principale de ce théorème est satisfaite d'après un résultat de contrôlabilité à zéro pour le système (1.69).

Contrôles insensibilisants pour le système de Boussinesq sans contrôle sur l'équation de la température

Ici on présente une extension du problème traité dans le Paragraphe 1.2.3. Concrètement, on s'intéresse à l'existence de contrôles insensibilisants pour le système de Boussinesq en n'agissant que sur l'équation du fluide ($v_0 \equiv 0$ dans (1.37)). Ceci a été proposé comme problème ouvert dans [48]. De plus, le contrôle v peut être choisi tel que $v_{i_0} \equiv 0$ $(0 < i_0 < N)$.

Ici on peut considérer une sentinelle comme celle de la Remarque 1.17, c'est-à-dire :

$$J_\tau(y,\theta) := \frac{1}{2}\iint_{\mathcal{O}_1\times(0,T)} |y|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\iint_{\mathcal{O}_2\times(0,T)} |\theta|^2 \,\mathrm{d}x\,\mathrm{d}t,$$

où $\mathcal{O}_1, \mathcal{O}_2 \subset \Omega$ sont les observatoires. Le domaine de contrôle est tel que $\omega \cap \mathcal{O}_1 \neq \emptyset$. D'après la Proposition 1.14, le système en cascade associé est le suivant :

$$\begin{cases} w_t - \Delta w + (w\cdot\nabla)w + \nabla p_0 = f + v\mathbb{1}_\omega + r\,e_N, & \nabla\cdot w = 0 \quad \text{dans } Q,\\ -z_t - \Delta z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + q\nabla r + \nabla p_1 = w\mathbb{1}_{\mathcal{O}_1}, & \nabla\cdot z = 0 \quad \text{dans } Q,\\ r_t - \Delta r + w\cdot\nabla r = f_0 & \text{dans } Q,\\ -q_t - \Delta q - w\cdot\nabla q = z_N + r\,\mathbb{1}_{\mathcal{O}_2} & \text{dans } Q,\\ w = z = 0, \quad r = q = 0 & \text{sur } \Sigma,\\ w(0) = y^0, \quad z(T) = 0, \quad r(0) = \theta^0, \quad q(T) = 0 & \text{dans } \Omega. \end{cases} \tag{1.72}$$

Ici, on cherche un contrôle v, avec $v_{i_0} \equiv 0$ $(0 < i_0 < N)$, tel que $z(0) = 0$ et $q(0) = 0$ dans Ω.

Le système adjoint associé à ce problème est donné par

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_\varphi = g^\varphi + \psi\,\mathbb{1}_{\mathcal{O}_1}, \quad \nabla\cdot\varphi = 0 & \text{dans } Q,\\ \psi_t - \Delta\psi + \nabla\pi_\psi = g^\psi + \sigma e_N, \quad \nabla\cdot\psi = 0 & \text{dans } Q,\\ -\phi_t - \Delta\phi = g^\phi + \varphi_N + \sigma\,\mathbb{1}_{\mathcal{O}_2} & \text{dans } Q,\\ \sigma_t - \Delta\sigma = g^\sigma & \text{dans } Q,\\ \varphi = \psi = 0, \quad \phi = \sigma = 0 & \text{sur } \Sigma,\\ \varphi(T) = 0, \quad \psi(0) = \psi^0, \quad \phi(T) = 0, \quad \sigma(0) = \sigma^0 & \text{dans } \Omega. \end{cases} \tag{1.73}$$

La seule différence par rapport à (1.43) est la présence de deux observatoires $\mathcal{O}_1$ et $\mathcal{O}_2$. Pour ce système, on montre une inégalité de Carleman du type

$$\iint_Q \rho_1(t)\left(|\varphi|^2 + |\psi|^2 + |\phi|^2 + |\sigma|^2\right)\mathrm{d}x\,\mathrm{d}t \le C\left\|\rho_2(t)(g^\varphi, g^\psi, g^\phi, g^\sigma)\right\|_X^2 + C\iint_{\omega\times(0,T)} \rho_3(t)\Big((N-2)|\varphi_{j_0}|^2 + |\varphi_N|^2\Big)\,\mathrm{d}x\,\mathrm{d}t, \tag{1.74}$$

où $\rho_k(t)$ sont des fonctions poids de type exponentielle qui ne sont pas dégénérées en $t = T$, $j_0 \in \{1,\ldots,N-1\}\setminus\{i_0\}$ et X est un espace de Banach approprié.

On explique brièvement comment montrer (1.74). Dans une première étape, on suit les idées de (1.44) mais sans utiliser l'équation de ϕ pour obtenir une inégalité comme

$$\iint_Q \rho_1\Big(|\varphi|^2 + |\psi|^2 + |\sigma|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \le C\left\|\rho_2\,(g^\varphi, g^\psi, g^\sigma)\right\|_X^2 + (N-2)C\iint_{\omega\times(0,T)} \rho_3\,|\varphi_{j_0}|^2\,\mathrm{d}x\,\mathrm{d}t + C\iint_{\omega\times(0,T)} \rho_4\,|\varphi_N|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{1.75}$$

Pour obtenir une telle inégalité nous avons appliqué une inégalité de Carleman à l'équation de la chaleur satisfaite par la variable $\mathcal{D}\sigma$ définie dans (1.48). Ceci justifie le terme global de σ à gauche de (1.75). Le terme local en σ est estimé à l'aide de la relation :

$$\begin{aligned}\mathcal{D}\sigma =&(-\Delta\partial_t^2 + \Delta^3)\varphi_N - (\Delta g_N^{\varphi})_t + \Delta^2 g_N^{\varphi} + (\partial_N \nabla\cdot g^{\varphi})_t\\ &- \Delta(\partial_N\nabla\cdot g^{\varphi}) - \Delta g_N^{\psi} + \partial_N\nabla\cdot g^{\psi}.\end{aligned}$$

Cette identité a lieu dans $\omega \cap \mathcal{O}_1$.

Ensuite, on ajoute le terme de ϕ à gauche de (1.75) à travers une inégalité d'énergie pour l'équation satisfaite par $\bar{\phi} := \rho_5(t)\,\phi$, où $\rho_5(t) = \exp(-C/t^{10})$ (qui est croissant et positif). À savoir

$$\begin{aligned}\iint_Q \rho_5^2|\phi|^2\,\mathrm{d}x\,\mathrm{d}t \leq C\iint_Q \rho_5^2|g^{\phi}|^2\,\mathrm{d}x + C\iint_Q \rho_5^2|\varphi_N|^2\,\mathrm{d}x\,\mathrm{d}t\\ + C\iint_{\mathcal{O}_2\times(0,T)} \rho_5^2|\sigma|^2\,\mathrm{d}x\,\mathrm{d}t - \iint_Q \rho_5\rho_5'|\phi|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

En choisissant C pour que $\rho_5^2 \leq \rho_1$ on en déduit (1.74). Notons que pour obtenir cette inégalité d'énergie, le fait que $\phi(T) = 0$ est essentiel.

Ceci fait l'objet d'un travail en cours.

1.3.2 Perspectives et problèmes ouverts

Dans ce paragraphe, on mentionne quelques commentaires et problèmes ouverts qui posent des lignes de recherche futures. Par rapport à la première partie de ce mémoire, on peut mentionner :

– Le Chapitre 2 établit un résultat de contrôlabilité à zéro pour le système de Navier-Stokes via des contrôles ayant une composante nulle. Une première question naturelle qui surgit est si l'on peut obtenir la contrôlabilité aux trajectoires comme dans [34] ou [35], bien entendu pour un domaine de contrôle ω quelconque. L'équation adjointe du système linéarisé autour d'une trajectoire $\bar{y}$ est donnée par

$$-\varphi_t - \Delta\varphi - D\varphi\,\bar{y} + \nabla\pi = g,$$

où $D\varphi := \nabla\varphi + \nabla\varphi^t$. La méthode que l'on utilise pour enlever une des composantes du contrôle a besoin de découpler les équations satisfaites par les différentes composantes de φ. Malheureusement, la présence de $D\varphi\,\bar{y}$ rend difficile d'appliquer notre méthode dans cette situation.
Cette remarque s'étend au système de Boussinesq traité dans le Chapitre 3 pour obtenir un résultat de contrôlabilité aux trajectoires $(\bar{y}, \bar{p}, \bar{\theta})$.

– En ce qui concerne le système de Boussinesq (Chapitre 3), on peut se demander si notre résultat reste vrai si l'on ne contrôle pas dans l'équation de la chaleur. Même dans le cas d'un contrôle sans composante nulle dans l'équation du fluide, ce résultat semble difficile à montrer. La propriété de continuation unique associée pour un problème linéaire ne semble pas claire non plus. Notons que la stratégie présentée dans le Paragraphe 1.3.1 (la partie concernant le système de Boussinesq) ne marche pas ici, car la condition initiale du système adjoint n'est pas nulle en général.

– Par rapport au nombre de composantes nulles des contrôles, récemment la contrôlabilité locale à zéro du système de Navier-Stokes tridimensionnel avec des contrôles ayant deux composantes nulles a été démontrée par J.-M. Coron et P. Lissy dans [24] en utilisant la méthode du retour (voir [16, 17, 19] ou [20, Chapitre 6]). On pourrait penser à se servir de cette méthode pour augmenter la quantité de composantes nulles des contrôles des systèmes traités dans les Chapitres 3 et 4. En particulier, ce serait intéressant de voir si le résultat que nous avons obtenu pour le système de Boussinesq reste vrai si l'on ne contrôle que l'équation de la température ou s'il existe des contrôles insensibilisants pour le système de Navier-Stokes ayant deux composantes nulles.
  Concernant le problème d'insensibilisation pour le système de Boussinesq (Chapitre 4), la méthode de [24] pourrait donner des pistes sur comment démontrer le résultat principal dans le cas tridimensionnel lorsque un seul contrôle dans l'équation de la température est autorisé. Ceci donnerait un résultat de contrôle pour un système de huit équations avec un seul contrôle.

En ce qui concerne la deuxième partie de ce manuscrit :

– Le problème principal qui reste ouvert est la contrôlabilité uniforme de (1.51) par rapport au coefficient de dispersion ε. Comme nous l'avons remarqué dans le Paragraphe 1.2.4, un tel résultat ne peut être obtenu que si $M > 0$. Un pas dans cette direction a été fait dans cette thèse en améliorant la constante d'observabilité (Théorème 1.19), mais il semble difficile de montrer une inégalité de dissipation exponentielle du type (1.61), même si T est arbitrairement grand. La condition à l'extrémité $x = 1$ de l'équation adjointe (1.57) nous fait obtenir de très mauvaises estimations par rapport à ε. Une possibilité pour palier à ce problème serait d'étudier la fonction $\phi := \varepsilon\varphi_{xx} - M\varphi$, qui satisfait la même équation que φ et $\phi_{|x=1} = 0$, mais les difficultés sont transmises à l'extrémité $x = 0$. Néanmoins nous avons pu démontrer que la norme L^2 de ϕ est dissipée indépendamment de ε (voir la Proposition 5.6) mais ceci ne suffit pas pour établir la contrôlabilité uniforme.

– Le résultat d'explosion du coût du contrôle (Théorème 1.21) quand $T < 1/|M|$ n'est pas optimal si $M > 0$. Cette restriction par rapport à T vient de l'absence d'une inégalité comme

$$\|\varphi_{|t=t_1}\|_{L^2(0,1)} \leq C\|\varphi_{|t=t_2}\|_{L^2(0,1)}, \quad 0 \leq t_1 \leq t_2 \leq T, \tag{1.76}$$

  où C est indépendant de ε. Une inégalité comme (1.76) nous permettrait de montrer le Théorème 1.21 pour tout $T < 1/M$ $(M > 0)$.

– L'inégalité de Carleman de la Proposition 1.20 est optimale par rapport à la puissance du poids $(m = 1/2)$ pour une équation de KdV. Le changement de variable $\phi := \varepsilon\varphi_{xx} - M\varphi$ est essentiel dans sa preuve. Il serait intéressant de trouver une méthode pour montrer une inégalité comme (1.59) sans passer par ce changement de variable.

Première partie

Some controllability results with a reduced number of scalar controls for systems of the Navier-Stokes kind

Chapitre 2

Local null controllability of the N-dimensional Navier-Stokes system with $N-1$ scalar controls in an arbitrary control domain

In this chapter we deal with the local null controllability of the N-dimensional Navier-Stokes system with internal controls having one vanishing component. The novelty is that no condition is imposed on the control domain.

This chapter is included in the paper [7] written in collaboration with S. Guerrero.

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2.1 Introduction

Let Ω be a nonempty bounded connected open subset of $\mathbb{R}^N$ ($N = 2$ or 3) of class $\mathcal{C}^\infty$. Let $T > 0$ and let $\omega \subset \Omega$ be a (small) nonempty open subset which is the control domain. We will use the notation $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$.

We will be concerned with the following controlled Navier-Stokes system :

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = v\mathbb{1}_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where v stands for the control which acts over the set ω.

The main objective of this chapter is to obtain the local null controllability of system (2.1) by means of $N-1$ scalar controls, i.e., we will prove the existence of a number $\delta > 0$ such that, for every $y^0 \in X$ (X is an appropriate Banach space) satisfying

$$\|y^0\|_X \leq \delta,$$

and every $i \in \{1, \dots, N\}$, we can find a control v in $L^2(\omega \times (0,T))^N$ with $v_i \equiv 0$ such that the corresponding solution to (2.1) satisfies

$$y(T) = 0 \text{ in } \Omega.$$

This result has been proved in [35] when $\overline{\omega}$ intersects the boundary of Ω. Here, we remove this geometric assumption and prove the null controllability result for any nonempty open set $\omega \subset \Omega$. A similar result was obtained in [23] for the Stokes system.

Let us recall the definition of some usual spaces in the context of incompressible fluids :

$$V = \{y \in H_0^1(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega\}$$

and

$$H = \{y \in L^2(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega,\ y \cdot n = 0 \text{ on } \partial\Omega\}.$$

Our main result is given in the following theorem :

Theorem 2.1. *Let $i \in \{1, \dots, N\}$. Then, for every $T > 0$ and $\omega \subset \Omega$, there exists $\delta > 0$ such that, for every $y^0 \in V$ satisfying*

$$\|y^0\|_V \leq \delta,$$

we can find a control $v \in L^2(\omega \times (0,T))^N$, with $v_i \equiv 0$, and a corresponding solution (y,p) to (2.1) *such that*

$$y(T) = 0,$$

i.e., the nonlinear system (2.1) *is locally null controllable by means of $N-1$ scalar controls for an arbitrary control domain.*

Remark 2.2. *For the sake of simplicity, we have taken the initial condition in a more regular space than usual. However, following the same arguments as in [34] and [35], we can get the same result by considering $y^0 \in H$ for $N = 2$ and $y^0 \in H \cap L^4(\Omega)^3$ for $N = 3$.*

To prove Theorem 2.1, we follow a standard approach (see for instance [34], [35] and [51]). We first deduce a null controllability result for a linear system associated to (2.1) :

$$\begin{cases} y_t - \Delta y + \nabla p = f + v\mathbb{1}_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{2.2}$$

where f will be taken to decrease exponentially to zero in T. We first prove a suitable Carleman estimate for the adjoint system of (2.2) (see (2.6) below). This will provide existence (and uniqueness) to a variational problem, from which we define a solution (y,p,v) to (2.2) such that $y(T) = 0$ in Ω and $v_i = 0$. Moreover, this solution is such that $e^{C/(T-t)}(y,v) \in L^2(Q)^N \times L^2(\omega \times (0,T))^N$ for some $C > 0$.

Finally, by means of an inverse mapping theorem, we deduce the null controllability for the nonlinear system.

This chapter is organized as follows. In section 2.2, we establish all the technical results needed to deal with the controllability problems. In section 2.3, we deal with the null controllability of the linear system (2.2). Finally, in section 2.4 we give the proof of Theorem 2.1.

2.2 Some previous results

In this section we will mainly prove a Carleman estimate for the adjoint system of (2.2). In order to do so, we are going to introduce some weight functions. Let ω_0 be a nonempty open subset of $\mathbb{R}^N$ such that $\overline{\omega_0} \subset \omega$ and $\eta \in \mathcal{C}^2(\overline{\Omega})$ such that

$$|\nabla\eta| > 0 \text{ in } \overline{\Omega} \setminus \omega_0, \quad \eta > 0 \text{ in } \Omega \text{ and } \eta \equiv 0 \text{ on } \partial\Omega. \tag{2.3}$$

The existence of such a function η is given in [39]. Let also $\ell \in \mathcal{C}^\infty([0,T])$ be a positive function in $(0,T)$ satisfying

$$\begin{aligned} &\ell(t) = t \quad \text{ if } t \in [0, T/4], \quad \ell(t) = T - t \quad \text{ if } t \in [3T/4, T], \\ &\ell(t) \leq \ell(T/2), \text{ for all } t \in [0,T]. \end{aligned} \tag{2.4}$$

Then, for all $\lambda \geq 1$ we consider the following weight functions :

$$\begin{aligned} &\alpha(x,t) := \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\ell^8(t)}, && \xi(x,t) := \frac{e^{\lambda\eta(x)}}{\ell^8(t)}, \\ &\alpha^*(t) := \max_{x\in\overline{\Omega}} \alpha(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - 1}{\ell^8(t)}, && \xi^*(t) := \min_{x\in\overline{\Omega}} \xi(x,t) = \frac{1}{\ell^8(t)}, \\ &\widehat{\alpha}(t) := \min_{x\in\overline{\Omega}} \alpha(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\|\eta\|_\infty}}{\ell^8(t)}, && \widehat{\xi}(t) := \max_{x\in\overline{\Omega}} \xi(x,t) = \frac{e^{\lambda\|\eta\|_\infty}}{\ell^8(t)}. \end{aligned} \tag{2.5}$$

These exact weight functions were considered in [52].

We consider now a backwards nonhomogeneous system associated to the Stokes equation :

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi = g & \text{in } Q, \\ \nabla\cdot\varphi = 0 & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T & \text{in } \Omega, \end{cases} \tag{2.6}$$

where $g \in L^2(Q)^N$ and $\varphi^T \in H$. Our Carleman estimate is given in the following proposition.

Proposition 2.3. *There exists a constant λ_0, such that for any $\lambda > \lambda_0$ there exist two constants $C(\lambda) > 0$ and $s_0(\lambda) > 0$ such that for any $i \in \{1, \dots, N\}$, any $g \in L^2(Q)^N$ and any $\varphi^T \in H$, the solution of (2.6) satisfies*

$$\begin{aligned} s^4 \iint_Q e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2 dx\,dt \leq C \Bigg(&\iint_Q e^{-3s\alpha^*}|g|^2 dx\,dt \\ &+ s^7 \sum_{j=1, j\neq i}^{N} \iint_{\omega\times(0,T)} e^{-2s\widehat{\alpha}-3s\alpha^*}(\widehat{\xi})^7|\varphi_j|^2 dx\,dt \Bigg) \end{aligned} \tag{2.7}$$

for every $s \geq s_0$.

The proof of inequality (2.7) is based on the arguments in [23], [34] and a Carleman inequality for parabolic equations with non-homogeneous boundary conditions proved in [52]. In [23], the authors take advantange of the fact that the Laplacian of the pressure is zero, but this is not the case here. Some arrangements of equation (2.6) have to be made in order to follow the same strategy. More details are given below.

Before giving the proof of Proposition 2.3, we present some technical results. We first present a Carleman inequality proved in [52] for parabolic equations with nonhomogeneous boundary conditions. To this end, let us introduce the equation

$$u_t - \Delta u = f_0 + \sum_{j=1}^{N} \partial_j f_j \text{ in } Q, \tag{2.8}$$

where $f_0, f_1, \ldots, f_N \in L^2(Q)$. We have the following result.

Lemma 2.4. *There exists a constant $\widehat{\lambda}_0$ only depending on Ω, ω_0, η and ℓ such that for any $\lambda > \widehat{\lambda}_0$ there exist two constants $C(\lambda) > 0$ and $\widehat{s}(\lambda)$, such that for every $s \geq \widehat{s}$ and every $u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ satisfying (2.8), we have*

$$\begin{aligned}
\frac{1}{s}\iint_Q e^{-2s\alpha}\frac{1}{\xi}|\nabla u|^2 dx\,dt + s\iint_Q e^{-2s\alpha}\xi|u|^2 dx\,dt \\
\leq C\Bigg(s^{-\frac{1}{2}}\|e^{-s\alpha}\xi^{-\frac{1}{4}}u\|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} + s^{-\frac{1}{2}}\|e^{-s\alpha}\xi^{-\frac{1}{8}}u\|^2_{L^2(\Sigma)} \\
+ \frac{1}{s^2}\iint_Q e^{-2s\alpha}\frac{1}{\xi^2}|f_0|^2 dx\,dt + \sum_{j=1}^{N}\iint_Q e^{-2s\alpha}|f_j|^2 dx\,dt \\
+ s\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi|u|^2 dx\,dt\Bigg).
\end{aligned} \tag{2.9}$$

Recall that

$$\|u\|_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} = \left(\|u\|^2_{H^{1/4}(0,T;L^2(\partial\Omega))} + \|u\|^2_{L^2(0,T;H^{1/2}(\partial\Omega))}\right)^{1/2}.$$

The next technical result is a particular case of Lemma 3 in [23].

Lemma 2.5. *There exists $\widehat{\lambda}_1 > 0$ and $C > 0$ depending only on Ω, ω_0, η and ℓ such that, for every $u \in L^2(0,T;H^1(\Omega))$,*

$$\begin{aligned}
s^3\lambda^2\iint_Q e^{-2s\alpha}\xi^3|u|^2 dx\,dt \\
\leq C\left(s\iint_Q e^{-2s\alpha}\xi|\nabla u|^2 dx\,dt + s^3\lambda^2\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|u|^2 dx\,dt\right),
\end{aligned} \tag{2.10}$$

for every $\lambda \geq \widehat{\lambda}_1$ and every $s \geq C$.

Remark 2.6. *In [23], slightly different weight functions are used to prove Lemma 2.5. Indeed, the authors take $\ell(t) = t(T-t)$. However, this does not change the result since the important property is that ℓ goes to 0 algebraically when t tends to 0 and T.*

The next lemma can be readily deduced from the corresponding result for parabolic equations in [39].

Lemma 2.7. *Let $\zeta(x) = \exp(\lambda\eta(x))$ for $x \in \Omega$. Then, there exist $\widehat{\lambda}_2 > 0$ and $C > 0$ depending only on Ω, ω_0 and η such that, for every $u \in H^1_0(\Omega) \cap H^2(\Omega)$,*

$$\begin{aligned}
\tau^6\lambda^8\int_\Omega e^{2\tau\zeta}\zeta^6|u|^2 dx + \tau^4\lambda^6\int_\Omega e^{2\tau\zeta}\zeta^4|\nabla u|^2 dx \\
\leq C\left(\tau^3\lambda^4\int_\Omega e^{2\tau\zeta}\zeta^3|\Delta u|^2 dx + \tau^6\lambda^8\int_{\omega_0} e^{2\tau\zeta}\zeta^6|u|^2 dx\right),
\end{aligned} \tag{2.11}$$

for every $\lambda \geq \widehat{\lambda}_2$ and every $\tau \geq C$.

The final technical result concerns the regularity of the solutions to the Stokes system that can be found in [59] (see also [76]).

Lemma 2.8. *For every $T > 0$ and every $F \in L^2(Q)^N$, there exists a unique solution $u \in L^2(0,T;H^2(\Omega)^N) \cap H^1(0,T;H) \cap L^\infty(0,T;V)$ to the Stokes system*

$$\begin{cases} u_t - \Delta u + \nabla p = F & \text{in } Q, \\ \nabla \cdot u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = 0 & \text{in } \Omega, \end{cases}$$

for some $p \in L^2(0,T;H^1(\Omega))$, and there exists a constant $C > 0$ depending only on Ω such that

$$\|u\|^2_{L^2(0,T;H^2(\Omega)^N)} + \|u\|^2_{H^1(0,T;L^2(\Omega)^N)} + \|u\|^2_{L^\infty(0,T;V)} \leq C\|F\|^2_{L^2(Q)^N}. \tag{2.12}$$

Furthermore, if $F \in L^2(0,T;H^2(\Omega)^N) \cap H^1(0,T;L^2(\Omega)^N)$ and satisfies the following compatibility condition :

$$\nabla p_F = F(0) \text{ on } \partial\Omega,$$

where p_F is any solution of the Neumann boundary-value problem

$$\begin{cases} \Delta p_F = \nabla \cdot F(0) & \text{in } \Omega, \\ \dfrac{\partial p_F}{\partial n} = F(0) \cdot n & \text{on } \partial\Omega. \end{cases}$$

Then $u \in L^2(0,T;H^4(\Omega)^N) \cap H^1(0,T;H^2(\Omega)^N)$ and there exists a constant $C > 0$ depending only on Ω such that

$$\|u\|^2_{L^2(0,T;H^4(\Omega)^N)} + \|u\|^2_{H^1(0,T;H^2(\Omega)^N)} \leq C\Big(\|F\|^2_{L^2(0,T;H^2(\Omega)^N)} + \|F\|^2_{H^1(0,T;L^2(\Omega)^N)}\Big). \tag{2.13}$$

2.2.1 Proof of Proposition 2.3

Without any lack of generality, we treat the case of $N = 2$ and $i = 2$. The arguments can be easily extended to the general case. We follow the ideas of [23]. In that paper, the arguments are based on the fact that $\Delta\pi = 0$, which is not the case here (recall that π appears in (2.6)). For this reason, let us first introduce (w, q) and (z, r), the solutions of the following systems :

$$\begin{cases} -w_t - \Delta w + \nabla q = \rho g & \text{in } Q, \\ \nabla \cdot w = 0 & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(T) = 0 & \text{in } \Omega, \end{cases} \tag{2.14}$$

and

$$\begin{cases} -z_t - \Delta z + \nabla r = -\rho' \varphi & \text{in } Q, \\ \nabla \cdot z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T) = 0 & \text{in } \Omega, \end{cases} \tag{2.15}$$

where $\rho(t) = e^{-\frac{3}{2}s\alpha^*}$. Adding (2.14) and (2.15), we see that $(w + z, q + r)$ solves the same system as $(\rho\varphi, \rho\pi)$, where (φ, π) is the solution to (2.6). By uniqueness of the Stokes system we have

$$\rho\varphi = w + z \text{ and } \rho\pi = q + r. \tag{2.16}$$

For system (2.14) we will use the regularity estimate (2.12), namely

$$\|w\|^2_{L^2(0,T;H^2(\Omega)^2)} + \|w\|^2_{H^1(0,T;L^2(\Omega)^2)} \leq C\|\rho g\|^2_{L^2(Q)^2}, \tag{2.17}$$

and for system (2.15) we will use the ideas of [23]. Using the divergence free condition on the equation of (2.15), we see that

$$\Delta r = 0 \text{ in } Q.$$

Then, we apply the operator $\nabla\Delta = (\partial_1\Delta, \partial_2\Delta)$ to the equation satisfied by z_1 and we denote $\psi := \nabla\Delta z_1$. We then have

$$-\psi_t - \Delta\psi = -\nabla(\Delta(\rho'\varphi_1)) \text{ in } Q.$$

We apply Lemma 2.4 to this equation and we obtain

$$\begin{aligned} I(s;\psi) := \frac{1}{s}\iint_Q e^{-2s\alpha}\frac{1}{\xi}|\nabla\psi|^2 \mathrm{d}x\,\mathrm{d}t + s\iint_Q e^{-2s\alpha}\xi|\psi|^2\mathrm{d}x\,\mathrm{d}t \\ \leq C\left(s^{-\frac{1}{2}}\|e^{-s\alpha}\xi^{-\frac{1}{4}}\psi\|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)^2} + s^{-\frac{1}{2}}\|e^{-s\alpha}\xi^{-\frac{1}{8}}\psi\|^2_{L^2(\Sigma)^2}\right. \\ \left. + \iint_Q e^{-2s\alpha}|\rho'|^2|\Delta\varphi_1|^2\mathrm{d}x\,\mathrm{d}t + s\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi|\psi|^2\mathrm{d}x\,\mathrm{d}t\right), \end{aligned} \tag{2.18}$$

for every $\lambda \geq \widehat{\lambda}_0$ and $s \geq \widehat{s}$.

We divide the rest of the proof in several steps :

- In Step 1, using Lemmas 2.5 and 2.7, we estimate global integrals of z_1 and z_2 by the left-hand side of (2.18).
- In Step 2, we deal with the boundary terms in (2.18).
- In Step 3, we estimate all the local terms by a local term of φ_1 and $\varepsilon\, I(s;\varphi)$ to conclude the proof.

Now, let us choose $\lambda_0 = \max\{\widehat{\lambda}_0, \widehat{\lambda}_1, \widehat{\lambda}_2\}$ so that Lemmas 2.5 and 2.7 can be applied and fix $\lambda \geq \lambda_0$. In the following, C will denote a generic constant depending on Ω, ω and λ.

Step 1. Estimate of z_1. We use Lemma 2.5 with $u = \Delta z_1$:

$$\begin{aligned} s^3\iint_Q e^{-2s\alpha}\xi^3|\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t \\ \leq C\left(s\iint_Q e^{-2s\alpha}\xi|\psi|^2\mathrm{d}x\,\mathrm{d}t + s^3\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t\right), \end{aligned} \tag{2.19}$$

for every $s \geq C$.

Now, we apply Lemma 2.7 with $u = z_1 \in H^1_0(\Omega)\cap H^2(\Omega)$ and we get, for almost everywhere $t \in (0,T)$:

$$\begin{aligned} \tau^6\int_\Omega e^{2\tau\zeta}\zeta^6|z_1|^2dx + \tau^4\int_\Omega e^{2\tau\zeta}\zeta^4|\nabla z_1|^2dx \\ \leq C\left(\tau^3\int_\Omega e^{2\tau\zeta}\zeta^3|\Delta z_1|^2dx + \tau^6\int_{\omega_0} e^{2\tau\zeta}\zeta^6|z_1|^2dx\right), \end{aligned}$$

for every $\tau \geq C$. Now we take

$$\tau = \frac{s}{\ell^8(t)}$$

for s large enough so we have $\tau \geq C$. This yields to, for almost everywhere $t \in (0,T)$,

$$s^6 \int_\Omega e^{2s\xi}\xi^6|z_1|^2dx + s^4 \int_\Omega e^{2s\xi}\xi^4|\nabla z_1|^2dx \leq C\left(s^3\int_\Omega e^{2s\xi}\xi^3|\Delta z_1|^2dx + s^6\int_{\omega_0} e^{2s\xi}\xi^6|z_1|^2dx\right),$$

for every $s \geq C$. We multiply this inequality by

$$\exp\left(-2s\frac{e^{2\lambda\|\eta\|_\infty}}{\ell^8(t)}\right),$$

and we integrate in $(0,T)$ to obtain

$$s^6 \iint_Q e^{-2s\alpha}\xi^6|z_1|^2 \mathrm{d}x\,\mathrm{d}t + s^4 \iint_Q e^{-2s\alpha}\xi^4|\nabla z_1|^2\mathrm{d}x\,\mathrm{d}t \leq C\left(s^3\iint_Q e^{-2s\alpha}\xi^3|\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t + s^6\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^6|z_1|^2\mathrm{d}x\,\mathrm{d}t\right),$$

for every $s \geq C$. Combining this with (2.19) we get the following estimate for z_1 :

$$\begin{aligned} s^6 \iint_Q e^{-2s\alpha}\xi^6|z_1|^2dxdt + s^4 \iint_Q e^{-2s\alpha}\xi^4|\nabla z_1|^2dxdt \\ + s^3\iint_Q e^{-2s\alpha}\xi^3|\Delta z_1|^2dxdt \leq C\left(s\iint_Q e^{-2s\alpha}\xi|\psi|^2\mathrm{d}x\,\mathrm{d}t\right. \\ \left.+s^3\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t + s^6\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^6|z_1|^2\mathrm{d}x\,\mathrm{d}t\right), \end{aligned} \tag{2.20}$$

for every $s \geq C$.

Estimate of z_2. Now we will estimate a term in z_2 by the left-hand side of (2.20). From the divergence free condition on z we find

$$\begin{aligned} s^4 \iint_Q e^{-2s\alpha^*}(\xi^*)^4|\partial_2 z_2|^2\mathrm{d}x\,\mathrm{d}t &= s^4 \iint_Q e^{-2s\alpha^*}(\xi^*)^4|\partial_1 z_1|^2\mathrm{d}x\,\mathrm{d}t \\ &\leq s^4 \iint_Q e^{-2s\alpha}\xi^4|\nabla z_1|^2\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.21}$$

Since $z_2|_{\partial\Omega} = 0$ and Ω is bounded, we have that

$$\int_\Omega |z_2|^2dx \leq C(\Omega)\int_\Omega |\partial_2 z_2|dx,$$

and because α^* and ξ^* do not depend on x, we also have

$$s^4 \iint_Q e^{-2s\alpha^*}(\xi^*)^4|z_2|^2\mathrm{d}x\,\mathrm{d}t \leq C(\Omega)s^4 \iint_Q e^{-2s\alpha^*}(\xi^*)^4|\partial_2 z_2|^2\mathrm{d}x\,\mathrm{d}t.$$

Combining this with (2.21) we obtain

$$s^4 \iint_Q e^{-2s\alpha^*}(\xi^*)^4|z_2|^2\mathrm{d}x\,\mathrm{d}t \leq Cs^4 \iint_Q e^{-2s\alpha}\xi^4|\nabla z_1|^2\mathrm{d}x\,\mathrm{d}t. \tag{2.22}$$

Now, observe that by (2.16), (2.17) and the fact that $s^2 e^{-2s\alpha}(\xi^*)^{9/4}$ is bounded we can estimate the third term in the right-hand side of (2.18). Indeed,

$$\begin{aligned}\iint_Q e^{-2s\alpha}|\rho'|^2|\Delta\varphi_1|^2 \mathrm{d}x\,\mathrm{d}t &= \iint_Q e^{-2s\alpha}|\rho'|^2|\rho|^{-2}|\Delta(\rho\varphi_1)|^2\mathrm{d}x\,\mathrm{d}t \\ &\leq C\left(s^2\iint_Q e^{-2s\alpha}(\xi^*)^{9/4}|\Delta w_1|^2\mathrm{d}x\,\mathrm{d}t + s^2\iint_Q e^{-2s\alpha}(\xi^*)^{9/4}|\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t\right) \\ &\leq C\left(\|\rho g\|^2_{L^2(Q)^2} + s^2\iint_Q e^{-2s\alpha}(\xi^*)^3|\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t\right).\end{aligned}$$

Putting together (2.18), (2.20), (2.22) and this last inequality we have for the moment

$$\begin{aligned}& s^6\iint_Q e^{-2s\alpha}\xi^6|z_1|^2dxdt + s^4\iint_Q e^{-2s\alpha^*}(\xi^*)^4|z_2|^2dxdt \\ &\quad + s^3\iint_Q e^{-2s\alpha}\xi^3|\Delta z_1|^2dxdt + s\iint_Q e^{-2s\alpha}\xi|\psi|^2dxdt + \frac{1}{s}\iint_Q e^{-2s\alpha}\frac{1}{\xi}|\nabla\psi|^2dxdt \\ &\quad\leq C\Big(s^{-\frac12}\|e^{-s\alpha}\xi^{-\frac14}\psi\|^2_{H^{\frac14,\frac12}(\Sigma)^2} + s^{-\frac12}\|e^{-s\alpha}\xi^{-\frac18}\psi\|^2_{L^2(\Sigma)^2} + \|\rho g\|^2_{L^2(Q)^2} \\ &\qquad + s\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi|\psi|^2\mathrm{d}x\,\mathrm{d}t + s^3\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t \\ &\qquad + s^6\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^6|z_1|^2\mathrm{d}x\,\mathrm{d}t\Big),\end{aligned} \tag{2.23}$$

for every $s \geq C$.

Step 2. In this step we deal with the boundary terms in (2.23).

First, we treat the second boundary term in (2.23). We consider a function $\theta(x) \in \mathcal{C}^2(\bar{\Omega})$ such that $\nabla\theta\cdot n > c > 0$ on $\partial\Omega$, where $n(x)$ is the outward unit normal to Ω at $x \in \partial\Omega$. Performing integration by parts, we have

$$\iint_\Sigma e^{-2s\alpha^*}|\psi|^2(\nabla\theta\cdot n)\,\mathrm{d}x = \iint_Q e^{-2s\alpha^*}|\psi|^2\Delta\theta\,\mathrm{d}x + 2\iint_Q e^{-2s\alpha^*}\psi\nabla\psi\cdot\nabla\theta\,\mathrm{d}x.$$

Therefore, from Cauchy-Schwarz's inequality, we find

$$\begin{aligned}\|e^{-s\alpha^*}\psi\|^2_{L^2(\Sigma)^2} \leq C\Big(&\|e^{-s\alpha^*}\psi\|^2_{L^2(Q)^2} \\ &+ \|s^{\frac12}e^{-s\alpha^*}(\xi^*)^{\frac12}\psi\|_{L^2(Q)^2}\|s^{-\frac12}e^{-s\alpha^*}(\xi^*)^{-\frac12}\nabla\psi\|_{L^2(Q)^4}\Big),\end{aligned} \tag{2.24}$$

and using the properties of the weight functions (see (2.5)) we have

$$\|e^{-s\alpha^*}\psi\|^2_{L^2(\Sigma)^2} \leq C\left(s\iint_Q e^{-2s\alpha}\xi|\psi|^2\mathrm{d}x\,\mathrm{d}t + \frac{1}{s}\iint_Q e^{-2s\alpha}\frac{1}{\xi}|\nabla\psi|^2\mathrm{d}x\,\mathrm{d}t\right).$$

Thus, $\|e^{-s\alpha^*}\psi\|^2_{L^2(\Sigma)^2}$ is bounded by the left-hand side of (2.23). On the other hand,

$$s^{-\frac12}\|e^{-s\alpha}\xi^{-\frac18}\psi\|^2_{L^2(\Sigma)^2} \leq Cs^{-\frac12}\|e^{-s\alpha}\psi\|^2_{L^2(\Sigma)^2},$$

and we can absorb $s^{-\frac12}\|e^{-s\alpha}\psi\|^2_{L^2(\Sigma)^2}$ by taking s large enough.

Now we treat the first boundary term in the right-hand side of (2.23). We will use regularity estimates to prove that z_1 multiplied by a certain weight function is regular enough. First, let us observe that from (2.16) we readily have

$$s^4 \iint_Q e^{-2s\alpha^*}(\xi^*)^4|\rho|^2|\varphi|^2 \mathrm{d}x\,\mathrm{d}t \\ \leq 2s^4 \iint_Q e^{-2s\alpha^*}(\xi^*)^4|w|^2 \mathrm{d}x\,\mathrm{d}t + 2s^4 \iint_Q e^{-2s\alpha^*}(\xi^*)^4|z|^2 \mathrm{d}x\,\mathrm{d}t.$$

Using the regularity estimate (2.17) for w we have

$$s^4 \iint_Q e^{-2s\alpha^*}(\xi^*)^4|\rho|^2|\varphi|^2 \mathrm{d}x\,\mathrm{d}t \leq C\left(\|\rho g\|^2_{L^2(Q)^2} + s^4 \iint_Q e^{-2s\alpha^*}(\xi^*)^4|z|^2 \mathrm{d}x\,\mathrm{d}t \right), \quad (2.25)$$

thus the term $\|s^2 e^{-s\alpha^*}(\xi^*)^2\rho\varphi\|^2_{L^2(Q)^2}$ is bounded by the left-hand side of (2.23) and $\|\rho g\|^2_{L^2(Q)^2}$.

We define now

$$\widetilde{z} := s e^{-s\alpha^*}(\xi^*)^{7/8} z, \quad \widetilde{r} := s e^{-s\alpha^*}(\xi^*)^{7/8} r.$$

From (2.15) we see that $(\widetilde{z}, \widetilde{r})$ is the solution of the Stokes system :

$$\begin{cases} -\widetilde{z}_t - \Delta\widetilde{z} + \nabla\widetilde{r} = -s e^{-s\alpha^*}(\xi^*)^{7/8}\rho'\varphi - (s e^{-s\alpha^*}(\xi^*)^{7/8})_t z & \text{in } Q, \\ \nabla\cdot\widetilde{z} = 0 & \text{in } Q, \\ \widetilde{z} = 0 & \text{on } \Sigma, \\ \widetilde{z}(T) = 0 & \text{in } \Omega. \end{cases}$$

Taking into account that

$$|\alpha^*_t| \leq C(\xi^*)^{9/8},\ |\rho'| \leq C s\rho(\xi^*)^{9/8}$$

and the regularity estimate (2.12) we have

$$\|\widetilde{z}\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)} \leq C\left(\|s^2 e^{-s\alpha^*}(\xi^*)^2\rho\varphi\|^2_{L^2(Q)^2} + \|s^2 e^{-s\alpha^*}(\xi^*)^2 z\|^2_{L^2(Q)^2} \right),$$

thus, from (2.25), $\|s e^{-s\alpha^*}(\xi^*)^{7/8} z\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)}$ is bounded by the left-hand side of (2.23) and $\|\rho g\|^2_{L^2(Q)^2}$. From (2.16), (2.17) and this last inequality we have that

$$\|s e^{-s\alpha^*}(\xi^*)^{7/8}\rho\varphi\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)} \\ \leq C\left(\|\rho g\|^2_{L^2(Q)^2} + \|\widetilde{z}\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)} \right),$$

and thus $\|s e^{-s\alpha^*}(\xi^*)^{7/8}\rho\varphi\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)}$ is bounded by the left-hand side of (2.23) and $\|\rho g\|^2_{L^2(Q)^2}$.

Next, let

$$\widehat{z} := e^{-s\alpha^*}(\xi^*)^{-1/4} z, \quad \widehat{r} := e^{-s\alpha^*}(\xi^*)^{-1/4} r.$$

From (2.15), $(\widehat{z}, \widehat{r})$ is the solution of the Stokes system :

$$\begin{cases} -\widehat{z}_t - \Delta\widehat{z} + \nabla\widehat{r} = -e^{-s\alpha^*}(\xi^*)^{-1/4}\rho'\varphi - (e^{-s\alpha^*}(\xi^*)^{-1/4})_t z & \text{in } Q, \\ \nabla\cdot\widehat{z} = 0 & \text{in } Q, \\ \widehat{z} = 0 & \text{on } \Sigma, \\ \widehat{z}(T) = 0 & \text{in } \Omega. \end{cases}$$

From the previous estimates, it is not difficult to see that the right-hand side of this system is in $L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)$ and, from the fact that it vanishes at $t = T$, it is clear that it satisfies the compatibility condition of Lemma 2.8. Thus, using the regularity estimate (2.13), we have

$$\|\widehat{z}\|^2_{L^2(0,T;H^4(\Omega)^2)\cap H^1(0,T;H^2(\Omega)^2)} \leq C\left(\|se^{-s\alpha^*}(\xi^*)^{7/8}\rho\varphi\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)}\right.$$
$$\left.+\|se^{-s\alpha^*}(\xi^*)^{7/8}z\|^2_{L^2(0,T;H^2(\Omega)^2)\cap H^1(0,T;L^2(\Omega)^2)}\right).$$

In particular, $e^{-s\alpha^*}(\xi^*)^{-1/4}\psi \in L^2(0,T;H^1(\Omega)^2)\cap H^1(0,T;H^{-1}(\Omega)^2)$ (recall that $\psi = \nabla\Delta z_1$) and

$$\|e^{-s\alpha^*}(\xi^*)^{-1/4}\psi\|^2_{L^2(0,T;H^1(\Omega)^2)} \text{ and } \|e^{-s\alpha^*}(\xi^*)^{-1/4}\psi\|^2_{H^1(0,T;H^{-1}(\Omega)^2)} \tag{2.26}$$

are bounded by the left-hand side of (2.23) and $\|\rho g\|^2_{L^2(Q)^2}$.

To end this step, we use the following trace inequality (see [66], for instance)

$$\begin{aligned}s^{-1/2}\|e^{-s\alpha}\xi^{-\frac14}\psi\|^2_{H^{\frac14,\frac12}(\Sigma)^2} &= s^{-1/2}\|e^{-s\alpha^*}(\xi^*)^{-\frac14}\psi\|^2_{H^{\frac14,\frac12}(\Sigma)^2}\\ &\leq C\,s^{-1/2}\left(\|e^{-s\alpha^*}(\xi^*)^{-1/4}\psi\|^2_{L^2(0,T;H^1(\Omega)^2)}\right.\\ &\qquad\left.+\|e^{-s\alpha^*}(\xi^*)^{-1/4}\psi\|^2_{H^1(0,T;H^{-1}(\Omega)^2)}\right).\end{aligned}$$

By taking s large enough in (2.23), the boundary term

$$s^{-1/2}\|e^{-s\alpha}\xi^{-\frac14}\psi\|^2_{H^{\frac14,\frac12}(\Sigma)^2}$$

can be absorbed by the terms in (2.26) and step 2 is finished.

Thus, at this point we have

$$\begin{aligned}s^4\iint_Q e^{-2s\alpha^*}(\xi^*)^4|\rho|^2|\varphi|^2\mathrm{d}x\,\mathrm{d}t + s^3\iint_Q e^{-2s\alpha}\xi^3|\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t\\ + s\iint_Q e^{-2s\alpha}\xi|\nabla\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t + \frac{1}{s}\iint_Q e^{-2s\alpha}\frac{1}{\xi}|\Delta^2 z_1|^2\mathrm{d}x\,\mathrm{d}t\\ \leq C\left(\|\rho g\|^2_{L^2(Q)^2} + s^6\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^6|z_1|^2\mathrm{d}x\,\mathrm{d}t\right.\\ \left.+s\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi|\nabla\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t + s^3\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t\right),\end{aligned} \tag{2.27}$$

for every $s \geq C$.

Step 3. In this step we estimate the two last local terms in the right-hand side of (2.27) in terms of local terms of z_1 and the left-hand side of (2.27) multiplied by small constants. Finally, we make the final arrangements to obtain (2.7).

We start with the term $\nabla\Delta z_1$ and we follow a standard approach. Let ω_1 be an open subset such that $\omega_0 \Subset \omega_1 \Subset \omega$ and let $\rho_1 \in \mathcal{C}^2_c(\omega_1)$ with $\rho_1 \equiv 1$ in ω_0 and $\rho_1 \geq 0$. Then, by integrating by parts we get

$$\begin{aligned}s\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi|\nabla\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t &\leq s\iint_{\omega_1\times(0,T)}\rho_1 e^{-2s\alpha}\xi|\nabla\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t\\ = -s\iint_{\omega_1\times(0,T)}\rho_1 e^{-2s\alpha}\xi\Delta^2 z_1\Delta z_1\mathrm{d}x\,\mathrm{d}t &+ \frac{s}{2}\iint_{\omega_1\times(0,T)}\Delta(\rho_1 e^{-2s\alpha}\xi)|\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Using Cauchy-Schwarz's inequality for the first term and

$$|\Delta(\rho_1 e^{-2s\alpha}\xi)| \leq Cs^2 e^{-2s\alpha}\xi^3,\ s \geq C$$

for the second one, we obtain for every $\varepsilon_1 > 0$

$$s \iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi|\nabla\Delta z_1|^2 \mathrm{d}x\,\mathrm{d}t$$
$$\leq \frac{\varepsilon_1}{s}\iint_{\omega_1\times(0,T)} e^{-2s\alpha}\frac{1}{\xi}|\Delta^2 z_1|^2 \mathrm{d}x\,\mathrm{d}t + C(\varepsilon_1)s^3\iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t,$$

for every $s \geq C$. Choosing ε_1 small enough, we can absorb the first term in the right-hand side of this inequality by the left-hand side of (2.27).

Let us now estimate the terms concerning Δz_1. Let $\rho_2 \in \mathcal{C}_c^2(\omega)$ with $\rho_2 \equiv 1$ in ω_1 and $\rho_2 \geq 0$. Then, by integrating by parts we get

$$s^3\iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t \leq s^3\iint_{\omega\times(0,T)} \rho_2 e^{-2s\alpha}\xi^3|\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t$$
$$= 2s^3\iint_{\omega\times(0,T)} \nabla(\rho_2 e^{-2s\alpha}\xi^3)\nabla\Delta z_1\cdot z_1\mathrm{d}x\,\mathrm{d}t + s^3\iint_{\omega\times(0,T)} \Delta(\rho_2 e^{-2s\alpha}\xi^3)\Delta z_1\cdot z_1\mathrm{d}x\,\mathrm{d}t$$
$$+ s^3\iint_{\omega\times(0,T)} \rho_2 e^{-2s\alpha}\xi^3\Delta^2 z_1\cdot z_1\mathrm{d}x\,\mathrm{d}t.$$

Using

$$|\nabla(\rho_2 e^{-2s\alpha}\xi^3)| \leq Cse^{-2s\alpha}\xi^4,\ s\geq C,$$

for the first term in the right-hand side of this last inequality,

$$|\Delta(\rho_2 s^3 e^{-2s\alpha}\xi^3)| \leq Cs^5 e^{-2s\alpha}\xi^5,\ s\geq C,$$

for the second one and Cauchy-Schwarz's inequality we obtain for every $\varepsilon_2 > 0$

$$s^3\iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^3|\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t$$
$$\leq \varepsilon_2\left(\frac{1}{s}\iint_{\omega\times(0,T)} e^{-2s\alpha}\frac{1}{\xi}|\Delta^2 z_1|^2\mathrm{d}x\,\mathrm{d}t + s\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi|\nabla\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t\right.$$
$$\left. +s^3\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^3|\Delta z_1|^2\mathrm{d}x\,\mathrm{d}t\right) + C(\varepsilon_2)s^7\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^7|z_1|^2\mathrm{d}x\,\mathrm{d}t,$$

for every $s \geq C$. We choose ε_2 small enough in order to absorb the first three integrals by the left-hand side of (2.27).

Finally, from (2.16) and (2.17) we readily obtain

$$s^7\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^7|z_1|^2\mathrm{d}x\,\mathrm{d}t$$
$$\leq 2s^7\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^7|\rho|^2|\varphi_1|^2\mathrm{d}x\,\mathrm{d}t + 2s^7\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^7|w_1|^2\mathrm{d}x\,\mathrm{d}t$$
$$\leq 2s^7\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^7|\rho|^2|\varphi_1|^2\mathrm{d}x\,\mathrm{d}t + C\|\rho g\|^2_{L^2(Q)^2}.$$

This concludes the proof of Proposition 2.3.

2.3 Null controllability of the linear system

Here we are concerned with the null controllability of the system

$$\begin{cases} y_t - \Delta y + \nabla p = f + v\mathbb{1}_\omega & \text{in } Q, \\ \nabla\cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y^0 & \text{in } \Omega, \end{cases} \tag{2.28}$$

where $y^0 \in V$, f is in an appropriate weighted space and the control $v \in L^2(\omega \times (0,T))^N$ is such that $v_i = 0$ for some $i \in \{1,\dots,N\}$.

Before dealing with the null controllability of (2.28), we will deduce a new Carleman inequality with weights not vanishing at $t=0$. To this end, let us introduce the following weight functions :

$$\begin{aligned}
&\beta(x,t) := \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\widetilde{\ell}^8(t)}, && \gamma(x,t) := \frac{e^{\lambda\eta(x)}}{\widetilde{\ell}^8(t)},\\
&\beta^*(t) := \max_{x\in\overline{\Omega}} \beta(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - 1}{\widetilde{\ell}^8(t)}, && \gamma^*(t) := \min_{x\in\overline{\Omega}} \gamma(x,t) = \frac{1}{\widetilde{\ell}^8(t)},\\
&\widehat{\beta}(t) := \min_{x\in\overline{\Omega}} \beta(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\|\eta\|_\infty}}{\widetilde{\ell}^8(t)}, && \widehat{\gamma}(t) := \max_{x\in\overline{\Omega}} \gamma(x,t) = \frac{e^{\lambda\|\eta\|_\infty}}{\widetilde{\ell}^8(t)},
\end{aligned} \tag{2.29}$$

where

$$\widetilde{\ell}(t) = \begin{cases} \|\ell\|_\infty & 0 \le t \le T/2,\\ \ell(t) & T/2 < t \le T. \end{cases}$$

Lemma 2.9. *Let $i \in \{1,\dots,N\}$ and let s and λ be like in Proposition 2.3. Then, there exists a constant $C > 0$ (depending on s and λ) such that every solution φ of* (2.6) *satisfies :*

$$\begin{aligned}
&\iint_Q e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2\,dx\,dt + \|\varphi(0)\|^2_{L^2(\Omega)^N}\\
&\qquad \le C\left(\iint_Q e^{-3s\beta^*}|g|^2\,dx\,dt + \sum_{j=1,j\neq i}^{N}\iint_{\omega\times(0,T)} e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7|\varphi_j|^2\,dx\,dt\right).
\end{aligned} \tag{2.30}$$

Proof. We start by an a priori estimate for the Stokes system (2.6). To do this, we introduce a function $\nu \in \mathcal{C}^1([0,T])$ such that

$$\nu \equiv 1 \text{ in } [0,T/2], \quad \nu \equiv 0 \text{ in } [3T/4,T].$$

We easily see that $(\nu\varphi, \nu\pi)$ satisfies

$$\begin{cases}
-(\nu\varphi)_t - \Delta(\nu\varphi) + \nabla(\nu\pi) = \nu g - \nu'\varphi & \text{in } Q,\\
\nabla\cdot(\nu\varphi) = 0 & \text{in } Q,\\
(\nu\varphi) = 0 & \text{on } \Sigma,\\
(\nu\varphi)(T) = 0 & \text{in } \Omega,
\end{cases}$$

thus we have the energy estimate

$$\|\nu\varphi\|^2_{L^2(0,T;H^1(\Omega)^N)} + \|\nu\varphi\|^2_{L^\infty(0,T;L^2(\Omega)^N)} \le C\Big(\|\nu g\|^2_{L^2(Q)^N} + \|\nu'\varphi\|^2_{L^2(Q)^N}\Big),$$

from which we readily obtain

$$\|\varphi\|^2_{L^2(0,T/2;L^2(\Omega)^N)} + \|\varphi(0)\|^2_{L^2(\Omega)^N} \le C\Big(\|g\|^2_{L^2(0,3T/4;L^2(\Omega)^N)} + \|\varphi\|^2_{L^2(T/2,3T/4;L^2(\Omega)^N)}\Big).$$

From this last inequality, and the fact that

$$e^{-3s\beta^*} \ge C > 0,\ \forall t \in [0,3T/4] \quad \text{and} \quad e^{-5s\beta^*}(\gamma^*)^4 \ge C > 0,\ \forall t \in [T/2,3T/4]$$

we have

$$\iint_{\Omega\times(0,T/2)} e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2\mathrm{d}x\,\mathrm{d}t + \|\varphi(0)\|^2_{L^2(\Omega)^N}$$
$$\leq C\left(\iint_{\Omega\times(0,3T/4)} e^{-3s\beta^*}|g|^2\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega\times(T/2,3T/4)} e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2\mathrm{d}x\,\mathrm{d}t\right). \quad (2.31)$$

Note that, since $(\alpha,\xi) = (\beta,\gamma)$ in $\Omega\times(T/2,T)$, we have :

$$\iint_{\Omega\times(T/2,T)} e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2\mathrm{d}x\,\mathrm{d}t = \iint_{\Omega\times(T/2,T)} e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2\mathrm{d}x\,\mathrm{d}t$$
$$\leq C\iint_Q e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2\mathrm{d}x\,\mathrm{d}t,$$

and by the Carleman inequality of Proposition 2.3

$$\iint_{\Omega\times(T/2,T)} e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2\mathrm{d}x\,\mathrm{d}t$$
$$\leq C\left(\iint_Q e^{-3s\alpha^*}|g|^2\mathrm{d}x\,\mathrm{d}t + \sum_{j=1,j\neq i}^{N}\iint_{\omega\times(0,T)} e^{-2s\widehat{\alpha}-3s\alpha^*}(\widehat{\xi})^7|\varphi_j|^2\mathrm{d}x\,\mathrm{d}t\right).$$

Now, it is easy to see that

$$e^{-3s\alpha^*}\leq e^{-3s\beta^*} \quad\text{and}\quad e^{-2s\widehat{\alpha}-3s\alpha^*}(\widehat{\xi})^7 \leq C_1 \leq C_2 e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7 \quad \forall t\in[0,T/2],$$

for some $C_1, C_2 > 0$, and

$$e^{-3s\alpha^*} = e^{-3s\beta^*} \quad\text{and}\quad e^{-2s\widehat{\alpha}-3s\alpha^*}(\widehat{\xi})^7 = e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7 \quad \forall t\in[T/2,T].$$

From this and the previous inequality, we obtain

$$\iint_{\Omega\times(T/2,T)} e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2\mathrm{d}x\,\mathrm{d}t$$
$$\leq C\left(\iint_Q e^{-3s\beta^*}|g|^2\mathrm{d}x\,\mathrm{d}t + \sum_{j=1,j\neq i}^{N}\iint_{\omega\times(0,T)} e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7|\varphi_j|^2\mathrm{d}x\,\mathrm{d}t\right),$$

which, together with (2.31), yields (2.30). □

Now we will prove the null controllability of (2.28). Actually, we will prove the existence of a solution for this problem in an appropriate weighted space.

Let us set

$$\mathcal{L}y := y_t - \Delta y$$

and let us introduce the space, for $N = 2$ or 3 and $i\in\{1,\dots,N\}$,

$$\begin{aligned} E^i_N := \{\,(y,p,v) :\ & e^{3/2s\beta^*}y\in L^2(Q)^N,\quad e^{s\widehat{\beta}+3/2s\beta^*}\widehat{\gamma}^{-7/2}v\mathbb{1}_\omega\in L^2(Q)^N,\quad v_i\equiv 0,\\ & e^{3/2s\beta^*}(\gamma^*)^{-9/8}y\in L^2(0,T;H^2(\Omega)^N)\cap L^\infty(0,T;V),\\ & e^{5/2s\beta^*}(\gamma^*)^{-2}(\mathcal{L}y+\nabla p - v\mathbb{1}_\omega)\in L^2(Q)^N\,\}. \end{aligned}$$

It is clear that E^i_N is a Banach space for the following norm :

$$\begin{aligned}\|(y,p,v)\|_{E^i_N} := \Big(&\|e^{3/2s\beta^*}y\|^2_{L^2(Q)^N} + \|e^{s\widehat{\beta}+3/2s\beta^*}\widehat{\gamma}^{-7/2}v\mathbb{1}_\omega\|^2_{L^2(Q)^N}\\ &+\|e^{3/2s\beta^*}(\gamma^*)^{-9/8}y\|^2_{L^2(0,T;H^2(\Omega)^N)} + \|e^{3/2s\beta^*}(\gamma^*)^{-9/8}y\|^2_{L^\infty(0,T;V)}\\ &+\|e^{5/2s\beta^*}(\gamma^*)^{-2}(\mathcal{L}y+\nabla p - v\mathbb{1}_\omega)\|^2_{L^2(Q)^N}\Big)^{1/2}\end{aligned}$$

Remark 2.10. *Observe in particular that $(y,p,v) \in E_N^i$ implies $y(T)=0$ in Ω. Moreover, the functions belonging to this space possesses the interesting following property :*

$$e^{5/2s\beta^*}(\gamma^*)^{-2}(y\cdot\nabla)y \in L^2(Q)^N.$$

Proposition 2.11. *Let $i\in\{1,\dots,N\}$. Assume that*

$$y^0 \in V \quad \textit{and} \quad e^{5/2s\beta^*}(\gamma^*)^{-2}f \in L^2(Q)^N.$$

Then, we can find a control v such that the associated solution (y,p) to (2.28) satisfies $(y,p,v)\in E_N^i$. In particular, $v_i\equiv 0$ and $y(T)=0$.

Proof. The proof of this proposition is very similar to the one of Proposition 2 in [34] and Proposition 1 in [35], so we will just give the main ideas.

Following the arguments in [39] and [51], we introduce the space

$$P_0 := \{\,(\chi,\sigma)\in \mathcal{C}^2(\overline{Q})^{N+1} :\ \nabla\cdot\chi = 0 \text{ in } Q, \quad \chi=0 \text{ on } \Sigma\,\}$$

and we consider the following variational problem :

$$a((\widehat{\chi},\widehat{\sigma}),(\chi,\sigma)) = \langle G,(\chi,\sigma)\rangle \quad \text{for all } (\chi,\sigma)\in P_0, \tag{2.32}$$

where we have used the notations

$$\begin{aligned} a((\widehat{\chi},\widehat{\sigma}),(\chi,\sigma)) := &\iint_Q e^{-3s\beta^*}\,(\mathcal{L}^*\widehat{\chi}+\nabla\widehat{\sigma})\cdot(\mathcal{L}^*\chi+\nabla\sigma)\,dx\,dt \\ &+ \sum_{j=1,j\neq i}^{N}\iint_{\omega\times(0,T)} e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7\,\widehat{\chi}_j\,\chi_j\,dx\,dt, \\ \langle G,(\chi,\sigma)\rangle := &\iint_Q f\cdot\chi\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega y^0\cdot\chi(0)\,dx \end{aligned}$$

and $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$, i.e.

$$\mathcal{L}^*\chi := -\chi_t - \Delta\chi.$$

It is clear that $a(\cdot\,,\cdot): P_0\times P_0\mapsto\mathbb{R}$ is a symmetric, definite positive bilinear form on P_0. We denote by P the completion of P_0 for the norm induced by $a(\cdot\,,\cdot)$. Then $a(\cdot\,,\cdot)$ is well-defined, continuous and again definite positive on P. Furthermore, in view of the Carleman estimate (2.30), the linear form $(\chi,\sigma)\mapsto\langle G,(\chi,\sigma)\rangle$ is well-defined and continuous on P. Hence, from Lax-Milgram's lemma, we deduce that the variational problem

$$\left\{\begin{array}{l} a((\widehat{\chi},\widehat{\sigma}),(\chi,\sigma)) = \langle G,(\chi,\sigma)\rangle \\ \text{for all } (\chi,\sigma)\in P, \quad (\widehat{\chi},\widehat{\sigma})\in P, \end{array}\right. \tag{2.33}$$

possesses exactly one solution $(\widehat{\chi},\widehat{\sigma})$.

Let $\widehat{y}$ and $\widehat{v}$ be given by

$$\left\{\begin{array}{ll} \widehat{y} := e^{-3s\beta^*}(\mathcal{L}^*\widehat{\chi}+\nabla\widehat{\sigma}), & \text{in } Q, \\ \widehat{v}_j := -e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7\,\widehat{\chi}_j \quad (j\neq i), \quad \widehat{v}_i := 0 & \text{in } \omega\times(0,T). \end{array}\right.$$

Then, it is readily seen that they satisfy

$$\iint_Q e^{3s\beta^*}|\widehat{y}|^2dxdt + \sum_{j=1,j\neq i}^{N}\iint_{\omega\times(0,T)} e^{2s\widehat{\beta}+3s\beta^*}\widehat{\gamma}^{-7}|\widehat{v}_j|^2dxdt = a((\widehat{\chi},\widehat{\sigma}),(\widehat{\chi},\widehat{\sigma})) < \infty$$

and also that $\widehat{y}$ is, together with some pressure $\widehat{p}$, the weak solution (belonging to $L^2(0,T;V)\cap L^\infty(0,T;H)$) of the Stokes system (2.28) for $v=\widehat{v}$.

It only remains to check that

$$e^{3/2s\beta^*}(\gamma^*)^{-9/8}\widehat{y}\in L^2(0,T;H^2(\Omega)^N)\cap L^\infty(0,T;V).$$

To this end, we define the functions

$$y^* := e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,\widehat{y},\quad p^* := e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,\widehat{p}$$

and

$$f^* := e^{3/2s\beta^*}(\gamma^*)^{-9/8}(f+\widehat{v}1\!\!1_\omega).$$

Then (y^*,p^*) satisfies

$$\begin{cases} \mathcal{L}y^*+\nabla p^* = f^* + (e^{3/2s\beta^*}(\gamma^*)^{-9/8})_t\,\widehat{y} & \text{in } Q,\\ \nabla\cdot y^* = 0 & \text{in } Q,\\ y^*=0 & \text{on } \Sigma,\\ y^*(0) = e^{3/2s\beta^*(0)}(\gamma^*(0))^{-9/8}y^0 & \text{in } \Omega. \end{cases} \tag{2.34}$$

From the fact that $f^*+(e^{3/2s\beta^*}(\gamma^*)^{-9/8})_t\,\widehat{y}\in L^2(Q)^N$ and $y^0\in V$, we have indeed

$$y^*\in L^2(0,T;H^2(\Omega)^N)\cap L^\infty(0,T;V)$$

(see (2.12)). This ends the proof of Proposition 2.11. □

2.4 Proof of Theorem 2.1

In this section we give the proof of Theorem 2.1 using similar arguments to those in [51] (see also [34] and [35]). The result of null controllability for the linear system (2.28) given by Proposition 2.11 will allow us to apply an inverse mapping theorem, namely (see [3]),

Theorem 2.12. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces and let $\mathcal{A}:\mathcal{B}_1\to\mathcal{B}_2$ satisfy $\mathcal{A}\in\mathcal{C}^1(\mathcal{B}_1;\mathcal{B}_2)$. Assume that $b_1\in\mathcal{B}_1$, $\mathcal{A}(b_1)=b_2$ and that $\mathcal{A}'(b_1):\mathcal{B}_1\to\mathcal{B}_2$ is surjective. Then, there exists $\delta>0$ such that, for every $b'\in\mathcal{B}_2$ satisfying $\|b'-b_2\|_{\mathcal{B}_2}<\delta$, there exists a solution of the equation*

$$\mathcal{A}(b)=b',\quad b\in\mathcal{B}_1.$$

We apply this theorem setting, for some given $i\in\{1,\dots,N\}$,

$$\mathcal{B}_1 := E_N^i,$$

$$\mathcal{B}_2 := L^2(e^{5/2s\beta^*}(\gamma^*)^{-2}(0,T);L^2(\Omega)^N)\times V$$

and the operator

$$\mathcal{A}(y,p,v) := (\mathcal{L}y+(y\cdot\nabla)y+\nabla p - v1\!\!1_\omega,\ y(0))$$

for $(y,p,v)\in E_N^i$.

In order to apply Theorem 2.12, it remains to check that the operator $\mathcal{A}$ is of class $\mathcal{C}^1(\mathcal{B}_1;\mathcal{B}_2)$. Indeed, notice that all the terms in $\mathcal{A}$ are linear, except for $(y\cdot\nabla)y$. We will prove that the bilinear operator

$$((y^1,p^1,v^1),(y^2,p^2,v^2))\to(y^1\cdot\nabla)y^2$$

is continuous from $\mathcal{B}_1 \times \mathcal{B}_1$ to $L^2(e^{5/2s\beta^*}(\gamma^*)^{-2}(0,T);L^2(\Omega)^N)$. To do this, notice that $e^{3/2s\beta^*}(\gamma^*)^{-9/8}y \in L^2(0,T;H^2(\Omega)^N) \cap L^\infty(0,T;V)$ for any $(y,p,v) \in B_1$, so we have

$$e^{3/2s\beta^*}(\gamma^*)^{-9/8}y \in L^2(0,T;L^\infty(\Omega)^N)$$

and

$$\nabla(e^{3/2s\beta^*}(\gamma^*)^{-9/8}y) \in L^\infty(0,T;L^2(\Omega)^N).$$

Consequently, we obtain

$$\begin{aligned}
&\|e^{5/2s\beta^*}(\gamma^*)^{-2}(y^1\cdot\nabla)y^2\|_{L^2(Q)^N}\\
&\leq C\|(e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,y^1\cdot\nabla)e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,y^2\|_{L^2(Q)^N}\\
&\leq C\|e^{3/2s\beta^*}(\gamma^*)^{-9/8}y^1\|_{L^2(0,T;L^\infty(\Omega)^N)}\,\|e^{3/2s\beta^*}(\gamma^*)^{-9/8}y^2\|_{L^\infty(0,T;H^1(\Omega)^N)}.
\end{aligned}$$

Notice that $\mathcal{A}'(0,0,0):\mathcal{B}_1\to\mathcal{B}_2$ is given by

$$\mathcal{A}'(0,0,0)(y,p,v) = (\mathcal{L}y+\nabla p, y(0)), \quad \text{for all } (y,p,v)\in\mathcal{B}_1,$$

so this functional is surjective in view of the null controllability result for the linear system (2.28) given by Proposition 2.11.

We are now able to apply Theorem 2.12 for $b_1=(0,0,0)$ and $b_2=(0,0)$. In particular, this gives the existence of a positive number δ such that, if $\|y(0)\|_V \leq \delta$, then we can find a control v satisfying $v_i \equiv 0$, for some given $i\in\{1,\dots,N\}$, such that the associated solution (y,p) to (2.1) satisfies $y(T)=0$ in Ω.

This concludes the proof of Theorem 2.1.

Chapitre 3

Local controllability of the N-dimensional Boussinesq system with $N-1$ scalar controls in an arbitrary control domain

In this chapter we deal with the local exact controllability to a particular class of trajectories of the N-dimensional Boussinesq system with internal controls having 2 vanishing components. The main novelty is that no condition is imposed on the control domain.

This chapter is included in [6].

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3.1 Introduction

Let Ω be a nonempty bounded connected open subset of $\mathbb{R}^N$ ($N = 2$ or 3) of class $\mathcal{C}^\infty$. Let $T > 0$ and let $\omega \subset \Omega$ be a (small) nonempty open subset which is the control domain. We will use the notation $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$.

We will be concerned with the following controlled Boussinesq system :

$$\begin{cases} y_t - \Delta y + (y\cdot\nabla)y + \nabla p = v\mathbb{1}_\omega + \theta\, e_N & \text{in } Q, \\ \theta_t - \Delta\theta + y\cdot\nabla\theta = v_0\mathbb{1}_\omega & \text{in } Q, \\ \nabla\cdot y = 0 & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad \theta(0) = \theta^0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where

$$e_N = \begin{cases} (0,1) & \text{if } N = 2, \\ (0,0,1) & \text{if } N = 3 \end{cases}$$

stands for the gravity vector field, $y = y(x,t)$ represents the velocity of the particules of the fluid, $\theta = \theta(x,t)$ their temperature and $(v_0, v) = (v_0, v_1, \ldots, v_N)$ stands for the control which acts over the set ω.

Let us recall the definition of some usual spaces in the context of incompressible fluids :

$$V = \{y \in H_0^1(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega\}$$

and

$$H = \{y \in L^2(\Omega)^N : \nabla \cdot y = 0 \text{ in } \Omega,\, y \cdot n = 0 \text{ on } \partial\Omega\}.$$

This chapter concerns the local exact controllability to the trajectories of system (3.1) at time $t = T$ with a reduced number of controls. To introduce this concept, let us consider $(\bar{y}, \bar{\theta})$ (together with some pressure $\bar{p}$) a trajectory of the following uncontrolled Boussinesq system :

$$\begin{cases} \bar{y}_t - \Delta\bar{y} + (\bar{y}\cdot\nabla)\bar{y} + \nabla\bar{p} = \bar{\theta}\, e_N & \text{in } Q, \\ \bar{\theta}_t - \Delta\bar{\theta} + \bar{y}\cdot\nabla\bar{\theta} = 0 & \text{in } Q, \\ \nabla\cdot\bar{y} = 0 & \text{in } Q, \\ \bar{y} = 0, \quad \bar{\theta} = 0 & \text{on } \Sigma, \\ \bar{y}(0) = \bar{y}^0, \quad \bar{\theta}(0) = \bar{\theta}^0 & \text{in } \Omega. \end{cases} \tag{3.2}$$

We say that the local exact controllability to the trajectories $(\bar{y}, \bar{\theta})$ holds if there exists a number $\delta > 0$ such that if $\|(y^0, \theta^0) - (\bar{y}^0, \bar{\theta}^0)\|_X \le \delta$ (X is an appropriate Banach space), there exist controls $(v_0, v) \in L^2(\omega \times (0,T))^{N+1}$ such that the corresponding solution (y, θ) to system (3.1) matches $(\bar{y}, \bar{\theta})$ at time $t = T$, i.e.,

$$y(T) = \bar{y}(T) \quad \text{and} \quad \theta(T) = \bar{\theta}(T) \text{ in } \Omega. \tag{3.3}$$

The first results concerning this problem were obtained in [40] and [38], with $N+1$ scalar controls acting in the whole boundary of Ω and with $N+1$ scalar controls acting in ω when Ω is a torus, respectively. Later, in [44], the author proved the local exact controllability for less regular trajectories $(\bar{y}, \bar{\theta})$ in an open bounded set and for an arbitrary control domain. Namely, the trajectories were supposed to satisfy

$$(\bar{y}, \bar{\theta}) \in L^\infty(Q)^{N+1}, \quad (\bar{y}_t, \bar{\theta}_t) \in L^2(0,T; L^r(\Omega))^{N+1}, \tag{3.4}$$

with $r > 1$ if $N = 2$ and $r > 6/5$ if $N = 3$.

In [35], the authors proved that local exact controllability can be achieved with $N-1$ scalar controls acting in ω when $\overline{\omega}$ intersects the boundary of Ω and (3.4) is satisfied. More precisely, we can find controls v_0 and v, with $v_N \equiv 0$ and $v_k \equiv 0$ for some $k < N$ (k is determined by some geometric assumption on ω, see [35] for more details), such that the corresponding solution to (3.1) satisfies (3.3).

In this chapter, we remove this geometric assumption on ω and consider a target trajectory of the form $(0, \bar{p}, \bar{\theta})$, i.e.,

$$\begin{cases} \nabla\bar{p} = \bar{\theta}\, e_N & \text{in } Q, \\ \bar{\theta}_t - \Delta\bar{\theta} = 0 & \text{in } Q, \\ \bar{\theta} = 0 & \text{on } \Sigma, \\ \bar{\theta}(0) = \bar{\theta}^0 & \text{in } \Omega, \end{cases} \tag{3.5}$$

where we assume

$$\bar{\theta} \in L^\infty(0,T; W^{3,\infty}(\Omega)) \text{ and } \nabla\bar{\theta}_t \in L^\infty(Q)^N. \tag{3.6}$$

The main result of this chapter is given in the following theorem.

Theorem 3.1. *Let $i < N$ be a positive integer and $(\bar{p}, \bar{\theta})$ a solution to* (3.5) *satisfying* (3.6)*. Then, for every $T > 0$ and $\omega \subset \Omega$, there exists $\delta > 0$ such that for every $(y^0, \theta^0) \in V \times H_0^1(\Omega)$ satisfying*

$$\|(y^0, \theta^0) - (0, \bar{\theta}^0)\|_{V \times H_0^1} \leq \delta,$$

we can find controls $v^0 \in L^2(\omega \times (0,T))$ and $v \in L^2(\omega \times (0,T))^N$, with $v_i \equiv 0$ and $v_N \equiv 0$, such that the corresponding solution to (3.1) *satisfies* (3.3)*, i.e.,*

$$y(T) = 0 \quad \text{and} \quad \theta(T) = \bar{\theta}(T) \text{ in } \Omega. \tag{3.7}$$

Remark 3.2. *Notice that when $N = 2$ we only need to control the temperature equation.*

Remark 3.3. *It would be interesting to know if the local controllability to the trajectories with $N - 1$ scalar controls holds for $\bar{y} \neq 0$ and ω as in Theorem 3.1. However, up to our knowledge, this is an open problem even for the case of the Navier-Stokes system.*

Remark 3.4. *One could also try to just control the movement equation, that is, $v_0 \equiv 0$ in* (3.1)*. However, this system does not seem to be controllable. To justify this, let us consider the control problem*

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla) y + \nabla p = v \mathbb{1}_\omega + \theta\, e_N & \text{in } Q, \\ \theta_t - \Delta \theta + y \cdot \nabla \theta = 0 & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0, \quad \nabla \theta \cdot n = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad \theta(0) = \theta^0 & \text{in } \Omega, \end{cases}$$

where we have homogeneous Neumann boundary conditions for the temperature. Integrating in Q, integration by parts gives

$$\int_\Omega \theta(T)\, dx = \int_\Omega \theta_0\, dx,$$

so we cannot expect in general null controllability.

Some recent works have been developed for controllability problems with reduced number of controls. For instance, in [23] the authors proved the null controllability for the Stokes system with $N - 1$ scalar controls, and in [7] the local null controllability was proved for the Navier-Stokes system with the same number of controls.

To prove Theorem 3.1 we follow a standard approach introduced in [39] and [51] (see also [34]). We first deduce a null controllability result for the linear system

$$\begin{cases} y_t - \Delta y + \nabla p = f + v \mathbb{1}_\omega + \theta\, e_N & \text{in } Q, \\ \theta_t - \Delta \theta + y \cdot \nabla \bar{\theta} = f_0 + v_0 \mathbb{1}_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad \theta(0) = \theta^0 & \text{in } \Omega, \end{cases} \tag{3.8}$$

where f and f_0 will be taken to decrease exponentially to zero in $t = T$.

The main tool to prove this null controllability result for system (3.8) is a suitable Carleman estimate for the solutions of the adjoint system

$$\begin{cases} -\varphi_t - \Delta \varphi + \nabla \pi = g - \psi \nabla \bar{\theta} & \text{in } Q, \\ -\psi_t - \Delta \psi = g_0 + \varphi_N & \text{in } Q, \\ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \varphi = 0, \quad \psi = 0 & \text{on } \Sigma, \\ \varphi(T) = \varphi^T, \quad \psi(T) = \psi^T & \text{in } \Omega, \end{cases} \tag{3.9}$$

where $g \in L^2(Q)^N$, $g_0 \in L^2(Q)$, $\varphi^T \in H$ and $\psi^T \in L^2(\Omega)$. In fact, this inequality is of the form

$$\iint_Q \widetilde{\rho}_1(t)(|\varphi|^2 + |\psi|^2)\,\mathrm{d}x\,\mathrm{d}t \le C\left(\iint_Q \widetilde{\rho}_2(t)(|g|^2 + |g_0|^2)\,\mathrm{d}x\,\mathrm{d}t \right.$$
$$\left. + \iint_{\omega\times(0,T)} \widetilde{\rho}_3(t)|\varphi_j|^2\,\mathrm{d}x\,\mathrm{d}t + \iint_{\omega\times(0,T)} \widetilde{\rho}_4(t)|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right), \quad (3.10)$$

if $N = 3$, and of the form

$$\iint_Q \widetilde{\rho}_1(t)(|\varphi|^2 + |\psi|^2)\,\mathrm{d}x\,\mathrm{d}t \le C\left(\iint_Q \widetilde{\rho}_2(t)(|g|^2 + |g_0|^2)\mathrm{d}x\,\mathrm{d}t + \iint_{\omega\times(0,T)} \widetilde{\rho}_4(t)|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\right),$$

if $N = 2$, where $j = 1$ or 2 and $\widetilde{\rho}_k(t)$ are positive smooth weight functions (see inequalities (3.14) and (3.15) below). From these estimates, we can find a solution (y, θ, v, v_0) of (3.8) with the same decreasing properties as f and f_0. In particular, $(y(T), \theta(T)) = (0, 0)$ and $v_i = v_N = 0$.

We conclude the controllability result for the nonlinear system by means of an inverse mapping theorem.

The rest of the chapter is organized as follows. In section 3.2, we prove a Carleman inequality of the form (3.10) for system (3.9). In section 3.3, we deal with the null controllability of the linear system (3.8). Finally, in section 3.4 we give the proof of Theorem 3.1.

3.2 Carleman estimate for the adjoint system

In this section we will prove a Carleman estimate for the adjoint system (3.9). In order to do so, we are going to introduce some weight functions. Let ω_0 be a nonempty open subset of $\mathbb{R}^N$ such that $\overline{\omega_0} \subset \omega$ and $\eta \in \mathcal{C}^2(\overline{\Omega})$ such that

$$|\nabla\eta| > 0 \text{ in } \overline{\Omega}\setminus\omega_0,\ \eta > 0 \text{ in } \Omega \text{ and } \eta \equiv 0 \text{ on } \partial\Omega. \quad (3.11)$$

The existence of such a function η is given in [39]. Let also $\ell \in \mathcal{C}^\infty([0, T])$ be a positive function satisfying

$$\begin{aligned} &\ell(t) = t \quad \forall t \in [0, T/4], \quad \ell(t) = T - t \quad \forall t \in [3T/4, T],\\ &\ell(t) \le \ell(T/2) \quad \forall t \in [0, T]. \end{aligned} \quad (3.12)$$

Then, for all $\lambda \ge 1$ we consider the following weight functions :

$$\begin{aligned} &\alpha(x,t) := \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\ell^8(t)}, && \xi(x,t) := \frac{e^{\lambda\eta(x)}}{\ell^8(t)},\\ &\alpha^*(t) := \max_{x\in\overline{\Omega}} \alpha(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - 1}{\ell^8(t)}, && \xi^*(t) := \min_{x\in\overline{\Omega}} \xi(x,t) = \frac{1}{\ell^8(t)},\\ &\widehat{\alpha}(t) := \min_{x\in\overline{\Omega}} \alpha(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\|\eta\|_\infty}}{\ell^8(t)}, && \widehat{\xi}(t) := \max_{x\in\overline{\Omega}} \xi(x,t) = \frac{e^{\lambda\|\eta\|_\infty}}{\ell^8(t)}. \end{aligned} \quad (3.13)$$

Our Carleman estimate is given in the following proposition.

Proposition 3.5. *Assume $N = 3$, $\omega \subset \Omega$ and $(\bar{p}, \bar{\theta})$ satisfies (3.6). There exists a constant λ_0, such that for any $\lambda \ge \lambda_0$ there exist two constants $C(\lambda) > 0$ and $s_0(\lambda) > 0$ such that* *for any $j \in \{1,2\}$, any $g \in L^2(Q)^3$, any $g_0 \in L^2(Q)$, any $\varphi^T \in H$ and any $\psi^T \in L^2(\Omega)$, the solution of* (3.9) *satisfies*

$$
\begin{aligned}
s^4 \iint_Q e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2 dx\,dt + s^5 \iint_Q e^{-5s\alpha^*}(\xi^*)^5|\psi|^2 dx\,dt \\
\leq C\left(\iint_Q e^{-3s\alpha^*}(|g|^2+|g_0|^2)dx\,dt + s^7\iint_{\omega\times(0,T)} e^{-2s\widehat{\alpha}-3s\alpha^*}(\widehat{\xi})^7|\varphi_j|^2 dx\,dt\right. \\
\left. + s^{12}\iint_{\omega\times(0,T)} e^{-4s\widehat{\alpha}-s\alpha^*}(\widehat{\xi})^{49/4}|\psi|^2 dx\,dt\right) \quad (3.14)
\end{aligned}
$$

for every $s \geq s_0$.

For the sake of completeness, let us also state this result for the 2-dimensional case.

Proposition 3.6. *Assume $N = 2$, $\omega \subset \Omega$ and $(\bar{p}, \bar{\theta})$ satisfies* (3.6)*. There exists a constant λ_0, such that for any $\lambda \geq \lambda_0$ there exist two constants $C(\lambda) > 0$ and $s_0(\lambda) > 0$ such that for any $g \in L^2(Q)^2$, any $g_0 \in L^2(Q)$, any $\varphi^T \in H$ and any $\psi^T \in L^2(\Omega)$, the solution of* (3.9) *satisfies*

$$
\begin{aligned}
s^4 \iint_Q e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2 dx\,dt + s^5 \iint_Q e^{-5s\alpha^*}(\xi^*)^5|\psi|^2 dx\,dt \\
\leq C\left(\iint_Q e^{-3s\alpha^*}(|g|^2+|g_0|^2)dx\,dt + s^{12}\iint_{\omega\times(0,T)} e^{-4s\widehat{\alpha}-s\alpha^*}(\widehat{\xi})^{49/4}|\psi|^2 dx\,dt\right) \quad (3.15)
\end{aligned}
$$

for every $s \geq s_0$.

To prove Proposition 3.5 we will follow the ideas of [23] and [35] (see also [7]). An important point in the proof of the Carleman inequality established in [23] is that the laplacian of the pressure in the adjoint system is zero. In [7], a decomposition of the solution was made, so that we can essentially concentrate in a solution where the laplacian of the pressure is zero. For system (3.9) this will not be possible because of the coupling term $\psi\nabla\bar{\theta}$. However, under hypothesis (3.6) we can follow the same ideas to obtain (3.14). All the details are given below.

3.2.1 Technical results

Let us present now the technical results needed to prove Carleman inequalities (3.14) and (3.15). These results were already stated in Chapter 2, but we include them here again for the sake of completeness.

Let us begin with a Carleman inequality for parabolic equations with nonhomogeneous boundary conditions proved in [52]. Consider the equation

$$
u_t - \Delta u = F_0 + \sum_{j=1}^{N} \partial_j F_j \quad \text{in } Q, \qquad (3.16)
$$

where $F_0, F_1, \ldots, F_N \in L^2(Q)$. We have the following result.

Lemma 3.7. *There exists a constant $\widehat{\lambda}_0$ only depending on Ω, ω_0, η and ℓ such that for any $\lambda > \widehat{\lambda}_0$ there exist two constants $C(\lambda) > 0$ and $\widehat{s}(\lambda)$, such that for every $s \geq \widehat{s}$ and* *every $u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ satisfying* (3.16)*, we have*

$$\frac{1}{s}\iint_Q e^{-2s\alpha}\frac{1}{\xi}|\nabla u|^2 dx\,dt + s\iint_Q e^{-2s\alpha}\xi|u|^2 dx\,dt \le C\left(s\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi|u|^2 dx\,dt \right.$$
$$+ s^{-1/2}\left\|e^{-s\alpha}\xi^{-1/4}u\right\|^2_{H^{\frac14,\frac12}(\Sigma)} + s^{-1/2}\left\|e^{-s\alpha}\xi^{-1/8}u\right\|^2_{L^2(\Sigma)}$$
$$\left. + s^{-2}\iint_Q e^{-2s\alpha}\xi^{-2}|F_0|^2 dx\,dt + \sum_{j=1}^{N}\iint_Q e^{-2s\alpha}|F_j|^2 dx\,dt\right). \quad (3.17)$$

Recall that

$$\|u\|_{H^{\frac14,\frac12}(\Sigma)} = \left(\|u\|^2_{H^{1/4}(0,T;L^2(\partial\Omega))} + \|u\|^2_{L^2(0,T;H^{1/2}(\partial\Omega))}\right)^{1/2}.$$

Remark 3.8. *The usual notation for this space is actually $H^{\frac12,\frac14}(\Sigma)$ (see [66], for instance). However, we follow the same notation used in [52].*

The next technical result is a particular case of Lemma 3 in [23].

Lemma 3.9. *There exists a constant $\widehat{\lambda}_1$ such that for any $\lambda \ge \widehat{\lambda}_1$ there exists $C > 0$ depending only on λ, Ω, ω_0, η and ℓ such that, for every $u \in L^2(0,T;H^1(\Omega))$,*

$$s^3\iint_Q e^{-2s\alpha}\xi^3|u|^2 dx\,dt$$
$$\le C\left(s\iint_Q e^{-2s\alpha}\xi|\nabla u|^2 dx\,dt + s^3\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|u|^2 dx\,dt\right), \quad (3.18)$$

for every $s \ge C$.

The next lemma is an estimate concerning the Laplace operator :

Lemma 3.10. *There exists a constant $\widehat{\lambda}_2$ such that for any $\lambda \ge \widehat{\lambda}_2$ there exists $C > 0$ depending only on λ, Ω, ω_0, η and ℓ such that, for every $u \in L^2(0,T;H^1_0(\Omega))$,*

$$s^6\iint_Q e^{-2s\alpha}\xi^6|u|^2 dx\,dt + s^4\iint_Q e^{-2s\alpha}\xi^4|\nabla u|^2 dx\,dt$$
$$\le C\left(s^3\iint_Q e^{-2s\alpha}\xi^3|\Delta u|^2 dx\,dt + s^6\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^6|u|^2 dx\,dt\right), \quad (3.19)$$

for every $s \ge C$.

Inequality (3.19) comes from the classical result in [39] for parabolic equations applied to the laplacian with parameter $s/\ell^8(t)$ (see Lemma 2.7 in Chapter 2). Then, multiplying by $\exp(-2se^{2\lambda\|\eta\|_\infty}/\ell^8(t))$ and integrating in $(0,T)$ we obtain (3.19). Details can be found in [23], [7] or Paragraph 2.2.1 in Chapter 2.

The last technical result concerns the regularity of the solutions to the Stokes system that can be found in [59] (see also [76]).

Lemma 3.11. *For every $T > 0$ and every $F \in L^2(Q)^N$, there exists a unique solution*

$$u \in L^2(0,T;H^2(\Omega)^N) \cap H^1(0,T;H) \cap L^\infty(0,T;V)$$

to the Stokes system

$$\begin{cases} u_t - \Delta u + \nabla p = F & \textit{in } Q, \\ \nabla \cdot u = 0 & \textit{in } Q, \\ u = 0 & \textit{on } \Sigma, \\ u(0) = 0 & \textit{in } \Omega, \end{cases}$$

for some $p \in L^2(0,T;H^1(\Omega))$, *and there exists a constant* $C > 0$ *depending only on* Ω *such that*

$$\|u\|^2_{L^2(0,T;H^2(\Omega)^N)} + \|u\|^2_{H^1(0,T;L^2(\Omega)^N)} + \|u\|^2_{L^\infty(0,T;V)} \le C\|F\|^2_{L^2(Q)^N}. \tag{3.20}$$

Furthermore, assume that $F \in L^2(0,T;H^2(\Omega)^N) \cap H^1(0,T;L^2(\Omega)^N)$ *and satisfies the following compatibility condition :*

$$\nabla p_F = F(0) \ \textit{on } \partial\Omega,$$

where p_F *is any solution of the Neumann boundary-value problem*

$$\begin{cases} \Delta p_F = \nabla \cdot F(0) & \textit{in } \Omega, \\ \dfrac{\partial p_F}{\partial n} = F(0) \cdot n & \textit{on } \partial\Omega. \end{cases}$$

Then, $u \in L^2(0,T;H^4(\Omega)^N) \cap H^1(0,T;H^2(\Omega)^N)$ *and there exists a constant* $C > 0$ *depending only on* Ω *such that*

$$\|u\|^2_{L^2(0,T;H^4(\Omega)^N)} + \|u\|^2_{H^1(0,T;H^2(\Omega)^N)} \le C\Big(\|F\|^2_{L^2(0,T;H^2(\Omega)^N)} + \|F\|^2_{H^1(0,T;L^2(\Omega)^N)}\Big). \tag{3.21}$$

From now on, we set $N = 3$, $i = 2$ and $j = 1$, i.e., we consider a control for the movement equation in (3.1) (and (3.8)) of the form $v = (v_1, 0, 0)$. The arguments can be easily adapted to the general case by interchanging the roles of i and j.

3.2.2 Proof of Proposition 3.5

Let us introduce (w, π_w), (z, π_z) and $\widetilde{\psi}$, the solutions of the following systems :

$$\begin{cases} -w_t - \Delta w + \nabla \pi_w = \rho\, g & \text{in } Q, \\ \nabla \cdot w = 0 & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(T) = 0 & \text{in } \Omega, \end{cases} \tag{3.22}$$

$$\begin{cases} -z_t - \Delta z + \nabla \pi_z = -\rho' \varphi - \widetilde{\psi} \nabla \bar{\theta} & \text{in } Q, \\ \nabla \cdot z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T) = 0 & \text{in } \Omega, \end{cases} \tag{3.23}$$

and

$$\begin{cases} -\widetilde{\psi}_t - \Delta \widetilde{\psi} = \rho\, g_0 + \rho\, \varphi_3 - \rho' \psi & \text{in } Q, \\ \widetilde{\psi} = 0 & \text{on } \Sigma, \\ \widetilde{\psi}(T) = 0 & \text{in } \Omega, \end{cases} \tag{3.24}$$

where $\rho(t) = e^{-\frac{3}{2} s \alpha^*}$. Adding (3.22) and (3.23), we see that $(w + z, \pi_w + \pi_z, \widetilde{\psi})$ solves the same system as $(\rho\, \varphi, \rho\, \pi, \rho\, \psi)$, where (φ, π, ψ) is the solution to (3.9). By uniqueness of the Cauchy problem we have

$$\rho\, \varphi = w + z, \quad \rho\, \pi = \pi_w + \pi_z \quad \text{and} \quad \rho\, \psi = \widetilde{\psi}. \tag{3.25}$$

Applying the divergence operator to (3.23) we see that $\Delta\pi_z = -\nabla\cdot(\widetilde{\psi}\nabla\bar{\theta})$. We apply now the operator $\nabla\Delta = (\partial_1\Delta, \partial_2\Delta, \partial_3\Delta)$ to the equations satisfied by z_1 and z_3. We then have

$$\begin{aligned} -(\nabla\Delta z_1)_t - \Delta(\nabla\Delta z_1) &= \nabla\left(\partial_1\nabla\cdot(\widetilde{\psi}\nabla\bar{\theta}) - \Delta(\widetilde{\psi}\partial_1\bar{\theta}) - \rho'\Delta\varphi_1\right) \text{ in } Q,\\ -(\nabla\Delta z_3)_t - \Delta(\nabla\Delta z_3) &= \nabla\left(\partial_3\nabla\cdot(\widetilde{\psi}\nabla\bar{\theta}) - \Delta(\widetilde{\psi}\partial_3\bar{\theta}) - \rho'\Delta\varphi_3\right) \text{ in } Q. \end{aligned} \tag{3.26}$$

To the equations in (3.26), we apply the Carleman inequality in Lemma 3.7 with $u = \nabla\Delta z_k$ for $k = 1,3$ to obtain

$$\begin{aligned} &\sum_{k=1,3}\left[\frac{1}{s}\iint_Q e^{-2s\alpha}\frac{1}{\xi}|\nabla\nabla\Delta z_k|^2 \mathrm{d}x\,\mathrm{d}t + s\iint_Q e^{-2s\alpha}\xi|\nabla\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t\right]\\ &\leq C\left(\sum_{k=1,3}\left[s\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi|\nabla\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t + s^{-1/2}\left\|e^{-s\alpha^*}(\xi^*)^{-1/8}\nabla\Delta z_k\right\|^2_{L^2(\Sigma)^3}\right.\right.\\ &\quad\left. + s^{-1/2}\left\|e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k\right\|^2_{H^{\frac14,\frac12}(\Sigma)^3} + \iint_Q e^{-2s\alpha}|\rho'|^2|\Delta\varphi_k|^2\mathrm{d}x\,\mathrm{d}t\right]\\ &\quad\left. + \iint_Q e^{-2s\alpha}\Big(\sum_{k,l=1}^3|\partial^2_{kl}\widetilde{\psi}|^2 + |\nabla\widetilde{\psi}|^2 + |\widetilde{\psi}|^2\Big)\mathrm{d}x\,\mathrm{d}t\right), \end{aligned} \tag{3.27}$$

for every $s \geq C$, where C depends also on $\|\bar{\theta}\|_{L^\infty(0,T;W^{3,\infty}(\Omega))}$.

Now, by Lemma 3.9 with $u = \Delta z_k$ for $k = 1,3$ we have

$$\begin{aligned} &\sum_{k=1,3} s^3\iint_Q e^{-2s\alpha}\xi^3|\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t\\ &\quad\leq C\sum_{k=1,3}\left(s\iint_Q e^{-2s\alpha}\xi|\nabla\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t + s^3\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^3|\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t\right), \end{aligned} \tag{3.28}$$

for every $s \geq C$, and by Lemma 3.10 with $u = z_k$ for $k = 1,3$:

$$\begin{aligned} &\sum_{k=1,3}\left[s^4\iint_Q e^{-2s\alpha}\xi^4|\nabla z_k|^2\mathrm{d}x\,\mathrm{d}t + s^6\iint_Q e^{-2s\alpha}\xi^6|z_k|^2\mathrm{d}x\,\mathrm{d}t\right]\\ &\quad\leq C\sum_{k=1,3}\left[s^3\iint_Q e^{-2s\alpha}\xi^3|\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t + s^6\iint_{\omega_0\times(0,T)} e^{-2s\alpha}\xi^6|z_k|^2\mathrm{d}x\,\mathrm{d}t\right], \end{aligned} \tag{3.29}$$

for every $s \geq C$.

Combining (3.27), (3.28) and (3.29) and considering a nonempty open set ω_1 such that $\omega_0 \Subset \omega_1 \Subset \omega$ we obtain after some integration by parts (for details, see Step 3 in Paragraph 2.2.1)

$$\begin{aligned} &\sum_{k=1,3}\left[\frac{1}{s}\iint_Q e^{-2s\alpha}\frac{1}{\xi}|\nabla\nabla\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t + s\iint_Q e^{-2s\alpha}\xi|\nabla\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t\right.\\ &\left. + s^3\iint_Q e^{-2s\alpha}\xi^3|\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t + s^4\iint_Q e^{-2s\alpha}\xi^4|\nabla z_k|^2\mathrm{d}x\,\mathrm{d}t + s^6\iint_Q e^{-2s\alpha}\xi^6|z_k|^2\mathrm{d}x\,\mathrm{d}t\right]\\ &\leq C\left(\sum_{k=1,3}\left[s^7\iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^7|z_k|^2\mathrm{d}x\,\mathrm{d}t + s^{-1/2}\left\|e^{-s\alpha^*}(\xi^*)^{-1/8}\nabla\Delta z_k\right\|^2_{L^2(\Sigma)^3}\right.\right. \end{aligned}$$

$$+s^{-1/2}\left\|e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k\right\|^2_{H^{\frac14,\frac12}(\Sigma)^3}+\iint_Q e^{-2s\alpha}|\rho'|^2|\Delta\varphi_k|^2\mathrm{d}x\,\mathrm{d}t\Bigg]$$
$$+\iint_Q e^{-2s\alpha}(\sum_{k,l=1}^3|\partial^2_{kl}\widetilde{\psi}|^2+|\nabla\widetilde{\psi}|^2+|\widetilde{\psi}|^2)\mathrm{d}x\,\mathrm{d}t\Bigg),\quad(3.30)$$

for every $s\geq C$.

Notice that from the identities in (3.25), the regularity estimate (3.20) for w and $|\rho'|^2\leq Cs^2\rho^2(\xi)^{9/4}$ we obtain for $k=1,3$

$$\begin{aligned}\iint_Q e^{-2s\alpha}|\rho'|^2|\Delta\varphi_k|^2\mathrm{d}x\,\mathrm{d}t&=\iint_Q e^{-2s\alpha}|\rho'|^2\rho^{-2}|\Delta(\rho\varphi_k)|^2\mathrm{d}x\,\mathrm{d}t\\&\leq Cs^2\iint_Q e^{-2s\alpha}\xi^{9/4}|\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t+Cs^2\iint_Q e^{-2s\alpha}\xi^{9/4}|\Delta w|^2\mathrm{d}x\,\mathrm{d}t\\&\leq Cs^2\iint_Q e^{-2s\alpha}\xi^3|\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t+C\|\rho\, g\|^2_{L^2(Q)^3},\end{aligned}$$

where we have also used the fact that $s^2e^{-2s\alpha}\xi^{9/4}$ is bounded and $1\leq C\xi^{3/4}$ in Q.

Now, from $z_{|\Sigma}=0$ and the divergence free condition we readily have (notice that α^* and ξ^* do not depend on x)

$$\begin{aligned}s^4\iint_Q e^{-2s\alpha^*}(\xi^*)^4|z_2|^2\mathrm{d}x\,\mathrm{d}t&\leq Cs^4\iint_Q e^{-2s\alpha^*}(\xi^*)^4|\partial_2 z_2|^2\mathrm{d}x\,\mathrm{d}t\\&\leq Cs^4\iint_Q e^{-2s\alpha}\xi^4(|\nabla z_1|^2+|\nabla z_3|^2)\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Using these two last estimates in (3.30), we get

$$\begin{aligned}I(s,z):=\sum_{k=1,3}\Bigg[&\frac1s\iint_Q e^{-2s\alpha}\frac1\xi|\nabla\nabla\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t+s\iint_Q e^{-2s\alpha}\xi|\nabla\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t\\&+s^3\iint_Q e^{-2s\alpha}\xi^3|\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t+s^4\iint_Q e^{-2s\alpha}\xi^4|\nabla z_k|^2\mathrm{d}x\,\mathrm{d}t\\&+s^6\iint_Q e^{-2s\alpha}\xi^6|z_k|^2\mathrm{d}x\,\mathrm{d}t\Bigg]+s^4\iint_Q e^{-2s\alpha^*}(\xi^*)^4|z_2|^2\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

$$\leq C\Bigg(\sum_{k=1,3}\Bigg[s^7\iint_{\omega_1\times(0,T)}e^{-2s\alpha}\xi^7|z_k|^2\mathrm{d}x\,\mathrm{d}t+s^{-1/2}\left\|e^{-s\alpha^*}(\xi^*)^{-1/8}\nabla\Delta z_k\right\|^2_{L^2(\Sigma)^3}$$
$$+s^{-1/2}\left\|e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k\right\|^2_{H^{\frac14,\frac12}(\Sigma)^3}\Bigg]+\|\rho\, g\|^2_{L^2(Q)^3}$$
$$+\iint_Q e^{-2s\alpha}(\sum_{k,l=1}^3|\partial^2_{kl}\widetilde{\psi}|^2+|\nabla\widetilde{\psi}|^2+|\widetilde{\psi}|^2)\mathrm{d}x\,\mathrm{d}t\Bigg),\quad(3.31)$$

for every $s\geq C$.

For equation (3.24), we use the classical Carleman inequality for the heat equation (see for example [39]) : there exists $\widehat{\lambda}_3>0$ such that for any $\lambda>\widehat{\lambda}_3$ there exists

$C(\lambda, \Omega, \omega_1, \|\bar{\theta}\|_{L^\infty(0,T;W^{3,\infty}(\Omega))}) > 0$ such that

$$\begin{aligned}J(s,\widetilde{\psi}) := s \iint_Q e^{-2s\alpha}\xi(|\widetilde{\psi}_t|^2 + \sum_{k,l=1}^{3} |\partial^2_{kl}\widetilde{\psi}|^2)\mathrm{d}x\,\mathrm{d}t + s^3 \iint_Q e^{-2s\alpha}\xi^3|\nabla\widetilde{\psi}|^2\mathrm{d}x\,\mathrm{d}t \\ + s^5 \iint_Q e^{-2s\alpha}\xi^5|\widetilde{\psi}|^2\mathrm{d}x\,\mathrm{d}t \leq C\left(s^2 \iint_Q e^{-2s\alpha}\xi^2\rho^2(|g_0|^2 + |\varphi_3|^2)\mathrm{d}x\,\mathrm{d}t \right. \\ \left. + s^2 \iint_Q e^{-2s\alpha}\xi^2|\rho'|^2|\rho|^{-2}|\widetilde{\psi}|^2\mathrm{d}x\,\mathrm{d}t + s^5 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^5|\widetilde{\psi}|^2\mathrm{d}x\,\mathrm{d}t\right), \quad (3.32)\end{aligned}$$

for every $s \geq C$.

We choose λ_0 in Proposition 3.5 (and Proposition 3.6) to be $\lambda_0 := \max\{\widehat{\lambda}_0, \widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3\}$ and we fix $\lambda \geq \lambda_0$.

Combining inequalities (3.31) and (3.32), and taking into account that $s^2 e^{-2s\alpha}\xi^2\rho^2$ is bounded, the identities in (3.25), estimate (3.20) for w and $|\rho'| \leq Cs(\xi^*)^{9/8}\rho$ we have

$$\begin{aligned}I(s,z) + J(s,\widetilde{\psi}) \leq C \Bigg(\|\rho\, g\|^2_{L^2(Q)^3} + \|\rho\, g_0\|^2_{L^2(Q)} + s^5 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^5|\widetilde{\psi}|^2\mathrm{d}x\,\mathrm{d}t \\ + \sum_{k=1,3} \Bigg[s^{-1/2} \left\| e^{-s\alpha^*}(\xi^*)^{-1/8}\nabla\Delta z_k \right\|^2_{L^2(\Sigma)^3} + s^{-1/2} \left\| e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k \right\|^2_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)^3} \\ + s^7 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^7|z_k|^2\mathrm{d}x\,\mathrm{d}t \Bigg] \Bigg), \quad (3.33)\end{aligned}$$

for every $s \geq C$.

It remains to treat the boundary terms of this inequality and to eliminate the local term in z_3.

Estimate of the boundary terms. First, we treat the first boundary term in (3.33). Notice that, since α^* and ξ^* do not depend on x, we can readily get by integration by parts (see (2.24)), for $k = 1, 3$,

$$\begin{aligned}\|e^{-s\alpha^*}\nabla\Delta z_k\|^2_{L^2(\Sigma)^3} \leq C\Big(\|e^{-s\alpha^*}\nabla\Delta z_k\|^2_{L^2(Q)^3} \\ + \|s^{1/2}e^{-s\alpha^*}(\xi^*)^{1/2}\nabla\Delta z_k\|_{L^2(Q)^3}\|s^{-1/2}e^{-s\alpha^*}(\xi^*)^{-1/2}\nabla\nabla\Delta z_k\|_{L^2(Q)^9}\Big),\end{aligned}$$

and using the properties of the weight functions (see (3.13)) we get

$$\|e^{-s\alpha^*}\nabla\Delta z_k\|^2_{L^2(\Sigma)^3} \leq C\left(s\iint_Q e^{-2s\alpha}\xi|\nabla\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t + \frac{1}{s}\iint_Q e^{-2s\alpha}\frac{1}{\xi}|\nabla\nabla\Delta z_k|^2\mathrm{d}x\,\mathrm{d}t \right).$$

From this estimate, we find that $\|e^{-s\alpha^*}\nabla\Delta z_k\|^2_{L^2(\Sigma)^3}$ is bounded by $I(s,z)$. On the other hand, we can bound the first boundary term as follows :

$$s^{-1/2}\left\|e^{-s\alpha^*}(\xi^*)^{-1/8}\nabla\Delta z_k\right\|^2_{L^2(\Sigma)^3} \leq Cs^{-1/2}\left\|e^{-s\alpha^*}\nabla\Delta z_k\right\|^2_{L^2(\Sigma)^3}.$$

Therefore, the first boundary terms can be absorbed by taking s large enough.

Now we treat the second boundary term in the right-hand side of (3.33). We will use regularity estimates to prove that z_1 and z_3 multiplied by a certain weight function are regular enough. First, let us observe that from (3.25) and the regularity estimate (3.20) for w we readily have

$$\|s^2 e^{-s\alpha^*}(\xi^*)^2 \rho\, \varphi\|^2_{L^2(Q)^3} \leq C\left(I(s,z) + \|\rho\, g\|^2_{L^2(Q)^3}\right). \tag{3.34}$$

We define now

$$\widetilde{z} := s e^{-s\alpha^*}(\xi^*)^{7/8} z, \quad \widetilde{\pi}_z := s e^{-s\alpha^*}(\xi^*)^{7/8}\pi_z.$$

From (3.23) we see that $(\widetilde{z}, \widetilde{\pi}_z)$ is the solution of the Stokes system :

$$\begin{cases} -\widetilde{z}_t - \Delta \widetilde{z} + \nabla \widetilde{\pi}_z = R_1 & \text{in } Q, \\ \nabla \cdot \widetilde{z} = 0 & \text{in } Q, \\ \widetilde{z} = 0 & \text{on } \Sigma, \\ \widetilde{z}(T) = 0 & \text{in } \Omega, \end{cases} \tag{3.35}$$

where $R_1 := -s e^{-s\alpha^*}(\xi^*)^{7/8}\rho' \varphi - s e^{-s\alpha^*}(\xi^*)^{7/8}\widetilde{\psi}\nabla\bar{\theta} - (s e^{-s\alpha^*}(\xi^*)^{7/8})_t z$. Taking into account that $|\alpha^*_t| \leq C(\xi^*)^{9/8}$, $|\rho'| \leq Cs(\xi^*)^{9/8}\,\rho$, (3.6) and (3.34) we have

$$\|R_1\|^2_{L^2(Q)^3} \leq C\left(I(s,z) + J(s,\widetilde{\psi}) + \|\rho g\|^2_{L^2(Q)^3}\right),$$

and therefore, by the regularity estimate (3.20) applied to (3.35), we obtain

$$\|\widetilde{z}\|^2_{L^2(0,T;H^2(\Omega)^3)\cap H^1(0,T;L^2(\Omega)^3)} \leq C\left(I(s,z) + J(s,\widetilde{\psi}) + \|\rho\, g\|^2_{L^2(Q)^3}\right). \tag{3.36}$$

Next, let

$$\widehat{z} := e^{-s\alpha^*}(\xi^*)^{-1/4} z, \quad \widehat{\pi}_z := e^{-s\alpha^*}(\xi^*)^{-1/4}\pi_z.$$

From (3.23), $(\widehat{z}, \widehat{\pi}_z)$ is the solution of the Stokes system :

$$\begin{cases} -\widehat{z}_t - \Delta \widehat{z} + \nabla \widehat{\pi}_z = R_2 & \text{in } Q, \\ \nabla \cdot \widehat{z} = 0 & \text{in } Q, \\ \widehat{z} = 0 & \text{on } \Sigma, \\ \widehat{z}(T) = 0 & \text{in } \Omega, \end{cases} \tag{3.37}$$

where $R_2 := -e^{-s\alpha^*}(\xi^*)^{-1/4}\rho'\varphi - e^{-s\alpha^*}(\xi^*)^{-1/4}\widetilde{\psi}\nabla\bar{\theta} - (e^{-s\alpha^*}(\xi^*)^{-1/4})_t z$. By the same arguments as before, and thanks to (3.36), we can easily prove that $R_2 \in L^2(0,T;H^2(\Omega)^3)\cap H^1(0,T;L^2(\Omega)^3)$ (for the first term in R_2, we use again (3.25) and (3.36)) and furthermore

$$\|R_2\|^2_{L^2(0,T;H^2(\Omega)^3)\cap H^1(0,T;L^2(\Omega)^3)} \leq C\left(I(s,z) + J(s,\widetilde{\psi}) + \|\rho\, g\|^2_{L^2(Q)^3}\right).$$

By the regularity estimate (3.21) applied to (3.37) (notice that the compatibility condition in Lemma 3.11 is satisfied since $R_2(T) = 0$), we have

$$\|\widehat{z}\|^2_{L^2(0,T;H^4(\Omega)^3)\cap H^1(0,T;H^2(\Omega)^3)} \leq C\left(I(s,z) + J(s,\widetilde{\psi}) + \|\rho\, g\|^2_{L^2(Q)^3}\right).$$

In particular, $e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k \in L^2(0,T;H^1(\Omega)^3)\cap H^1(0,T;H^{-1}(\Omega)^3)$ for $k = 1,3$ and

$$\sum_{k=1,3} \|e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k\|^2_{L^2(0,T;H^1(\Omega)^3)} + \|e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k\|^2_{H^1(0,T;H^{-1}(\Omega)^3)} \leq C\left(I(s,z) + J(s,\widetilde{\psi}) + \|\rho\, g\|^2_{L^2(Q)^3}\right). \tag{3.38}$$

To end this part, we use a trace inequality to estimate the second boundary term in the right-hand side of (3.33) (see [66], for instance) :

$$\sum_{k=1,3} s^{-1/2} \left\| e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k \right\|^2_{H^{\frac14,\frac12}(\Sigma)^3} \le C\, s^{-1/2} \sum_{k=1,3} \left[\left\| e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k \right\|^2_{L^2(0,T;H^1(\Omega)^3)} + \left\| e^{-s\alpha^*}(\xi^*)^{-1/4}\nabla\Delta z_k \right\|^2_{H^1(0,T;H^{-1}(\Omega)^3)} \right].$$

By taking s large enough in (3.33), the boundary terms $s^{-1/2}\|e^{-s\alpha}\xi^{-1/4}\nabla\Delta z_k\|^2_{H^{\frac14,\frac12}(\Sigma)^3}$ can be absorbed by the terms in the left-hand side of (3.38).

Thus, using (3.25) and (3.20) for w in the right-hand side of (3.33), we have for the moment

$$I(s,z) + J(s,\widetilde{\psi}) \le C \left(\|\rho\, g\|^2_{L^2(Q)^3} + \|\rho\, g_0\|^2_{L^2(Q)} + s^5 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^5|\widetilde{\psi}|^2 \mathrm{d}x\,\mathrm{d}t + s^7 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^7\rho^2|\varphi_1|^2\mathrm{d}x\,\mathrm{d}t + s^7 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^7\rho^2|\varphi_3|^2\mathrm{d}x\,\mathrm{d}t \right),$$

for every $s \ge C$. Furthermore, notice that using again (3.25), (3.20) for w and (3.36) we obtain from the previous inequality

$$s^2 \iint_Q e^{-2s\alpha^*}(\xi^*)^{7/4}\rho^2|\varphi_{3,t}|^2\mathrm{d}x\,\mathrm{d}t + \widetilde{I}(s,\rho\,\varphi) + J(s,\widetilde{\psi}) \le C \left(\|\rho\, g\|^2_{L^2(Q)^3} + \|\rho\, g_0\|^2_{L^2(Q)^3} + s^5 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^5|\widetilde{\psi}|^2\mathrm{d}x\,\mathrm{d}t + s^7 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^7\rho^2|\varphi_1|^2\mathrm{d}x\,\mathrm{d}t + s^7 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^7\rho^2|\varphi_3|^2\mathrm{d}x\,\mathrm{d}t \right), \quad (3.39)$$

for every $s \ge C$, where

$$\widetilde{I}(s,\rho\,\varphi) := \sum_{k=1,3} \left[s^3 \iint_Q e^{-2s\alpha}\xi^3\rho^2|\Delta\varphi_k|^2\mathrm{d}x\,\mathrm{d}t + s^4 \iint_Q e^{-2s\alpha}\xi^4\rho^2|\nabla\varphi_k|^2\mathrm{d}x\,\mathrm{d}t + s^6 \iint_Q e^{-2s\alpha}\xi^6\rho^2|\varphi_k|^2\mathrm{d}x\,\mathrm{d}t \right] + s^4 \iint_Q e^{-2s\alpha^*}(\xi^*)^4\rho^2|\varphi_2|^2\mathrm{d}x\,\mathrm{d}t.$$

Estimate of φ_3. We deal in this part with the last term in the right-hand side of (3.39). We introduce a function $\zeta_1 \in \mathcal{C}^2_0(\omega)$ such that $\zeta_1 \ge 0$ and $\zeta_1 = 1$ in ω_1, and using equation (3.24) we have

$$C\,s^7 \iint_{\omega_1\times(0,T)} e^{-2s\alpha}\xi^7\rho^2|\varphi_3|^2\mathrm{d}x\,\mathrm{d}t \le Cs^7 \iint_{\omega\times(0,T)} \zeta_1 e^{-2s\alpha}\xi^7\rho^2|\varphi_3|^2\mathrm{d}x\,\mathrm{d}t = Cs^7 \iint_{\omega\times(0,T)} \zeta_1 e^{-2s\alpha}\xi^7\rho\,\varphi_3(-\widetilde{\psi}_t - \Delta\widetilde{\psi} - \rho\, g_0 + \rho'\psi)\mathrm{d}x\,\mathrm{d}t,$$

and we integrate by parts in this last term, in order to estimate it by local integrals of $\widetilde{\psi}$, g_0 and $\varepsilon\, I(s,\rho\,\varphi)$. This approach was already introduced in [35].

We first integrate by parts in time taking into account that $e^{-2s\alpha(0)}\xi^7(0) = 0$ and $e^{-2s\alpha(T)}\xi^7(T) = 0$:

$$\begin{aligned}
&-Cs^7 \iint_{\omega\times(0,T)} \zeta_1 e^{-2s\alpha}\xi^7\rho\,\varphi_3\widetilde{\psi}_t \mathrm{d}x\,\mathrm{d}t \\
&\quad = Cs^7 \iint_{\omega\times(0,T)} \zeta_1 e^{-2s\alpha}\xi^7\rho\,\varphi_{3,t}\widetilde{\psi}\mathrm{d}x\,\mathrm{d}t + Cs^7 \iint_{\omega\times(0,T)} \zeta_1 (e^{-2s\alpha}\xi^7\rho)_t\,\varphi_3\widetilde{\psi}\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \varepsilon\left(s^2 \iint_{\omega\times(0,T)} e^{-2s\alpha^*}(\xi^*)^{7/4}\rho^2|\varphi_{3,t}|^2\mathrm{d}x\,\mathrm{d}t + \widetilde{I}(s,\rho\,\varphi)\right) \\
&\quad + C(\lambda,\varepsilon)\left(s^{12}\iint_{\omega\times(0,T)} e^{-4s\alpha+2s\alpha^*}\xi^{49/4}|\widetilde{\psi}|^2\mathrm{d}x\,\mathrm{d}t + s^{10}\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^{41/4}|\widetilde{\psi}|^2\mathrm{d}x\,\mathrm{d}t\right),
\end{aligned}$$

where we have used that

$$|(e^{-2s\alpha}\xi^7\rho)_t| \le Cse^{-2s\alpha}\xi^{65/8}\rho$$

and Young's inequality. Now we integrate by parts in space :

$$\begin{aligned}
&-Cs^7 \iint_{\omega\times(0,T)} \zeta_1 e^{-2s\alpha}\xi^7\rho\,\varphi_3\Delta\widetilde{\psi}\mathrm{d}x\,\mathrm{d}t = -Cs^7 \iint_{\omega\times(0,T)} \zeta_1 e^{-2s\alpha}\xi^7\rho\,\Delta\varphi_3\widetilde{\psi}\mathrm{d}x\,\mathrm{d}t \\
&\quad - 2Cs^7 \iint_{\omega\times(0,T)} \nabla(\zeta_1 e^{-2s\alpha}\xi^7)\cdot\rho\,\nabla\varphi_3\widetilde{\psi}\mathrm{d}x\,\mathrm{d}t - C\,s^7 \iint_{\omega\times(0,T)} \Delta(\zeta_1 e^{-2s\alpha}\xi^7)\rho\,\varphi_3\widetilde{\psi}\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \varepsilon\,\widetilde{I}(s,\rho\,\varphi) + C(\varepsilon)\,s^{12}\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^{12}|\widetilde{\psi}|^2\mathrm{d}x\,\mathrm{d}t,
\end{aligned}$$

where we have used that

$$\nabla(\zeta_1 e^{-2s\alpha}\xi^7) \le Cse^{-2s\alpha}\xi^8 \quad \text{and} \quad \Delta(\zeta_1 e^{-2s\alpha}\xi^7) \le Cs^2e^{-2s\alpha}\xi^9,$$

and Young's inequality.

Finally,

$$\begin{aligned}
&Cs^7 \iint_{\omega\times(0,T)} \zeta_1 e^{-2s\alpha}\xi^7\rho\,\varphi_3(-\rho\,g_0 + \rho'\,\psi)\mathrm{d}x\,\mathrm{d}t \\
&\le Cs^7 \iint_{\omega\times(0,T)} \zeta_1 e^{-2s\alpha}\xi^7\rho\,|\varphi_3|(\rho|g_0| + Cs\xi^{9/8}\,|\widetilde{\psi}|)\mathrm{d}x\,\mathrm{d}t \\
&\le \varepsilon\,\widetilde{I}(s,\rho\,\varphi) + C\left(s^8\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^8\rho^2\,|g_0|^2\mathrm{d}x\,\mathrm{d}t + s^{10}\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^{41/4}|\widetilde{\psi}|^2\mathrm{d}x\,\mathrm{d}t\right).
\end{aligned}$$

Setting $\varepsilon = 1/6$ and noticing that

$$e^{-2s\alpha} \le e^{-4s\alpha+2s\alpha^*} \text{ in } Q,$$

(see (3.13)) we obtain (3.14) from (3.39). This completes the proof of Proposition 3.5.

3.3 Null controllability of the linear system

Here we are concerned with the null controllability of the system

$$\begin{cases}
\mathcal{L}y + \nabla p = f + (v_1,0,0)\mathbb{1}_\omega + \theta\,e_3 & \text{in } Q,\\
\mathcal{L}\theta + y\cdot\nabla\bar{\theta} = f_0 + v_0\mathbb{1}_\omega & \text{in } Q,\\
\nabla\cdot y = 0 & \text{in } Q,\\
y = 0, \quad \theta = 0 & \text{on } \Sigma,\\
y(0) = y^0, \quad \theta(0) = \theta^0 & \text{in } \Omega,
\end{cases} \tag{3.40}$$

where $y^0 \in V$, $\theta^0 \in H^1_0(\Omega)$, f and f_0 are in appropriate weighted spaces, the controls v_0 and v_1 are in $L^2(\omega\times(0,T))$ and

$$\mathcal{L}q := q_t - \Delta q.$$

Before dealing with the null controllability of (3.40), we will deduce a Carleman inequality with weights not vanishing at $t=0$. To this end, let us introduce the following weight functions :

$$\begin{aligned}
&\beta(x,t) := \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\widetilde{\ell}^8(t)}, && \gamma(x,t) := \frac{e^{\lambda\eta(x)}}{\widetilde{\ell}^8(t)},\\
&\beta^*(t) := \max_{x\in\overline{\Omega}} \beta(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - 1}{\widetilde{\ell}^8(t)}, && \gamma^*(t) := \min_{x\in\overline{\Omega}} \gamma(x,t) = \frac{1}{\widetilde{\ell}^8(t)},\\
&\widehat{\beta}(t) := \min_{x\in\overline{\Omega}} \beta(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\|\eta\|_\infty}}{\widetilde{\ell}^8(t)}, && \widehat{\gamma}(t) := \max_{x\in\overline{\Omega}} \gamma(x,t) = \frac{e^{\lambda\|\eta\|_\infty}}{\widetilde{\ell}^8(t)},
\end{aligned} \tag{3.41}$$

where

$$\widetilde{\ell}(t) = \begin{cases} \|\ell\|_\infty & 0\le t\le T/2,\\ \ell(t) & T/2 < t \le T.\end{cases}$$

Lemma 3.12. *Assume $N=3$. Let s and λ be like in Proposition 3.5 and $(\bar p,\bar\theta)$ satisfy* (3.5)-(3.6). *Then, there exists a constant $C>0$ (depending on s, λ and $\bar\theta$) such that every solution (φ,π,ψ) of* (3.9) *satisfies :*

$$\begin{aligned}
&\iint_Q e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2 dx\,dt + \iint_Q e^{-5s\beta^*}(\gamma^*)^5|\psi|^2 dx\,dt + \|\varphi(0)\|^2_{L^2(\Omega)^3} + \|\psi(0)\|^2_{L^2(\Omega)}\\
&\quad\le C\Bigg(\iint_Q e^{-3s\beta^*}(|g|^2+|g_0|^2)dx\,dt + \iint_{\omega\times(0,T)} e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7|\varphi_1|^2 dx\,dt\\
&\qquad + \iint_{\omega\times(0,T)} e^{-4s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{49/4}|\psi|^2 dx\,dt\Bigg).
\end{aligned} \tag{3.42}$$

Let us also state this result for $N=2$.

Lemma 3.13. *Assume $N=2$. Let s and λ be like in Proposition 3.6 and $(\bar p,\bar\theta)$ satisfy* (3.5)-(3.6). *Then, there exists a constant $C>0$ (depending on s, λ and $\bar\theta$) such that every solution (φ,π,ψ) of* (3.9) *satisfies :*

$$\begin{aligned}
&\iint_Q e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2 dx\,dt + \iint_Q e^{-5s\beta^*}(\gamma^*)^5|\psi|^2 dx\,dt + \|\varphi(0)\|^2_{L^2(\Omega)^2} + \|\psi(0)\|^2_{L^2(\Omega)}\\
&\quad\le C\Bigg(\iint_Q e^{-3s\beta^*}(|g|^2+|g_0|^2)dx\,dt + \iint_{\omega\times(0,T)} e^{-4s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{49/4}|\psi|^2 dx\,dt\Bigg).
\end{aligned} \tag{3.43}$$

Proof of Lemma 3.12 : We start by an a priori estimate for system (3.9). To do this, we introduce a function $\nu\in\mathcal{C}^1([0,T])$ such that

$$\nu\equiv 1 \text{ in } [0,T/2],\quad \nu\equiv 0 \text{ in } [3T/4,T].$$

We easily see that $(\nu\varphi,\nu\pi,\nu\psi)$ satisfies

$$\begin{cases}
-(\nu\varphi)_t - \Delta(\nu\varphi) + \nabla(\nu\pi) = \nu\, g - (\nu\psi)\nabla\bar\theta - \nu'\varphi & \text{in } Q,\\
-(\nu\psi)_t - \Delta(\nu\psi) = \nu\, g_0 + \nu\,\varphi_3 - \nu'\psi & \text{in } Q,\\
\nabla\cdot(\nu\varphi) = 0 & \text{in } Q,\\
(\nu\varphi) = 0,\quad (\nu\psi) = 0 & \text{on } \Sigma,\\
(\nu\varphi)(T) = 0,\quad (\nu\psi)(T) = 0 & \text{in } \Omega,
\end{cases}$$

thus we have the energy estimate

$$\|\nu\varphi\|^2_{L^2(0,T;V)} + \|\nu\varphi\|^2_{L^\infty(0,T;H)} + \|\nu\psi\|^2_{L^2(0,T;H^1(\Omega))} + \|\nu\psi\|^2_{L^\infty(0,T;L^2(\Omega))} \\ \leq C(\|\nu g\|^2_{L^2(Q)^3} + \|\nu'\varphi\|^2_{L^2(Q)^3} + \|\nu g_0\|^2_{L^2(Q)} + \|\nu'\psi\|^2_{L^2(Q)}).$$

Using the properties of the function ν, we readily obtain

$$\|\varphi\|^2_{L^2(0,T/2;H)} + \|\varphi(0)\|^2_{L^2(\Omega)^3} + \|\psi\|^2_{L^2(0,T/2;L^2(\Omega))} + \|\psi(0)\|^2_{L^2(\Omega)} \\ \leq C\left(\|g\|^2_{L^2(0,3T/4;L^2(\Omega)^3)} + \|\varphi\|^2_{L^2(T/2,3T/4;L^2(\Omega)^3)} \right. \\ \left. + \|g_0\|^2_{L^2(0,3T/4;L^2(\Omega))} + \|\psi\|^2_{L^2(T/2,3T/4;L^2(\Omega))}\right).$$

From this last inequality, and the fact that

$$e^{-3s\beta^*} \geq C > 0,\ \forall t \in [0,3T/4] \quad \text{and} \quad e^{-5s\beta^*}(\gamma^*)^4, e^{-5s\beta^*}(\gamma^*)^5 \geq C > 0,\ \forall t \in [T/2,3T/4]$$

we have

$$\iint_{\Omega\times(0,T/2)} e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{\Omega\times(0,T/2)} e^{-5s\beta^*}(\gamma^*)^5|\psi|^2 \mathrm{d}x\,\mathrm{d}t \\ + \|\varphi(0)\|^2_{L^2(\Omega)^3} + \|\psi(0)\|^2_{L^2(\Omega)} \leq C\left(\iint_{\Omega\times(0,3T/4)} e^{-3s\beta^*}(|g|^2 + |g_0|^2)\mathrm{d}x\,\mathrm{d}t \right. \\ \left. + \iint_{\Omega\times(T/2,3T/4)} e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{\Omega\times(T/2,3T/4)} e^{-5s\beta^*}(\gamma^*)^5|\psi|^2 \mathrm{d}x\,\mathrm{d}t\right). \quad (3.44)$$

Note that the last two terms in (3.44) are bounded by the left-hand side of the Carleman inequality (3.14). Since $\alpha = \beta$ in $\Omega \times (T/2,T)$, we have :

$$\iint_{\Omega\times(T/2,T)} e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{\Omega\times(T/2,T)} e^{-5s\beta^*}(\gamma^*)^5|\psi|^2 \mathrm{d}x\,\mathrm{d}t \\ = \iint_{\Omega\times(T/2,T)} e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{\Omega\times(T/2,T)} e^{-5s\alpha^*}(\xi^*)^5|\psi|^2 \mathrm{d}x\,\mathrm{d}t \\ \leq \iint_Q e^{-5s\alpha^*}(\xi^*)^4|\varphi|^2 \mathrm{d}x\,\mathrm{d}t + \iint_Q e^{-5s\alpha^*}(\xi^*)^5|\psi|^2 \mathrm{d}x\,\mathrm{d}t.$$

Combining this with the Carleman inequality (3.14), we deduce

$$\iint_{\Omega\times(T/2,T)} e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{\Omega\times(T/2,T)} e^{-5s\beta^*}(\gamma^*)^5|\psi|^2 \mathrm{d}x\,\mathrm{d}t \\ \leq C\left(\iint_Q e^{-3s\alpha^*}(|g|^2 + |g_0|^2)\mathrm{d}x\,\mathrm{d}t + \iint_{\omega\times(0,T)} e^{-2s\widehat{\alpha}-3s\alpha^*}(\widehat{\xi})^7|\varphi_1|^2 \mathrm{d}x\,\mathrm{d}t \right. \\ \left. + \iint_{\omega\times(0,T)} e^{-4s\widehat{\alpha}-s\alpha^*}(\widehat{\xi})^{49/4}|\psi|^2 \mathrm{d}x\,\mathrm{d}t\right).$$

In this inequality, we use that for every $t \in [0,T/2]$ we have

$$e^{-3s\alpha^*} \leq e^{-3s\beta^*},$$

$$e^{-2s\widehat{\alpha}-3s\alpha^*}(\widehat{\xi})^7 \leq C_1 \leq C_2 e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7,$$

$$e^{-4s\widehat{\alpha}-s\alpha^*}(\widehat{\xi})^{49/4} \leq C_1 \leq C_2 e^{-4s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{49/4},$$

for some $C_1, C_2 > 0$, and for every $t \in [T/2, T]$:

$$e^{-3s\alpha^*} = e^{-3s\beta^*},$$
$$e^{-2s\widehat{\alpha}-3s\alpha^*}(\widehat{\xi})^7 = e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7,$$
$$e^{-4s\widehat{\alpha}-s\alpha^*}(\widehat{\xi})^{49/4} = e^{-4s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{49/4}.$$

Therefore, we get

$$\begin{aligned}
&\iint_{\Omega\times(T/2,T)} e^{-5s\beta^*}(\gamma^*)^4|\varphi|^2\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega\times(T/2,T)} e^{-5s\beta^*}(\gamma^*)^5|\psi|^2\mathrm{d}x\,\mathrm{d}t \\
&\quad\leq C\left(\iint_Q e^{-3s\beta^*}(|g|^2+|g_0|^2)\mathrm{d}x\,\mathrm{d}t + \iint_{\omega\times(0,T)} e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7|\varphi_1|^2\mathrm{d}x\,\mathrm{d}t\right. \\
&\qquad\qquad\left. + \iint_{\omega\times(0,T)} e^{-4s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{49/4}|\psi|^2\mathrm{d}x\,\mathrm{d}t\right),
\end{aligned}$$

which, together with (3.44), gives (3.42). □

Now we will prove the null controllability of (3.40). Actually, we will prove the existence of a solution for this problem in an appropriate weighted space. Let us introduce the space

$$\begin{aligned}
E := \{\,(y,p,v_1,\theta,v_0) :\ & e^{3/2s\beta^*}\,y \in L^2(Q)^3, \quad e^{s\widehat{\beta}+3/2s\beta^*}\widehat{\gamma}^{-7/2}\,(v_1,0,0)\mathbb{1}_\omega \in L^2(Q)^3, \\
& e^{3/2s\beta^*}\,\theta \in L^2(Q), \quad e^{2s\widehat{\beta}+1/2s\beta^*}\widehat{\gamma}^{-49/8}\,v_0\mathbb{1}_\omega \in L^2(Q), \\
& e^{3/2s\beta^*}(\gamma^*)^{-9/8}y \in L^2(0,T;H^2(\Omega)^3)\cap L^\infty(0,T;V), \\
& e^{3/2s\beta^*}(\gamma^*)^{-9/8}\theta \in L^2(0,T;H^2(\Omega))\cap L^\infty(0,T;H_0^1(\Omega)), \\
& e^{5/2s\beta^*}(\gamma^*)^{-2}(\mathcal{L}y+\nabla p-\theta e_3-(v_1,0,0)\mathbb{1}_\omega) \in L^2(Q)^3, \\
& e^{5/2s\beta^*}(\gamma^*)^{-5/2}(\mathcal{L}\theta+y\cdot\nabla\bar{\theta}-v_0\mathbb{1}_\omega) \in L^2(Q)\,\}.
\end{aligned}$$

It is clear that E is a Banach space for the following norm :

$$\begin{aligned}
\|(y,p,v_1,\theta,v_0)\|_E := \Big(&\|e^{3/2s\beta^*}\,y\|^2_{L^2(Q)^3} + \|e^{s\widehat{\beta}+3/2s\beta^*}\widehat{\gamma}^{-7/2}\,v_1\mathbb{1}_\omega\|^2_{L^2(Q)} \\
&+\|e^{3/2s\beta^*}\,\theta\|^2_{L^2(Q)} + \|e^{2s\widehat{\beta}+1/2s\beta^*}\widehat{\gamma}^{-49/8}\,v_0\mathbb{1}_\omega\|^2_{L^2(Q)} \\
&+\|e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,y\|^2_{L^2(0,T;H^2(\Omega)^3)} + \|e^{3/2s\beta^*}(\gamma^*)^{-9/8}y\|^2_{L^\infty(0,T;V)} \\
&+\|e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,\theta\|^2_{L^2(0,T;H^2(\Omega))} + \|e^{3/2s\beta^*}(\gamma^*)^{-9/8}\theta\|^2_{L^\infty(0,T;H_0^1)} \\
&+\|e^{5/2s\beta^*}(\gamma^*)^{-2}(\mathcal{L}y+\nabla p-\theta e_3-(v_1,0,0)\mathbb{1}_\omega)\|^2_{L^2(Q)^3} \\
&+\|e^{5/2s\beta^*}(\gamma^*)^{-5/2}(\mathcal{L}\theta+y\cdot\nabla\bar{\theta}-v_0\mathbb{1}_\omega)\|^2_{L^2(Q)}\Big)^{1/2}
\end{aligned}$$

Remark 3.14. *Observe in particular that $(y,p,v_1,\theta,v_0)\in E$ implies $y(T)=0$ and $\theta(T)=0$ in Ω. Moreover, the functions belonging to this space possess the interesting following property :*

$$e^{5/2s\beta^*}(\gamma^*)^{-2}(y\cdot\nabla)y \in L^2(Q)^3 \quad \text{and} \quad e^{5/2s\beta^*}(\gamma^*)^{-5/2}y\cdot\nabla\theta \in L^2(Q).$$

Proposition 3.15. *Assume $N=3$, $(\bar{p},\bar{\theta})$ satisfies (3.5)-(3.6) and*

$$y^0\in V, \quad \theta_0\in H_0^1(\Omega), \quad e^{5/2s\beta^*}(\gamma^*)^{-2}f\in L^2(Q)^3 \quad \text{and} \quad e^{5/2s\beta^*}(\gamma^*)^{-5/2}f_0\in L^2(Q).$$

Then, we can find controls v_1 and v_0 such that the associated solution (y,p,θ) to (3.40) satisfies $(y,p,v_1,\theta,v_0)\in E$. In particular, $y(T)=0$ and $\theta(T)=0$.

Proof : The proof of this proposition is very similar to the one of Proposition 2 in [44] (see also Proposition 2 in [34] and Proposition 3.3 in [7]), so we will just give the main ideas.

Following the arguments in [39] and [51], we introduce the space

$$P_0 := \{ (\chi, \sigma, \kappa) \in C^2(\overline{Q})^5 : \nabla \cdot \chi = 0 \text{ in } Q, \quad \chi = 0 \text{ on } \Sigma, \quad \kappa = 0 \text{ on } \Sigma\}$$

and we consider the following variational problem : find $(\widehat{\chi}, \widehat{\sigma}, \widehat{\kappa}) \in P_0$ such that

$$a((\widehat{\chi}, \widehat{\sigma}, \widehat{\kappa}), (\chi, \sigma, \kappa)) = \langle G, (\chi, \sigma, \kappa)\rangle \quad \forall (\chi, \sigma, \kappa) \in P_0, \tag{3.45}$$

where we have used the notations

$$a((\widehat{\chi}, \widehat{\sigma}, \widehat{\kappa}), (\chi, \sigma, \kappa)) := \iint_Q e^{-3s\beta^*} (\mathcal{L}^*\widehat{\chi} + \nabla\widehat{\sigma} + \widehat{\kappa}\nabla\bar{\theta}) \cdot (\mathcal{L}^*\chi + \nabla\sigma + \kappa\nabla\bar{\theta}) \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_Q e^{-3s\beta^*} (\mathcal{L}^*\widehat{\kappa} - \widehat{\chi}_3)(\mathcal{L}^*\kappa - \chi_3) \,\mathrm{d}x\,\mathrm{d}t + \iint_{\omega\times(0,T)} e^{-2s\widehat{\beta}-3s\beta^*} \widehat{\gamma}^7\, \widehat{\chi}_1\, \chi_1 \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{\omega\times(0,T)} e^{-4s\widehat{\beta}-s\beta^*} \widehat{\gamma}^{49/4}\, \widehat{\kappa}\, \kappa \,\mathrm{d}x\,\mathrm{d}t,$$

$$\langle G, (\chi, \sigma, \kappa)\rangle := \iint_Q f \cdot \chi \,\mathrm{d}x\,\mathrm{d}t + \iint_Q f_0\, \kappa \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega y^0 \cdot \chi(0)\, dx + \int_\Omega \theta^0\, \kappa(0)\, dx$$

and $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$, i.e.

$$\mathcal{L}^* q := -q_t - \Delta q.$$

It is clear that $a(\cdot\,,\cdot\,,\cdot) : P_0 \times P_0 \mapsto \mathbb{R}$ is a symmetric, definite positive bilinear form on P_0. We denote by P the completion of P_0 for the norm induced by $a(\cdot\,,\cdot\,,\cdot)$. Then $a(\cdot\,,\cdot\,,\cdot)$ is well-defined, continuous and again definite positive on P. Furthermore, in view of the Carleman estimate (3.42), the linear form $(\chi, \sigma, \kappa) \mapsto \langle G, (\chi, \sigma, \kappa)\rangle$ is well-defined and continuous on P. Hence, from Lax-Milgram's lemma, we deduce that the variational problem

$$\begin{cases} a((\widehat{\chi}, \widehat{\sigma}, \widehat{\kappa}), (\chi, \sigma, \kappa)) = \langle G, (\chi, \sigma, \kappa)\rangle \\ \forall (\chi, \sigma, \kappa) \in P, \quad (\widehat{\chi}, \widehat{\sigma}, \widehat{\kappa}) \in P, \end{cases}$$

possesses exactly one solution $(\widehat{\chi}, \widehat{\sigma}, \widehat{\kappa})$.

Let $\widehat{y}$, $\widehat{v}_1$, $\widehat{\theta}$ and $\widehat{v}_0$ be given by

$$\begin{cases} \widehat{y} := e^{-3s\beta^*}(\mathcal{L}^*\widehat{\chi} + \nabla\widehat{\sigma} + \widehat{\kappa}\nabla\bar{\theta}), & \text{in } Q, \\ \widehat{v}_1 := -e^{-2s\widehat{\beta}-3s\beta^*}\widehat{\gamma}^7\, \widehat{\chi}_1, & \text{in } \omega\times(0,T), \\ \widehat{\theta} := e^{-3s\beta^*}(\mathcal{L}^*\widehat{\kappa} - \widehat{\chi}_3), & \text{in } Q, \\ \widehat{v}_0 := -e^{-4s\widehat{\beta}-s\beta^*}\widehat{\gamma}^{49/4}\, \widehat{\kappa}, & \text{in } \omega\times(0,T). \end{cases}$$

Then, it is readily seen that they satisfy

$$\iint_Q e^{3s\beta^*} |\widehat{y}|^2 \mathrm{d}x\,\mathrm{d}t + \iint_Q e^{3s\beta^*} |\widehat{\theta}|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{\omega\times(0,T)} e^{2s\widehat{\beta}+3s\beta^*} \widehat{\gamma}^{-7} |\widehat{v}_1|^2 \mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{\omega\times(0,T)} e^{4s\widehat{\beta}+s\beta^*} \widehat{\gamma}^{-49/4} |\widehat{v}_0|^2 \mathrm{d}x\,\mathrm{d}t = a((\widehat{\chi}, \widehat{\sigma}, \widehat{\kappa}), (\widehat{\chi}, \widehat{\sigma}, \kappa)) < +\infty$$

and also that $(\widehat{y}, \widehat{\theta})$ is, together with some pressure $\widehat{p}$, the weak solution of the system (3.40) for $v_1 = \widehat{v}_1$ and $v_0 = \widehat{v}_0$.

It only remains to check that

$$e^{3/2s\beta^*}(\gamma^*)^{-9/8}\widehat{y} \in L^2(0,T;H^2(\Omega)^3) \cap L^\infty(0,T;V)$$

and

$$e^{3/2s\beta^*}(\gamma^*)^{-9/8}\widehat{\theta} \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$$

To this end, we define the functions

$$y^* := e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,\widehat{y}, \quad p^* := e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,\widehat{p}, \quad \theta^* := e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,\widehat{\theta},$$

$$f^* := e^{3/2s\beta^*}(\gamma^*)^{-9/8}(f + (\widehat{v}_1, 0, 0)\mathbb{1}_\omega) \quad \text{and} \quad f_0^* := e^{3/2s\beta^*}(\gamma^*)^{-9/8}(f_0 + \widehat{v}_0\mathbb{1}_\omega).$$

Then (y^*, p^*, θ^*) satisfies

$$\begin{cases} \mathcal{L}y^* + \nabla p^* = f^* + \theta^*\, e_3 + (e^{3/2s\beta^*}(\gamma^*)^{-9/8})_t\widehat{y} & \text{in } Q, \\ \mathcal{L}\theta^* + y^*\cdot\nabla\bar{\theta} = f_0^* + (e^{3/2s\beta^*}(\gamma^*)^{-9/8})_t\widehat{\theta} & \text{in } Q, \\ \nabla\cdot y^* = 0 & \text{in } Q, \\ y^* = 0, \quad \theta^* = 0 & \text{on } \Sigma, \\ y^*(0) = e^{3/2s\beta^*(0)}(\gamma^*(0))^{-9/8}y^0, & \text{in } \Omega, \\ \theta^*(0) = e^{3/2s\beta^*(0)}(\gamma^*(0))^{-9/8}\theta^0, & \text{in } \Omega. \end{cases}$$

From the fact that $f^* + (e^{3/2s\beta^*}(\gamma^*)^{-9/8})_t\,\widehat{y} \in L^2(Q)^3$, $f_0^* + (e^{3/2s\beta^*}(\gamma^*)^{-9/8})_t\widehat{\theta} \in L^2(Q)$, $y^0 \in V$ and $\theta^0 \in H_0^1(\Omega)$, we have indeed

$$y^* \in L^2(0,T;H^2(\Omega)^3) \cap L^\infty(0,T;V) \quad \text{and} \quad \theta^* \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$$

(see (3.20)). This ends the proof of Proposition 3.15. □

3.4 Proof of Theorem 3.1

In this section we give the proof of Theorem 3.1 using similar arguments to those in [51] (see also [34], [35], [44] and [7]). The result of null controllability for the linear system (3.40) given by Proposition 3.15 will allow us to apply an inverse mapping theorem. Namely, we will use the following theorem (see [3]).

Theorem 3.16. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces and let $\mathcal{A} : \mathcal{B}_1 \to \mathcal{B}_2$ satisfy $\mathcal{A} \in \mathcal{C}^1(\mathcal{B}_1;\mathcal{B}_2)$. Assume that $b_1 \in \mathcal{B}_1$, $\mathcal{A}(b_1) = b_2$ and that $\mathcal{A}'(b_1) : \mathcal{B}_1 \to \mathcal{B}_2$ is surjective. Then, there exists $\delta > 0$ such that, for every $b' \in \mathcal{B}_2$ satisfying $\|b' - b_2\|_{\mathcal{B}_2} < \delta$, there exists a solution of the equation*

$$\mathcal{A}(b) = b', \quad b \in \mathcal{B}_1.$$

Let us set

$$y := \widetilde{y}, \quad p := \bar{p} + \widetilde{p} \quad \text{and} \quad \theta := \bar{\theta} + \widetilde{\theta}.$$

Using (3.1) and (3.5) we obtain

$$\begin{cases} \widetilde{y}_t - \Delta\widetilde{y} + (\widetilde{y}\cdot\nabla)\widetilde{y} + \nabla\widetilde{p} = v\mathbb{1}_\omega + \widetilde{\theta}\, e_N & \text{in } Q, \\ \widetilde{\theta}_t - \Delta\widetilde{\theta} + \widetilde{y}\cdot\nabla\widetilde{\theta} + \widetilde{y}\cdot\nabla\bar{\theta} = v_0\mathbb{1}_\omega & \text{in } Q, \\ \nabla\cdot\widetilde{y} = 0 & \text{in } Q, \\ \widetilde{y} = 0, \quad \widetilde{\theta} = 0 & \text{on } \Sigma, \\ \widetilde{y}(0) = y^0, \quad \widetilde{\theta}(0) = \theta^0 - \bar{\theta}^0 & \text{in } \Omega. \end{cases} \tag{3.46}$$

Thus, we have reduced our problem to the local null controllability of the nonlinear system (3.46).

We apply Theorem 3.16 setting

$$\mathcal{B}_1 := E,$$

$$\mathcal{B}_2 := L^2(e^{5/2s\beta^*}(\gamma^*)^{-2}(0,T);L^2(\Omega)^3) \times V \times L^2(e^{5/2s\beta^*}(\gamma^*)^{-5/2}(0,T);L^2(\Omega)) \times H_0^1(\Omega)$$

and the operator

$$\begin{aligned}\mathcal{A}(\widetilde{y},\widetilde{p},v_1,\widetilde{\theta},v_0) := (&\mathcal{L}\widetilde{y} + (\widetilde{y}\cdot\nabla)\widetilde{y} + \nabla\widetilde{p} - \widetilde{\theta}\, e_3 - (v_1,0,0)\mathbb{1}_\omega,\ \widetilde{y}(0),\\ &\mathcal{L}\widetilde{\theta} + \widetilde{y}\cdot\nabla\widetilde{\theta} + \widetilde{y}\cdot\nabla\bar{\theta} - v_0\mathbb{1}_\omega,\ \widetilde{\theta}(0))\end{aligned}$$

for $(\widetilde{y},\widetilde{p},v_1,\widetilde{\theta},v_0) \in \mathcal{B}_1$.

In order to apply Theorem 3.16, it remains to check that the operator $\mathcal{A}$ is of class $\mathcal{C}^1(\mathcal{B}_1;\mathcal{B}_2)$. Indeed, notice that all the terms in $\mathcal{A}$ are linear, except for $(\widetilde{y}\cdot\nabla)\widetilde{y}$ and $\widetilde{y}\cdot\nabla\widetilde{\theta}$. We will prove that the bilinear operator

$$((y^1,p^1,v_1^1,\theta^1,v_0^1),(y^2,p^2,v_1^2,\theta^2,v_0^2)) \to (y^1\cdot\nabla)y^2$$

is continuous from $\mathcal{B}_1\times\mathcal{B}_1$ to $L^2(e^{5/2s\beta^*}(\gamma^*)^{-2}(0,T);L^2(\Omega)^3)$. To do this, notice that

$$e^{3/2s\beta^*}(\gamma^*)^{-9/8}y \in L^2(0,T;H^2(\Omega)^3)\cap L^\infty(0,T;V)$$

for any $(y,p,v_1,\theta,v_0)\in\mathcal{B}_1$, so we have

$$e^{3/2s\beta^*}(\gamma^*)^{-9/8}y \in L^2(0,T;L^\infty(\Omega)^3)$$

and

$$\nabla(e^{3/2s\beta^*}(\gamma^*)^{-9/8}y) \in L^\infty(0,T;L^2(\Omega)^3).$$

Consequently, we obtain

$$\begin{aligned}&\|e^{5/2s\beta^*}(\gamma^*)^{-2}(y^1\cdot\nabla)y^2\|_{L^2(Q)^3}\\ &\le C\|(e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,y^1\cdot\nabla)e^{3/2s\beta^*}(\gamma^*)^{-9/8}\,y^2\|_{L^2(Q)^3}\\ &\le C\|e^{3/2s\beta^*}(\gamma^*)^{-9/8}y^1\|_{L^2(0,T;L^\infty(\Omega)^3)}\,\|e^{3/2s\beta^*}(\gamma^*)^{-9/8}y^2\|_{L^\infty(0,T;V)}.\end{aligned}$$

In the same way, we can prove that the bilinear operator

$$((y^1,p^1,v_1^1,\theta^1,v_0^1),(y^2,p^2,v_1^2,\theta^2,v_0^2)) \to y^1\cdot\nabla\theta^2$$

is continuous from $\mathcal{B}_1\times\mathcal{B}_1$ to $L^2(e^{5/2s\beta^*}(\gamma^*)^{-5/2}(0,T);L^2(\Omega))$ just by taking into account that

$$e^{3/2s\beta^*}(\gamma^*)^{-9/8}\theta \in L^\infty(0,T;H_0^1(\Omega)),$$

for any $(y,p,v_1,\theta,v_0)\in\mathcal{B}_1$.

Notice that $\mathcal{A}'(0,0,0,0,0):\mathcal{B}_1\to\mathcal{B}_2$ is given by

$$\begin{aligned}\mathcal{A}'(0,0,0,0,0)(\widetilde{y},\widetilde{p},v_1,\widetilde{\theta},v_0) = (&\mathcal{L}\widetilde{y} + \nabla\widetilde{p} - \widetilde{\theta}\, e_3 - (v_1,0,0)\mathbb{1}_\omega,\ \widetilde{y}(0),\\ &\mathcal{L}\widetilde{\theta} + \widetilde{y}\cdot\nabla\bar{\theta} - v_0\mathbb{1}_\omega,\ \widetilde{\theta}(0)),\end{aligned}$$

for all $(\widetilde{y},\widetilde{p},v_1,\widetilde{\theta},v_0)\in\mathcal{B}_1$, so this functional is surjective in view of the null controllability result for the linear system (3.40) given by Proposition 3.15.

We are now able to apply Theorem 3.16 for $b_1 = (0,0,0,0,0)$ and $b_2 = (0,0,0,0)$. In particular, this gives the existence of a positive number $\delta > 0$ such that, if $\|\widetilde{y}(0),\widetilde{\theta}(0)\|_{V\times H_0^1(\Omega)} \le \delta$, then we can find controls v_1 and v_0 such that the associated solution $(\widetilde{y},\widetilde{p},\widetilde{\theta})$ to (3.46) satisfies $\widetilde{y}(T) = 0$ and $\widetilde{\theta}(T) = 0$ in Ω.

This concludes the proof of Theorem 3.1.

Chapitre 4

Insensitizing controls with vanishing components for the Boussinesq system

In this chapter we prove the existence of insensitizing controls for a viscous newtonian fluid wherein thermic effects are taken into account, the so called Boussinesq system. The proof relies on a standard approach introduced by Fursikov and Imanuvilov for the Navier-Stokes system which consists in solving a constrained extremal problem, and then on an inverse mapping theorem to deal with the nonlinear system. Furthermore, we use the coupling with the heat equation to get rid of two components of the control in the fluid equations.

This chapter is included in [8], which has been written in collaboration with S. Guerrero and M. Gueye.

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4.1 Introduction

Let Ω be a nonempty bounded connected open subset of $\mathbb{R}^N$ ($N = 2$ or 3) of class $\mathcal{C}^\infty$. Let $T > 0$ and let $\omega \subset \Omega$ be a (small) nonempty open subset which is the *control set*. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$. Let us also introduce another open set $\mathcal{O} \subset \Omega$ which is called the *observatory* or *observation set*.

Let us recall the definition of some usual spaces in the context of incompressible fluids :

$$\mathcal{V} = \{y \in \mathcal{C}_0^\infty(\Omega)^N \, : \, \nabla \cdot y = 0 \text{ in } \Omega\}.$$

We denote by H the closure of the space $\mathcal{V}$ in $L^2(\Omega)$ and by V its closure in $H_0^1(\Omega)$.

We introduce the following Boussinesq control system with incomplete data :

$$\begin{cases} y_t - \Delta y + (y \cdot \nabla)y + \nabla p = f + v\mathbb{1}_\omega + \theta\, e_N, \quad \nabla \cdot y = 0 & \text{in } Q, \\ \theta_t - \Delta\theta + y \cdot \nabla\theta = f_0 + v_0\mathbb{1}_\omega & \text{in } Q, \\ y = 0, \quad \theta = 0 & \text{on } \Sigma, \\ y(0) = y^0 + \tau\widehat{y}_0, \quad \theta(0) = \theta^0 + \tau\widehat{\theta}_0 & \text{in } \Omega. \end{cases} \tag{4.1}$$

Here,

$$e_N = \begin{cases} (0,1) & \text{if } N = 2, \\ (0,0,1) & \text{if } N = 3, \end{cases}$$

stands for the gravity vector field, $y = y(x,t)$ represents the velocity of the particules of the fluid, $\theta = \theta(x,t)$ their temperature, $(v_0, v) = (v_0, v_1, \ldots, v_N)$ stands for the control which acts over the set ω, $(f, f_0) \in L^2(Q)^{N+1}$ is a given externally applied force and the initial state $(y(0), \theta(0))$ is partially unknown in the following sense :

- $y^0 \in H$ and $\theta^0 \in L^2(\Omega)$ are known,
- $\widehat{y}_0 \in H$ and $\widehat{\theta}_0 \in L^2(\Omega)$ are unknown with $\|\widehat{y}_0\|_{L^2(\Omega)^N} = \left\|\widehat{\theta}_0\right\|_{L^2(\Omega)} = 1$, and
- τ is a small unknown real number.

We observe the solution of system (4.1) via some functional $J_\tau(y,\theta)$, which is called the *sentinel*. Here, the sentinel is given by the square of the local L^2-norm of the state variables :

$$J_\tau(y,\theta) := \frac{1}{2}\iint_{\mathcal{O}\times(0,T)} \left(|y|^2 + |\theta|^2\right) \mathrm{d}x\,\mathrm{d}t. \tag{4.2}$$

The insensitizing control problem is to find (v, v_0) such that the uncertainty in the initial data does not affect the measurement J_τ, at least at the first order, i.e.,

$$\left.\frac{\partial J_\tau(y,\theta)}{\partial\tau}\right|_{\tau=0} = 0 \quad \forall\,(\widehat{y}_0, \widehat{\theta}_0) \in L^2(\Omega)^{N+1} \text{ such that } \|\widehat{y}_0\|_{L^2(\Omega)^N} = \left\|\widehat{\theta}_0\right\|_{L^2(\Omega)} = 1. \tag{4.3}$$

If (4.3) holds, we say that v insensitizes the functional J_τ. This kind of problem was first considered by J.-L. Lions in [65].

The first results of existence of insensitizing controls were obtained for the heat equation in [5, 25]. Both papers are concerned with a sentinel of the kind (4.2).

A very close subject is controllability with controls having some vanishing components, which may be an interesting field from the applications point of view. The first results were obtained in [35] for the local exact controllability to the trajectories of the Navier-Stokes and Boussinesq system when the closure of the control set ω intersects the boundary of Ω. Later, this geometric assumption was removed for the Stokes system in [23], for the local null controllability of the Navier-Stokes system in [7] and in [6] for the Boussinesq system. Recently, the local null controllability of the three dimensional Navier-Stokes system with a control having two vanishing components has been obtained in [24].

As long as insensitivity results for fluids is concerned, the first result was obtained in [70, section 2.3], where the author establishes the existence of ε-insensitizing controls of the form $(v_1, v_2, 0)$ for the $3D$-Stokes system. Then, the existence of insensitizing controls for the Stokes system is proved in [45] and for the Navier-Stokes system in [49]. Finally, in [9], the existence of insensitizing controls for the Navier-Stokes system with one vanishing component was established. The main goal of this chapter is to establish the existence of insensitizing controls for the Boussinesq system (4.1) having two vanishing components, that is, $v_{i_0} \equiv 0$ for any given $0 < i_0 < N$ and $v_N \equiv 0$. Notice that if $N = 2$, this means $v \equiv 0$.

The particular form of the sentinel J_τ allows us to reformulate the insensitizing problem (4.1)-(4.3) as a controllability problem for a cascade system (see Proposition 1.14. See also [5] or [64], for instance). More precisely, condition (4.3) is equivalent to $(z(0), q(0)) = 0$ in Ω, where (z, q), together with (w, r), solves the following coupled system :

$$\begin{cases} w_t - \Delta w + (w \cdot \nabla) w + \nabla p_0 = f + v \mathbb{1}_\omega + r\, e_N, & \nabla \cdot w = 0 \quad \text{in } Q, \\ -z_t - \Delta z + (z \cdot \nabla^t) w - (w \cdot \nabla) z + q \nabla r + \nabla p_1 = w \mathbb{1}_{\mathcal{O}}, & \nabla \cdot z = 0 \quad \text{in } Q, \\ r_t - \Delta r + w \cdot \nabla r = f_0 + v_0 \mathbb{1}_\omega & \text{in } Q, \\ -q_t - \Delta q - w \cdot \nabla q = z_N + r\, \mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ w = z = 0, \quad r = q = 0 & \text{on } \Sigma, \\ w(0) = y^0, \quad z(T) = 0, \quad r(0) = \theta^0, \quad q(T) = 0 & \text{in } \Omega. \end{cases} \tag{4.4}$$

Here, (w, p_0, r) is the solution of system (4.1) for $\tau = 0$, the equation of (z, p_1, q) corresponds to a formal adjoint of the equation satisfied by the derivative of (y, θ) with respect to τ at $\tau = 0$ and we have denoted :

$$((z, \nabla^t) w)_i = \sum_{j=1}^{N} z_j \partial_i w_j \quad i = 1, \dots, N.$$

Our main result is stated in the following theorem, which expresses local null-controllability of (4.4) :

Theorem 4.1. *Let $0 < i_0 < N$ and $m \geq 10$ be a real number. Assume that $\omega \cap \mathcal{O} \neq \emptyset$, $y^0 \equiv 0$ and $\theta^0 \equiv 0$. Then, there exist $\delta > 0$ and $C > 0$, depending on ω, Ω, $\mathcal{O}$ and T, such that for any $f \in L^2(Q)^N$ and any $f_0 \in L^2(Q)$ satisfying $\|e^{C/t^m}(f, f_0)\|_{L^2(Q)^{N+1}} < \delta$, there exists a control $(v, v_0) \in L^2(Q)^{N+1}$ with $v_{i_0} \equiv v_N \equiv 0$ such that the corresponding solution (w, z, r, q) to (4.4) satisfies $z(0) = 0$ and $q(0) = 0$ in Ω.*

In particular, no control on the velocity equation is required when $N = 2$.

Remark 4.2. *The condition $\omega \cap \mathcal{O} \neq \emptyset$ has always been imposed as long as insensitizing controls are concerned. However, in [54], it has been proved that this is not a necessary condition for ε-insensitizing controls for some linear parabolic equations (see also [69]).*

Remark 4.3. *In [25], the author proved for the linear heat equation that we cannot expect insensitivity to hold for all initial data, except when the control acts everywhere in Ω. Thus, we shall assume that $y^0 \equiv 0$ and $\theta^0 \equiv 0$.*

To prove Theorem 4.1 we follow a standard approach introduced in [39] (see also [6], [9] and [51]). We first deduce a null controllability result for the linear system :

$$\begin{cases} w_t - \Delta w + \nabla p_0 = f^w + v\, \mathbb{1}_\omega + r e_N, & \nabla \cdot w = 0 \quad \text{in } Q, \\ -z_t - \Delta z + \nabla p_1 = f^z + w \mathbb{1}_{\mathcal{O}}, & \nabla \cdot z = 0 \quad \text{in } Q, \\ r_t - \Delta r = f^r + v_0\, \mathbb{1}_\omega & \text{in } Q, \\ -q_t - \Delta q = f^q + z_N + r \mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ w = z = 0, \quad r = q = 0 & \text{on } \Sigma, \\ w(0) = y^0, \quad z(T) = 0, \quad r(0) = \theta^0, \quad q(T) = 0 & \text{in } \Omega, \end{cases} \tag{4.5}$$

where f^w, f^z, f^r and f^q will be taken to decrease exponentially to zero at $t = 0$.

The main tool to prove this controllability result for system (4.5), and the second main result of this chapter, is a suitable Carleman estimate for the solutions of its adjoint system, namely,

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_\varphi = g^\varphi + \psi\,\mathbb{1}_{\mathcal{O}}, & \nabla\cdot\varphi = 0 & \text{in } Q,\\ \psi_t - \Delta\psi + \nabla\pi_\psi = g^\psi + \sigma e_N, & \nabla\cdot\psi = 0 & \text{in } Q,\\ -\phi_t - \Delta\phi = g^\phi + \varphi_N + \sigma\,\mathbb{1}_{\mathcal{O}} & & \text{in } Q,\\ \sigma_t - \Delta\sigma = g^\sigma & & \text{in } Q,\\ \varphi = \psi = 0, \quad \phi = \sigma = 0 & & \text{on } \Sigma,\\ \varphi(T) = 0, \quad \psi(0) = \psi^0, \quad \phi(T) = 0, \quad \sigma(0) = \sigma^0 & & \text{in } \Omega. \end{cases} \tag{4.6}$$

Here, $\psi^0 \in H$, $\sigma^0 \in L^2(\Omega)$ and g^φ, g^ψ, g^ϕ and g^σ will be taken with different regularity properties that will be detailed later on. In fact, this Carleman inequality is of the form

$$\begin{aligned} \iint_Q \widetilde{\rho}_1(t)\left(|\varphi|^2 + |\psi|^2 + |\phi|^2 + |\sigma|^2\right) \mathrm{d}x\,\mathrm{d}t \le C \left\|\widetilde{\rho}_2(t)(g^\varphi, g^\psi, g^\phi, g^\sigma)\right\|_X^2 \\ + (N-2)C \iint_{\omega\times(0,T)} \widetilde{\rho}_3(t)|\varphi_{j_0}|^2\,\mathrm{d}x\,\mathrm{d}t + C \iint_{\omega\times(0,T)} \widetilde{\rho}_4(t)|\phi|^2\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{4.7}$$

where $\widetilde{\rho}_k(t)$, $k \in \{1,2,3,4\}$ are positive weight functions, $j_0 \in \{1,\dots,N-1\}\setminus\{i_0\}$, $C > 0$ only depends on Ω, ω, $\mathcal{O}$ and T and X is a suitable Banach space. Observe that when $N = 2$ the local term in φ_{j_0} does not appear. This estimate is stated in Proposition 4.15.

Since the proof of (4.7) is quite technical, let us first give some insight about it. In particular, we are going to prove here the qualitative property corresponding to (4.7), that is to say

$$(N-2)\varphi_{j_0} \equiv \phi \equiv 0 \quad \text{in } \omega\times(0,T) \Rightarrow \psi(T) \equiv \sigma(T) \equiv 0 \quad \text{in } \Omega. \tag{4.8}$$

for the solutions of (4.6) with $(g^\varphi, g^\psi, g^\phi, g^\sigma) = (0,0,0,0)$.

Observe that, thanks to the backwards uniqueness for the heat and Stokes operator, $\psi(T) \equiv 0$ and $\sigma(T) \equiv 0$ in Ω implies that $\varphi \equiv \psi \equiv 0$ and $\phi \equiv \sigma \equiv 0$ in Q.

Remark that (4.8) is equivalent to the approximate controllability of system (4.5) with $(f^w, f^z, f^r, f^q) = (0,0,0,0)$

Proposition 4.4. *Assume that $\omega\cap\mathcal{O} \neq \emptyset$ and $0 < j_0 < N$. Then, the unique continuation property* (4.8) *holds.*

Remark 4.5. *The proof of this unique continuation result follows the steps of the proof of* (4.7)*. This is why we are going to divide its proof in several steps. In the lines below, we will make precise to which part of the proof of* (4.7) *corresponds each step.*

Proof of Proposition 4.4. Without loss of generality, we can assume that $\psi^0 \in \mathcal{V}$. Moreover, we can assume that $\sigma^0 \in \mathcal{C}^\infty(\overline{\Omega})$. Now let us introduce the differential operators

$$\mathcal{D} := \partial_1^2 + (N-2)\partial_2^2, \quad \mathcal{P} := \Delta^3 - \Delta\partial_t^2. \tag{4.9}$$

We divide the proof in several steps :

Step 1. First we prove that the solutions of (4.6) satisfy

$$\phi \equiv 0 \quad \text{in } \omega\times(0,T) \Rightarrow \mathcal{D}\sigma \equiv 0 \quad \text{in } Q.$$

Straightforward computations, using in particular that $\Delta\pi_\varphi = 0$ in $\mathcal{O}\times(0,T)$ and $\Delta\pi_\psi = \partial_N\sigma$ in Q, show that (for more details, see Section 4.3.3)

$$-\mathcal{P}\phi_t - \Delta\mathcal{P}\phi = \mathcal{D}\sigma \quad \text{in } \mathcal{O}\times(0,T).$$

Then, we deduce that $\mathcal{D}\sigma = 0$ in $(\omega\cap\mathcal{O})\times(0,T)$. Furthermore, since the equation of σ is homogeneous, $\mathcal{D}\sigma$ solves the heat equation

$$(\mathcal{D}\sigma)_t - \Delta(\mathcal{D}\sigma) = 0 \quad \text{in } Q.$$

Then, from the parabolic unique continuation (see [75]) we deduce that $\mathcal{D}\sigma = 0$ in Q.

Step 1 corresponds to estimate (4.48) in the proof of (4.7).

Step 2. Here, we observe that

$$\mathcal{D}\sigma \equiv 0 \quad \text{in } Q \Rightarrow \sigma \equiv 0 \quad \text{in } Q.$$

This fact is trivial from $\sigma = 0$ on Σ.

Step 2 corresponds to estimate (4.34) in the proof of (4.7).

Step 3.

$$\phi \equiv 0 \text{ in } \omega \times (0,T),\ \sigma \equiv 0 \text{ in } Q \Rightarrow \psi_N \equiv 0 \text{ in } Q.$$

From $\phi \equiv 0$ in $\omega \times (0,T)$ and $\sigma \equiv 0$ in Q we get $\varphi_N \equiv 0$ in $\omega \times (0,T)$ (from the equation of ϕ). Thus, from the equation satisfied by φ_N we have

$$\Delta\psi_N = -(\Delta\varphi_N)_t - \Delta(\Delta\varphi_N) = 0 \text{ in } (\omega \cap \mathcal{O}) \times (0,T).$$

Observe that $\Delta\psi_N$ satisfies the heat equation

$$(\Delta\psi_N)_t - \Delta(\Delta\psi_N) = 0 \text{ in } Q.$$

Making use of the parabolic unique continuation property we deduce that $\Delta\psi_N \equiv 0$ in Q. Then, the homogeneous Dirichlet boundary condition gives that $\psi_N \equiv 0$ in Q.

In dimension 2, this implies that $\psi \equiv 0$ in Q, since $\nabla \cdot \psi = 0$ in Q and $\psi|_\Sigma = 0$. This concludes the proof of Proposition 4.4 when $N = 2$. Observe that we did not use that $\varphi_{j_0} \equiv 0$ in $\omega \times (0,T)$.

Step 4. The last step will be to prove that if $\sigma \equiv 0$ in Q, then

$$\varphi_{j_0} \equiv \phi \equiv 0 \text{ in } \omega \times (0,T) \ \Rightarrow \psi \equiv 0 \text{ in } Q,$$

for $0 < j_0 < N$. Using the fact that $\Delta\pi_\varphi = 0$ in $\mathcal{O} \times (0,T)$, we find as before

$$\Delta\psi_{j_0} = -(\Delta\varphi_{j_0})_t - \Delta(\Delta\varphi_{j_0}) = 0 \text{ in } (\omega \cap \mathcal{O}) \times (0,T).$$

From $\sigma \equiv 0$ in Q, we have that $\Delta\pi_\psi \equiv 0$ in Q and so

$$(\Delta\psi_{j_0})_t - \Delta(\Delta\psi_{j_0}) = 0 \quad \text{in } Q. \tag{4.10}$$

Thus, the parabolic unique continuation property applied to (4.10) gives $\Delta\psi_{j_0} \equiv 0$ in Q. Consequently, $\psi_{j_0} \equiv 0$ in Q, since $\psi_{j_0|\Sigma} = 0$.

This, together with $\psi_N \equiv 0$ in Q and combined with $\nabla \cdot \psi = 0$ in Q and $\psi_{|\Sigma} = 0$ yield $\psi \equiv 0$ in Q. □

Remark 4.6. *Notice that after the first 3 steps, we have that* $\psi_3 \equiv 0$ *in* Q *and* $\sigma \equiv 0$ *in* Q *using only* $\phi \equiv 0$ *in* $\omega \times (0,T)$. *From previous results [26, 67], we can deduce that* $\psi \equiv 0$ *in* Q *under some particular generic geometric conditions, that is, we are able to obtain the approximate controllability of system* (4.1) *with just a control in the heat equation.*

The rest of the chapter is organized as follows. In section 4.2, we present some notation and the technical results we need. In section 4.3, we prove the observability inequality (4.7). In section 4.4, we prove a null controllability result for the linear system (4.5). Finally, by means of a inverse mapping theorem, we prove Theorem 4.1 in section 4.5.

4.2 Technical results and notations

In this section we introduce some notation and all the technical results needed in this work.

4.2.1 Some notations

We denote by $X_0 := L^2(Q)$ and $Y_0 := L^2(0,T;H)$. For n a positive integer we define the spaces X_n and Y_n as follows :

$$X_n := L^2(0,T;H^{2n}(\Omega)\cap H_0^1(\Omega))\cap H^n(0,T;L^2(\Omega)),$$

$$Y_n := L^2(0,T;H^{2n}(\Omega)^N\cap V)\cap H^n(0,T;L^2(\Omega)^N),$$

endowed with the norms

$$\|u\|^2_{X_n} := \|u\|^2_{L^2(0,T;H^{2n}(\Omega))} + \|u\|^2_{H^n(0,T;L^2(\Omega))}$$

and

$$\|u\|^2_{Y_n} := \|u\|^2_{L^2(0,T;H^{2n}(\Omega)^N)} + \|u\|^2_{H^n(0,T;L^2(\Omega)^N)},$$

respectively.

The following subspaces will be used only in section 4.4. First, for every positive integer n, we set

$$X_{n,0} := \{u\in X_n : [\mathcal{L}^k u]_{|\Sigma} = 0,\ [\mathcal{L}^k u](0) = 0,\ k = 0,\ldots,n-1\},$$

endowed with the equivalent norm (by Lemma 4.13),

$$\|u\|^2_{X_{n,0}} := \|\mathcal{L}^n u\|^2_{L^2(Q)}$$

where we have denoted by

$$\mathcal{L} := \partial_t - \Delta \quad \text{and} \quad \mathcal{L}^* := -\partial_t - \Delta$$

the heat operator and its adjoint, respectively.

Next, let

$$Y_{1,0} := \{u\in Y_1 :\ u(0) = 0\}$$

and

$$Y_{2,0} := \{u\in Y_1\cap L^2(0,T;H^4(\Omega)^N)\cap H^2(0,T;L^2(\Omega)^N) : (\mathcal{L}_H u)_{|\Sigma} = 0,\ (\mathcal{L}_H u)(0) = 0\}$$

endowed with the equivalent norm (by Lemma 4.14 with $u_0\equiv 0$)

$$\|u\|^2_{Y_{n,0}} := \|\mathcal{L}_H^n u\|^2_{L^2(Q)^N}, \quad n = 1,2.$$

Here, $\mathcal{L}_H := \partial_t - \mathcal{P}_L(\Delta)$, where $\mathcal{P}_L$ denotes the Leray projector over the space H, i.e. $\mathcal{P}_L : L^2(Q)^N \to L^2(Q)^N$, $\mathcal{P}_L u := u - \nabla p$, where $\Delta p = \nabla\cdot u$ in Ω and $\nabla p\cdot\vec{n} = u\cdot\vec{n}$ on $\partial\Omega$ (see [76, pages 16-18]).

This equivalence between norms is used later to obtain Lemma (4.16).

4.2.2 Carleman estimates

In this subsection we present some Carleman estimates needed to prove estimate (4.7). These inequalities have been proved in previous papers and we give precise references about where to find each one of them. Before we can establish these estimates, let us introduce some classical weight functions. Let ω_0 be a nonempty open subset of $\mathbb{R}^N$ such that $\omega_0 \Subset \omega \cap \mathcal{O}$ and $\eta \in \mathcal{C}^2(\overline{\Omega})$ such that

$$|\nabla\eta| > 0 \text{ in } \overline{\Omega\setminus\omega_0},\ \eta > 0 \text{ in } \Omega \text{ and } \eta \equiv 0 \text{ on } \partial\Omega.$$

The existence of such a function η is given in [39]. Let also $\ell \in C^\infty([0,T])$ be a positive function in $(0,T)$ satisfying

$$\begin{aligned}&\ell(t) = t \quad \forall\, t \in [0, T/4],\ \ell(t) = T - t \quad \forall\, t \in [3T/4, T],\\ &\ell(t) \leq \ell(T/2),\ \forall\, t \in [0,T].\end{aligned}$$

Then, for all $\lambda \geq 1$ and $m \geq 10$ we consider the following weight functions :

$$\begin{aligned}
&\alpha(x,t) := \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\ell^m(t)}, && \xi(x,t) := \frac{e^{\lambda\eta(x)}}{\ell^m(t)},\\
&\alpha^*(t) := \max_{x\in\overline{\Omega}} \alpha(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - 1}{\ell^m(t)}, && \xi^*(t) := \min_{x\in\overline{\Omega}} \xi(x,t) = \frac{1}{\ell^m(t)},\\
&\widehat{\alpha}(t) := \min_{x\in\overline{\Omega}} \alpha(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\|\eta\|_\infty}}{\ell^m(t)}, && \widehat{\xi}(t) := \max_{x\in\overline{\Omega}} \xi(x,t) = \frac{e^{\lambda\|\eta\|_\infty}}{\ell^m(t)}.
\end{aligned} \tag{4.11}$$

Notice that from (4.11), we obtain the following properties :

$$|\partial_t^n\alpha|,\ |\partial_t^n\xi| \leq C\xi^{(1+n/m)}, \quad |\partial_x^l\alpha|,\ |\partial_x^l\xi| \leq C\xi \tag{4.12}$$

where n is any nonnegative integer, l is a N-multi-index and $C > 0$ is a constant only depending on Ω, λ, η and ℓ. This property is also valid for the pairs (α^*, ξ^*) and $(\widehat{\alpha}, \widehat{\xi})$.

The first result is a Carleman inequality for the Stokes system with right-hand side in $L^2(Q)^N$ proved in [7, Proposition 2.1]. This estimate has the interesting property that the local term does not contain one of the components of the solution.

Lemma 4.7. *There exists a constant $\widehat{\lambda}_0 > 0$ such that for any $\lambda \geq \widehat{\lambda}_0$ there exists $C > 0$ depending only on λ, Ω, ω, η and ℓ such that for any $i \in \{1,\ldots,N\}$, any $g \in L^2(Q)^N$ and any $u^0 \in H$, the solution of*

$$\begin{cases} u_t - \Delta u + \nabla p = g, \quad \nabla\cdot u = 0 & \text{in } Q,\\ u = 0 & \text{on } \Sigma,\\ u(0) = u^0 & \text{in } \Omega,\end{cases} \tag{4.13}$$

satisfies

$$\begin{aligned}
I_1(u) := {}& s^2 \iint_Q e^{-11s\alpha^*}(\xi^*)^{2-2/m}|u_t|^2\,dx\,dt + s^3\sum_{\substack{j=1\\ j\neq i}}^N \iint_Q e^{-2s\alpha-9s\alpha^*}\xi^3|\Delta u_j|^2\,dx\,dt\\
&+ s^4\sum_{j=1,j\neq i}^N \iint_Q e^{-2s\alpha-9s\alpha^*}\xi^4|\nabla u_j|^2\,dx\,dt + s^4\iint_Q e^{-11s\alpha^*}(\xi^*)^4|u|^2dx\,dt\\
&\leq C\left(\iint_Q e^{-9s\alpha^*}|g|^2\,dx\,dt + s^7\sum_{\substack{j=1\\ j\neq i}}^N \iint_{\omega_0\times(0,T)} e^{-2s\widehat{\alpha}-9s\alpha^*}\widehat{\xi}^7|u_j|^2dx\,dt\right)
\end{aligned} \tag{4.14}$$

for every $s \geq C$.

Remark 4.8. *In [7], the weight functions α and ξ are given for $m = 8$, but the proof also holds for any $m \geq 8$. Additionally, the terms involving derivatives of u in the left-hand side of (4.14) do not appear explicitly in Proposition 2.1 of [7]. However, it is easily seen from its proof that these terms can be added. Finally, one should replace $\rho = e^{-3/2s\alpha*}$ (defined in section 2.1 right after (2.13) in reference [7]) by $\rho = e^{-9/2s\alpha*}$.*

Now, we present an estimate proved in [9, Proposition 3.2] (with $f = 0$ in that reference) similar to (4.14) with local terms of the Laplacian.

Lemma 4.9. *There exists a constant $\widehat{\lambda}_1$, such that for any $\lambda \geq \widehat{\lambda}_1$ there exists a constant $C(\lambda) > 0$ such that for any $i \in \{1, \dots, N\}$, any $u^0 \in H$ and any*

$$g \in L^2(0,T; H^3(\Omega)^N) \cap H^2(0,T; H^{-1}(\Omega)^N),$$

the solution of (4.13) *satisfies*

$$\begin{aligned} I_2(u) := \sum_{\substack{j=1\\ j\neq i}}^{N} \left[\iint_Q e^{-9s\alpha}(s^5\xi^5|\Delta u_j|^2 + s^3\xi^3|\nabla\Delta u_j|^2 + s\xi|\nabla\nabla\Delta u_j|^2 + s^{-1}\xi^{-1}|\nabla\nabla\nabla\Delta u_j|^2)dx\,dt \right] \\ +s^5 \iint_Q e^{-9s\alpha^*}(\xi^*)^5|u|^2dx\,dt \leq C\left(s^{5/2}\iint_Q e^{-9s\alpha^*}(\xi^*)^{3-2/m}|g|^2\,dx\,dt \right. \\ +s^{3/2}\int_0^T \left\| \left(e^{-9/2s\alpha^*}(\xi^*)^{1-3/(2m)}g\right)_t \right\|^2_{H^{-1}(\Omega)^N} dt + s^{-1/2}\int_0^T e^{-9s\alpha^*}(\xi^*)^{-5/m}\,\|g\|^2_{H^3(\Omega)^N}\,dt \\ +s^{-1/2}\int_0^T \left\| \left(e^{-9/2s\alpha^*}(\xi^*)^{-5/(2m)}g\right)_t \right\|^2_{H^1(\Omega)^N} dt + s^{-1/2}\int_0^T \left\| \left(e^{-9/2s\alpha^*}(\xi^*)^{-5/(2m)}g\right)_{tt} \right\|^2_{H^{-1}(\Omega)^N} dt \\ + \sum_{j=1, j\neq i}^{N} \iint_Q e^{-9s\alpha}|\nabla\Delta g_j|^2dx\,dt + \iint_Q e^{-9s\alpha}|\nabla\nabla(\nabla\cdot g)|^2dx\,dt \\ \left. +s^5\sum_{\substack{j=1\\ j\neq i}}^{N} \iint_{\omega_0\times(0,T)} e^{-9s\alpha}\xi^5|\Delta u_j|^2dx\,dt \right), \end{aligned}$$

for every $s \geq C$.

The third Carleman inequality we present applies to parabolic equations with non-homogeneous boundary conditions. It was proved in [52, Theorem 2.1] :

Lemma 4.10. *Let $f_0, f_1, \dots, f_N \in L^2(Q)$. There exists a constant $\widehat{\lambda}_2 > 0$ such that for any $\lambda \geq \widehat{\lambda}_2$ there exists $C > 0$ depending only on λ, Ω, ω_0, η and ℓ such that for every $u \in L^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega))$ satisfying*

$$u_t - \Delta u = f_0 + \sum_{j=1}^{N} \partial_j f_j \text{ in } Q,$$

we have

$$\begin{aligned} s^{-1}\iint_Q e^{-8s\alpha}\xi^{-1}|\nabla u|^2dx\,dt + s\iint_Q e^{-8s\alpha}\xi|u|^2dx\,dt \leq C\left(s\iint_{\omega_0\times(0,T)} e^{-8s\alpha}\xi|u|^2dx\,dt \right. \\ + s^{-1/2}\left\|e^{-4s\alpha^*}(\xi^*)^{-1/4}u\right\|^2_{H^{\frac14,\frac12}(\Sigma)} + s^{-1/2}\left\|e^{-4s\alpha^*}(\xi^*)^{-1/4+1/m}u\right\|^2_{L^2(\Sigma)} \\ \left. +s^{-2}\iint_Q e^{-8s\alpha}\xi^{-2}|f_0|^2dx\,dt + \sum_{j=1}^{N}\iint_Q e^{-8s\alpha}|f_j|^2dx\,dt \right), \end{aligned}$$

for every $s \geq C$.

Here,

$$\|u\|_{H^{\frac{1}{4},\frac{1}{2}}(\Sigma)} = \left(\|u\|^2_{H^{1/4}(0,T;L^2(\partial\Omega))} + \|u\|^2_{L^2(0,T;H^{1/2}(\partial\Omega))} \right)^{1/2}.$$

Remark 4.11. *Notice that the usual notation for this space is actually $H^{\frac{1}{2},\frac{1}{4}}(\Sigma)$ (see, for instance, [66]). However, we choose to follow the notation used in [52].*

The last estimate that we need is a technical result that can be found, along with its proof, in [23, Lemma 3].

Lemma 4.12. *Let $k \in \mathbb{R}$. There exists $C > 0$ depending only on Ω, ω_0, η and ℓ such that, for every $u \in L^2(0,T;H^1(\Omega))$,*

$$s^2 \iint_Q e^{-8s\alpha}\xi^{k+2}|u|^2 dx\, dt \leq C \left(\iint_Q e^{-8s\alpha}\xi^k |\nabla u|^2 dx\, dt + s^2 \iint_{\omega_0\times(0,T)} e^{-8s\alpha}\xi^{k+2}|u|^2 dx\, dt \right),$$

for every $s \geq C$.

4.2.3 Regularity results

Here, we state some regularity results concerning the heat and Stokes equations, respectively. The first one is (see for instance [58, Chapter 4]) :

Lemma 4.13. *For every $T > 0$ and every $f \in L^2(Q)$, there exists a unique solution*

$$u \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$$

to the heat equation

$$\begin{cases} u_t - \Delta u = f & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = 0 & \text{in } \Omega, \end{cases} \tag{4.15}$$

and there exists a constant $C > 0$ depending only on Ω such that

$$\|u\|^2_{L^2(0,T;H^2(\Omega))} + \|u\|^2_{H^1(0,T;L^2(\Omega))} \leq C\, \|f\|^2_{L^2(Q)}. \tag{4.16}$$

Furthermore, if $f \in X_n$ (n any nonnegative integer), the unique solution to the heat equation (4.15) *satisfies $u \in X_{n+1}$ and there exists a constant $C > 0$ depending only on Ω such that*

$$\|u\|^2_{X_{n+1}} \leq C\, \|f\|^2_{X_n}. \tag{4.17}$$

The second one can be found in [59, Theorem 6, pages 100-101] (see also [76]) :

Lemma 4.14. *For every $T > 0$, every $u^0 \in V$ and every $f \in L^2(Q)^N$, there exists a unique solution*

$$u \in L^2(0,T;H^2(\Omega)^N) \cap H^1(0,T;H) \cap L^\infty(0,T;V)$$

to the Stokes system

$$\begin{cases} u_t - \Delta u + \nabla p = f, \quad \nabla\cdot u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0) = u^0 & \text{in } \Omega, \end{cases}$$

for some $p \in L^2(0,T;H^1(\Omega))$, and there exists a constant $C>0$ depending only on Ω such that

$$\|u\|^2_{L^2(0,T;H^2(\Omega)^N)} + \|u\|^2_{H^1(0,T;L^2(\Omega)^N)} + \|u\|^2_{L^\infty(0,T;V)} \le C\left(\|f\|^2_{L^2(Q)^N} + \left\|u^0\right\|^2_V\right). \quad (4.18)$$

Moreover, if $f \in L^2(0,T;H^2(\Omega)^N)\cap H^1(0,T;L^2(\Omega)^N)$ and $u^0 \in H^3(\Omega)^N \cap V$ satisfy the compatibility condition :

$$\nabla \bar{p} = \Delta u^0 + f(0) \text{ on } \partial\Omega,$$

where $\bar{p}$ is any solution of the Neumann boundary-value problem

$$\begin{cases} \Delta \bar{p} = \nabla\cdot f(0) & \text{in } Q,\\ \dfrac{\partial \bar{p}}{\partial n} = \Delta u^0\cdot n + f(0)\cdot n & \text{on } \Sigma,\end{cases}$$

then $u \in Y_2$ and there exists a constant $C>0$ depending only on Ω such that

$$\|u\|^2_{Y_2} \le C\left(\|f\|^2_{Y_1} + \left\|u^0\right\|^2_{H^3(\Omega)}\right). \quad (4.19)$$

4.3 Carleman estimate for the adjoint system

In this section we prove a new Carleman estimate for the following coupled system :

$$\begin{cases} -\varphi_t - \Delta\varphi + \nabla\pi_\varphi = g^\varphi + \psi 1\!\!1_{\mathcal{O}}, \quad \nabla\cdot\varphi = 0 & \text{in } Q,\\ \psi_t - \Delta\psi + \nabla\pi_\psi = g^\psi + \sigma\, e_N, \quad \nabla\cdot\psi = 0 & \text{in } Q,\\ -\phi_t - \Delta\phi = g^\phi + \varphi_N + \sigma 1\!\!1_{\mathcal{O}} & \text{in } Q,\\ \sigma_t - \Delta\sigma = g^\sigma & \text{in } Q,\\ \varphi = \psi = 0, \quad \phi = \sigma = 0 & \text{on } \Sigma,\\ \varphi(T) = 0, \quad \psi(0) = \psi^0, \quad \phi(T) = 0, \quad \sigma(0) = \sigma^0 & \text{in } \Omega,\end{cases} \quad (4.20)$$

where $g^\varphi \in Y_0$, $g^\psi \in Y_2$, $g^\phi \in X_0$, $g^\sigma \in X_4$, $\psi^0 \in H$ and $\sigma^0 \in L^2(\Omega)$. It is given by the following proposition :

Proposition 4.15. *Assume that $\omega \cap \mathcal{O} \neq \emptyset$. Then, there exists a constant λ_0, such that for any $\lambda \ge \lambda_0$ there exists a constant $C>0$ depending only on λ, Ω, ω and ℓ such that for any $j_0 \in \{1,\dots,N-1\}$, any $g^\varphi \in L^2(Q)^N$, any $g^\psi \in Y_2$, any $g^\phi \in L^2(Q)$, any $g^\sigma \in X_4$, any $\psi^0 \in H$ and any $\sigma_0 \in L^2(\Omega)$, the solution $(\varphi,\psi,\phi,\sigma)$ of (4.20) satisfies*

$$\begin{aligned} s^4 \iint_Q e^{-11s\alpha^*}(\xi^*)^4|\varphi|^2 dx\,dt + s^5 \iint_Q e^{-9s\alpha^*}(\xi^*)^5|\psi|^2 dx\,dt \\ + s^3 \iint_Q e^{-12s\alpha^*}(\xi^*)^3|\phi|^2 dx\,dt + s^5 \iint_Q e^{-8s\alpha^*}(\xi^*)^5|\sigma|^2 dx\,dt \\ \le C\Bigg(s^{15}\left\|e^{-8s\alpha+4s\alpha^*}\xi^9 g^\varphi\right\|^2_{Y_0} + \left\|e^{-7/2s\alpha^*}\, g^\psi\right\|^2_{Y_2} \\ + s^{19}\left\|e^{-8s\alpha+4s\alpha^*}\xi^{11} g^\phi\right\|^2_{X_0} + \left\|e^{-7/2s\alpha^*}\, g^\sigma\right\|^2_{X_4} \\ +(N-2)s^{13}\iint_{\omega\times(0,T)} e^{-9s\alpha}\xi^{13}|\varphi_{j_0}|^2 dx\,dt + s^{26}\iint_{\omega\times(0,T)} e^{-18s\alpha+11s\alpha^*}\xi^{30}|\phi|^2 dx\,dt\Bigg), \end{aligned} \quad (4.21)$$

for every $s \ge C$.

For the sake of completeness, we treat the more general case of $N = 3$ (with $j_0 = 1$, for instance). The general idea is to combine suitable Carleman estimates for the heat and Stokes equations in (4.20). The proof is then divided in several parts :

- First, we deduce from Lemma 4.9 a Carleman estimate for ψ with local terms of $\Delta\psi_1$ and $\Delta\psi_3$. Using the coupling with the equation of φ, we estimate these terms by local terms of φ_1 and φ_3.
- Using the equation of ϕ, we estimate the local term of φ_3 by a local term of ϕ.
- Finally, to add σ to the left hand side of (4.21) and absorb all its global terms on the right- hand side, we prove a Carleman estimate for a certain operator of σ such that its global terms can be estimated by terms of the right-hand side of (4.21).

The details of these steps are the target of the following subsections.

4.3.1 Carleman estimate for (φ, ψ)

The first step is to apply Lemma 4.9 to the equation satisfied by ψ. However, before doing that, let us remark some simple (but useful) properties of the weight functions. Note that for every $a > 0$, and every $b, c \in \mathbb{R}$, the function $s^b e^{-as\alpha}\xi^c$ is bounded in Q. Furthermore, for any given $\varepsilon > 0$, we have

$$s^b e^{-as\alpha}\xi^c < \varepsilon, \tag{4.22}$$

for $s \geq C$, C a positive constant large enough. Taking this and (4.12) into account, we have, for example,

$$\begin{aligned}|(s^{3/4}e^{-9/2s\alpha^*}(\xi^*)^{1-3/(2m)}\sigma)_t| &= |(s^{1/2}e^{-4s\alpha^*}(\xi^*)^{1/2-2/m}\sigma\,(s^{1/4}e^{-1/2s\alpha^*}(\xi^*)^{1/2+1/(2m)}))_t| \\ &\leq \varepsilon(|s^{1/2}e^{-4s\alpha^*}(\xi^*)^{1/2-2/m}\sigma| + |(s^{1/2}e^{-4s\alpha^*}(\xi^*)^{1/2-2/m}\sigma)_t|), \quad (4.23)\end{aligned}$$

for every $s \geq C$, that is,

$$\left\|s^{3/4}e^{-9/2s\alpha^*}(\xi^*)^{1-3/(2m)}\sigma\right\|_{H^1(0,T;L^2(\Omega))} \leq \varepsilon\left\|s^{1/2}e^{-4s\alpha^*}(\xi^*)^{1/2-2/m}\sigma\right\|_{H^1(0,T;L^2(\Omega))}.$$

Observe that this argument is easily extended for derivatives of greater order.

Carleman estimate for ψ

Now, we apply Lemma 4.9 with $i = 2$, $u = \psi$, and $g = g^\psi + \sigma\, e_3$. The idea is to use (4.22) and the idea developed in (4.23) in order to accommodate the terms in a more convenient way for us. It is not difficult to check that we can obtain

$$\begin{aligned}I_2(\psi) \leq \varepsilon\Big(s\int_0^T &e^{-8s\alpha^*}(\xi^*)^{1-4/m}\|g^\psi + \sigma\, e_3\|^2_{H^3(\Omega)^3}\,\mathrm{d}t \\ &+ \int_0^T \left\|\left(s^{1/2}e^{-4s\alpha^*}(\xi^*)^{1/2-2/m}\,(g^\psi + \sigma\, e_3)\right)_t\right\|^2_{H^1(\Omega)^3}\mathrm{d}t \\ &+ \int_0^T \left\|\left(s^{1/2}e^{-4s\alpha^*}(\xi^*)^{1/2-2/m}\,(g^\psi + \sigma\, e_3)\right)_{tt}\right\|^2_{L^2(\Omega)^3}\mathrm{d}t\Big) \\ &+ Cs^5\iint_{\omega_0\times(0,T)} e^{-9s\alpha}\xi^5(|\Delta\psi_1|^2 + |\Delta\psi_3|^2)\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

for every $s \geq C$ (large enough), where $\varepsilon > 0$ is a constant to be chosen small enough later on. Furthermore, using the properties (4.22) and (4.23) again, we prove the following inequality :

$$I_2(\psi) \leq C \left\| e^{-7/2s\alpha^*} g^\psi \right\|^2_{Y_2} + \varepsilon \left\| s^{1/2} e^{-4s\alpha^*} (\xi^*)^{1/2-2/m} \sigma \right\|^2_{X_2} + Cs^5 \iint_{\omega_0\times(0,T)} e^{-9s\alpha}\xi^5(|\Delta\psi_1|^2 + |\Delta\psi_3|^2)\mathrm{d}x\,\mathrm{d}t, \quad (4.24)$$

for every $s \geq C$, and $\varepsilon > 0$ is to be chosen small enough later on. Notice that the weight functions and norms chosen for g^ψ and σ in inequality (4.24) are not sharp. Indeed, this is a technical choice for the sake of the arguments that will be employed in the rest of the chapter.

Carleman estimate for φ

Next, we apply Lemma 4.7 to the equation satisfied by φ, with $i = 2$, $u = \varphi$ and $g = g^\varphi + \psi\,\mathbb{1}_{\mathcal{O}}$. We obtain

$$I_1(\varphi) \leq C \iint_Q e^{-9s\alpha^*}|g^\varphi|^2\mathrm{d}x\,\mathrm{d}t + C \iint_{\mathcal{O}\times(0,T)} e^{-9s\alpha^*}|\psi|^2\mathrm{d}x\,\mathrm{d}t + Cs^7 \iint_{\omega_0\times(0,T)} e^{-2s\alpha-9s\alpha^*}\xi^7(|\varphi_1|^2 + |\varphi_3|^2)\mathrm{d}x\,\mathrm{d}t, \quad (4.25)$$

for every $s \geq C$. Noticing that the second integral in the right of this inequality is bounded by $C\,s^{-5}\,I_2(\psi)$, for some $C > 0$, we can combine inequalities (4.24) and (4.25) to get

$$I_1(\varphi) + I_2(\psi) \leq C \left\| e^{-9/2s\alpha^*} g^\varphi \right\|^2_{Y_0} + C \left\| e^{-7/2s\alpha^*} g^\psi \right\|^2_{Y_2} + \varepsilon \left\| s^{1/2} e^{-4s\alpha^*} (\xi^*)^{1/2-2/m} \sigma \right\|^2_{X_2}$$
$$+Cs^7 \iint_{\omega_0\times(0,T)} e^{-2s\alpha-9s\alpha^*}\xi^7(|\varphi_1|^2+|\varphi_3|^2)\mathrm{d}x\,\mathrm{d}t + Cs^5 \iint_{\omega_0\times(0,T)} e^{-9s\alpha}\xi^5(|\Delta\psi_1|^2+|\Delta\psi_3|^2)\mathrm{d}x\,\mathrm{d}t, \quad (4.26)$$

for every $s \geq C$. Now, using the coupling between φ and ψ we will estimate the last two local terms. This is the objective of the next subsection.

Estimation of $\Delta\psi_1$ and $\Delta\psi_3$

We look at the equations satisfied by φ_1 and φ_3 in the set $\mathcal{O} \times (0, T)$ to find

$$\Delta\psi_1 = -\Delta\varphi_{1,t} - \Delta^2\varphi_1 + \partial_1\nabla \cdot g^\varphi - \Delta g_1^\varphi,$$

$$\Delta\psi_3 = -\Delta\varphi_{3,t} - \Delta^2\varphi_3 + \partial_3\nabla \cdot g^\varphi - \Delta g_3^\varphi.$$

Here, we have used that $\Delta\pi_\varphi = \nabla \cdot g^\varphi$ in $\mathcal{O} \times (0, T)$ which follows directly from the free-divergence conditions for φ and ψ. Let now $\zeta \in \mathcal{C}_0^4(\widetilde{\omega})$ be a nonnegative function with $\zeta \equiv 1$ in ω_0 and $\omega_0 \Subset \widetilde{\omega} \Subset \omega \cap \mathcal{O}$.

For simplicity, we only treat the term concerning φ_3 in the last integral in (4.26), since the other term is quite similar. Using this last equality, we have

$$s^5 \iint_{\omega_0\times(0,T)} e^{-9s\alpha}\xi^5|\Delta\psi_3|^2\mathrm{d}x\,\mathrm{d}t \leq s^5 \iint_{\widetilde{\omega}\times(0,T)} \zeta(x)e^{-9s\alpha}\xi^5|\Delta\psi_3|^2\mathrm{d}x\,\mathrm{d}t$$
$$= s^5 \iint_{\widetilde{\omega}\times(0,T)} \zeta(x)e^{-9s\alpha}\xi^5\Delta\psi_3(-\Delta\varphi_{3,t} - \Delta^2\varphi_3 + \partial_3\nabla \cdot g^\varphi - \Delta g_3^\varphi)\mathrm{d}x\,\mathrm{d}t.$$

An integration by parts, in time and space, gives the following inequality :

$$
\begin{aligned}
& s^5 \iint_{\omega_0\times(0,T)} e^{-9s\alpha}\xi^5|\Delta\psi_3|^2 \mathrm{d}x\,\mathrm{d}t \\
& \leq s^5 \iint_{\widetilde{\omega}\times(0,T)} \zeta(x) e^{-9s\alpha}\xi^5(\Delta\psi_{3,t}-\Delta^2\psi_3)\Delta\varphi_3 \mathrm{d}x\,\mathrm{d}t \\
& + s^5 \iint_{\widetilde{\omega}\times(0,T)} \Delta\left((\zeta(x)e^{-9s\alpha}\xi^5)_t\Delta\psi_3 - \Delta(\zeta(x)e^{-9s\alpha}\xi^5)\Delta\psi_3 - 2\nabla(\zeta(x)e^{-9s\alpha}\xi^5)\cdot\nabla\Delta\psi_3\right)\varphi_3 \mathrm{d}x\,\mathrm{d}t \\
& + s^5 \iint_{\widetilde{\omega}\times(0,T)} \nabla\partial_3\left(\zeta(x)e^{-9s\alpha}\xi^5\Delta\psi_3\right)\cdot g^{\varphi}\mathrm{d}x\,\mathrm{d}t - s^5 \iint_{\widetilde{\omega}\times(0,T)} \Delta\left(\zeta(x)e^{-9s\alpha}\xi^5\Delta\psi_3\right) g_3^{\varphi}\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{4.27}
$$

Now we estimate the three lines of terms in the right-hand side of (4.27), which we call respectively L_1, L_2 and L_3. For the first one, we use the equation satisfied by $\Delta\psi_3$, namely

$$
\Delta\psi_{3,t} - \Delta^2\psi_3 = \Delta g_3^{\psi} - \partial_3\nabla\cdot g^{\psi} + (\partial_1^2+\partial_2^2)\sigma. \tag{4.28}
$$

This equation, together with some integration by parts, yields

$$
L_1 = s^5 \iint_{\widetilde{\omega}\times(0,T)} \Delta\left(\zeta(x)e^{-9s\alpha}\xi^5(\Delta g_3^{\psi} - \partial_3\nabla\cdot g^{\psi} + (\partial_1^2+\partial_2^2)\sigma)\right)\varphi_3 \mathrm{d}x\,\mathrm{d}t.
$$

A careful analysis of these terms, taking into account (4.12), (4.22) and using Young's inequality, gives

$$
\begin{aligned}
|L_1| \leq \; & Cs^{13}\iint_{\widetilde{\omega}\times(0,T)} e^{-9s\alpha}\xi^{13}|\varphi_3|^2\mathrm{d}x\,\mathrm{d}t + C\left\| e^{-7/2s\alpha^*} g^{\psi}\right\|_{Y_2}^2 \\
& + \varepsilon \left\| s^{1/2}e^{-4s\alpha^*}(\xi^*)^{1/2-2/m}\,\sigma\right\|_{X_2}^2 .
\end{aligned}
$$

A similar argument for L_2 and L_3 gives the following estimates

$$
|L_2| \leq \frac{1}{4C} I_2(\psi) + Cs^{13}\iint_{\widetilde{\omega}\times(0,T)} e^{-9s\alpha}\xi^{13}|\varphi_3|^2\mathrm{d}x\,\mathrm{d}t,
$$

$$
|L_3| \leq \frac{1}{4C} I_2(\psi) + Cs^{9}\iint_{\widetilde{\omega}\times(0,T)} e^{-9s\alpha}\xi^{9}|g^{\varphi}|^2\mathrm{d}x\,\mathrm{d}t.
$$

Combining these estimates for L_1, L_2 and L_3 with (4.27), together with the same computations for ψ_1, provide

$$
\begin{aligned}
I_1(\varphi) + I_2(\psi) \leq C & \left(\left\| s^{9/2}e^{-9/2s\alpha}\xi^{9/2}\,g^{\varphi}\right\|_{Y_0}^2 + \left\| e^{-7/2s\alpha^*}\,g^{\psi}\right\|_{Y_2}^2 \right) \\
& + \varepsilon\left\| s^{1/2}e^{-4s\alpha^*}(\xi^*)^{1/2-2/m}\,\sigma\right\|_{X_2}^2 + Cs^{13}\iint_{\widetilde{\omega}\times(0,T)} e^{-9s\alpha}\xi^{13}(|\varphi_1|^2+|\varphi_3|^2)\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{4.29}
$$

for every $s \geq C$.

Here, we have also used the properties of the weight functions to find a more compact expression. This ends this part of the proof of Proposition 4.15.

The next step is to eliminate the local term of φ_3. This is done in the next subsection.

4.3.2 Estimation of φ_3 and Carleman estimate for ϕ

As in the previous section, we look at the equation satisfied by ϕ in the set $\mathcal{O}\times(0,T)$:

$$\varphi_3 = -\phi_t - \Delta\phi - g^\phi - \sigma.$$

We consider again a nonnegative function $\zeta \in \mathcal{C}_0^2(\omega')$ such that $\zeta \equiv 1$ in $\widetilde{\omega}$ and $\widetilde{\omega} \Subset \omega' \Subset \omega \cap \mathcal{O}$, and perform integration by parts to obtain

$$\begin{aligned} s^{13} \iint_{\omega'\times(0,T)} \zeta(x) e^{-9s\alpha}\xi^{13}|\varphi_3|^2 \mathrm{d}x\,\mathrm{d}t &= s^{13}\iint_{\omega'\times(0,T)} \zeta(x)e^{-9s\alpha}\xi^{13}\varphi_3(-\phi_t - \Delta\phi - g^\phi - \sigma)\mathrm{d}x\,\mathrm{d}t \\ &= s^{13}\iint_{\omega'\times(0,T)} [\partial_t - \Delta]\left(\zeta(x)e^{-9s\alpha}\xi^{13}\varphi_3\right)\phi\,\mathrm{d}x\,\mathrm{d}t - s^{13}\iint_{\omega'\times(0,T)} \zeta(x)e^{-9s\alpha}\xi^{13}\varphi_3(g^\phi + \sigma)\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Let us call L_4 and L_5 the two integrals at the end of the last expression. A careful analysis of L_4, together with property (4.12) and Young's inequality, yield

$$|L_4| \le \frac{1}{2C} I_1(\varphi) + Cs^{26}\iint_{\omega'\times(0,T)} e^{-18s\alpha+11s\alpha^*}\xi^{30}|\phi|^2\,\mathrm{d}x\,\mathrm{d}t + Cs^{24}\iint_{\omega'\times(0,T)} e^{-16s\alpha+9s\alpha^*}\xi^{24}|\phi|^2\,\mathrm{d}x\,\mathrm{d}t,$$

for every $s \ge C$.

For the term concerning L_5, we apply again Young's inequality and property (4.22) :

$$\begin{aligned} |L_5| \le \frac{1}{2C} s^{13}\iint_{\omega'\times(0,T)} \zeta(x)e^{-9s\alpha}\xi^{13}|\varphi_3|^2\mathrm{d}x\,\mathrm{d}t + C\left\| s^{13/2}e^{-9/2s\alpha}\xi^{13/2}g^\phi\right\|^2_{X_0} \\ +\varepsilon\left\| s^{1/2}e^{-4s\alpha^*}(\xi^*)^{1/2-2/m}\,\sigma\right\|^2_{X_0}. \end{aligned}$$

From (4.11), we have $e^{-16s\alpha+9s\alpha^*} < e^{-18s\alpha+11s\alpha^*}$ in Q, so we obtain

$$\begin{aligned} s^{13}\iint_{\omega'\times(0,T)} \zeta(x)e^{-9s\alpha}\xi^{13}|\varphi_3|^2\mathrm{d}x\,\mathrm{d}t \le \frac{1}{2C}I_1(\varphi) + Cs^{26}\iint_{\omega'\times(0,T)} e^{-18s\alpha+11s\alpha^*}\xi^{30}|\phi|^2\,\mathrm{d}x\,\mathrm{d}t \\ + C\left\| s^{13/2}e^{-9/2s\alpha}\xi^{13/2}g^\phi\right\|^2_{X_0} + \varepsilon\left\| s^{1/2}e^{-4s\alpha^*}(\xi^*)^{1/2-2/m}\,\sigma\right\|^2_{X_0}, \end{aligned}$$

for every $s \ge C$, which plugged in (4.29) together with the fact that

$$s^{13}\iint_{\widetilde{\omega}\times(0,T)} e^{-9s\alpha}\xi^{13}|\varphi_3|^2\mathrm{d}x\,\mathrm{d}t \le s^{13}\iint_{\omega'\times(0,T)} \zeta(x)e^{-9s\alpha}\xi^{13}|\varphi_3|^2\mathrm{d}x\,\mathrm{d}t,$$

leads to

$$\begin{aligned} I_1(\varphi)+I_2(\psi) \le C\left\| s^{9/2}e^{-9/2s\alpha}\xi^{9/2}\,g^\varphi\right\|^2_{Y_0} + C\left\| e^{-7/2s\alpha^*}\,g^\psi\right\|^2_{Y_2} + C\left\| s^{13/2}e^{-9/2s\alpha}\xi^{13/2}g^\phi\right\|^2_{X_0} \\ +\varepsilon\left\| s^{1/2}e^{-4s\alpha^*}(\xi^*)^{1/2-2/m}\,\sigma\right\|^2_{X_2} + Cs^{13}\iint_{\widetilde{\omega}\times(0,T)} e^{-9s\alpha}\xi^{13}|\varphi_1|^2\mathrm{d}x\,\mathrm{d}t \\ + Cs^{26}\iint_{\omega'\times(0,T)} e^{-18s\alpha+11s\alpha^*}\xi^{30}|\phi|^2\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{4.30}$$

for every $s \ge C$.

To end this section of the proof, we will combine (4.30) with a Carleman inequality for ϕ. This will allow us to add the term with ϕ in the left hand side of (4.21). Namely, ϕ satisfies

$$\begin{aligned} I_3(\phi) &:= \iint_Q e^{-12s\alpha}(s^3\xi^3|\phi|^2 + s\xi|\nabla\phi|^2 + s^{-1}\xi^{-1}(|\phi_t|^2 + |\Delta\phi|^2))\,\mathrm{d}x\,\mathrm{d}t \\ &\le C\iint_Q e^{-12s\alpha}(|g^\phi|^2 + |\varphi_3|^2 + |\sigma|^2\mathbb{1}_{\mathcal{O}})\,\mathrm{d}x\,\mathrm{d}t + Cs^3\iint_{\omega\times(0,T)} e^{-12s\alpha}\xi^3|\phi|^2\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{4.31}$$

for every $s \geq C$. This is the classical Carleman estimate for the solutions of the heat equation with homogeneous Dirichlet boundary conditions (see, for instance, [39]). Notice that, taking (4.22) into account, the right-hand side of (4.31) is bounded by

$$C \left\| s^{13/2} e^{-9/2 s\alpha} \xi^{13/2} g^{\phi} \right\|_{X_0}^2 + \frac{1}{2} I_1(\varphi) + \varepsilon \left\| s^{1/2} e^{-4s\alpha^*} (\xi^*)^{1/2-2/m} \sigma \right\|_{X_2}^2 + C s^{26} \iint_{\omega\times(0,T)} e^{-18s\alpha+11s\alpha^*} \xi^{30} |\phi|^2 \,\mathrm{d}x\,\mathrm{d}t,$$

for every $s \geq C$. Thus, combining this bound with (4.30) and (4.31) we obtain

$$\begin{aligned} I_1(\varphi) + I_2(\psi) + I_3(\phi) \leq C \Bigg(& \left\| s^{9/2} e^{-9/2 s\alpha} \xi^{9/2} g^{\varphi} \right\|_{Y_0}^2 + \left\| e^{-7/2 s\alpha^*} g^{\psi} \right\|_{Y_2}^2 \\ & + \left\| s^{13/2} e^{-9/2 s\alpha} \xi^{13/2} g^{\phi} \right\|_{X_0}^2 \Bigg) + \varepsilon \left\| s^{1/2} e^{-4s\alpha^*} (\xi^*)^{1/2-2/m} \sigma \right\|_{X_2}^2 \\ + C s^{13} \iint_{\omega\times(0,T)} e^{-9s\alpha} \xi^{13} |\varphi_1|^2 \mathrm{d}x\,\mathrm{d}t & + C s^{26} \iint_{\omega\times(0,T)} e^{-18s\alpha+11s\alpha^*} \xi^{30} |\phi|^2 \,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{4.32}$$

for every $s \geq C$.

At this point, there are a few tasks left to do. First, we need to eliminate the global term of σ in the right-hand side of (4.32). This will be done combining this estimate with a suitable Carleman estimate for σ, provided that we choose the weight functions conveniently. This will add σ to the left-hand side and a local term of σ to the right-hand side that also needs to be eliminated. Normally, we would try to use again the coupling of σ with the other equations. However, this will make appear the pressure term of the equation of ψ, or again a local term of φ_3 in the equation of ϕ, which we would not be able to eliminate again. Therefore, a more careful analysis needs to be done to avoid these issues. Details are given in the next subsection.

4.3.3 Carleman estimate for σ and final computations

This final part of the proof of Proposition 4.15 is divided in three steps. In the first one, we look for an equation of σ which does not contain neither π_ψ nor φ_3. Then, we show a suitable Carleman inequality for σ based on this expression. Finally, we eliminate the local terms of σ and conclude.

Operator for σ

Let us look at system (4.20) in the set $\mathcal{O} \times (0,T)$. All the following computations will be seen in this set. Since $\Delta \pi_\varphi = \nabla \cdot g^{\varphi}$, we find

$$-(\Delta\varphi_3)_t - \Delta^2 \varphi_3 = \Delta g_3^{\varphi} - \partial_3 \nabla \cdot g^{\varphi} + \Delta\psi_3.$$

In the previous section, we found the equation satisfied by $\Delta\psi_3$ (see (4.28))

$$(\Delta\psi_3)_t - \Delta^2 \psi_3 = \Delta g_3^{\psi} - \partial_3 \nabla \cdot g^{\psi} + \mathcal{D}\sigma.$$

Here, we recall the notation used in (4.9), $\mathcal{D} := \partial_1^2 + \partial_2^2$. Combining these expressions, we can easily find the following relation between φ_3 and σ :

$$\mathcal{P}\varphi_3 = (\Delta g_3^{\varphi})_t - \Delta^2 g_3^{\varphi} - (\partial_3 \nabla \cdot g^{\varphi})_t + \Delta(\partial_3 \nabla \cdot g^{\varphi}) + \Delta g_3^{\psi} - \partial_3 \nabla \cdot g^{\psi} + \mathcal{D}\sigma,$$

where we have denoted $\mathcal{P} = -\Delta\partial_t^2 + \Delta^3$ (recall again (4.9)).

Finally, we apply this operator to the equation satisfied by ϕ and, combined with the last expression, we obtain

$$\begin{aligned} -(\mathcal{P}_t + \Delta\mathcal{P})\phi =& (\Delta g_3^\varphi)_t - \Delta^2 g_3^\varphi - \partial_3\nabla\cdot g_t^\varphi + \Delta(\partial_3\nabla\cdot g^\varphi) + \Delta g_3^\psi - \partial_3\nabla\cdot g^\psi \\ &+ \mathcal{P}g^\phi - \Delta g_t^\sigma - \Delta^2 g^\sigma + \mathcal{D}\sigma, \end{aligned} \tag{4.33}$$

where we have used the equation satisfied by σ to find $\mathcal{P}\sigma = -\Delta g_t^\sigma - \Delta^2 g^\sigma$.

The idea now is to prove a Carleman estimate for σ with a local term of $\mathcal{D}\sigma$ and use (4.33) to eliminate it.

Carleman estimate for $\mathcal{D}\sigma$

Here, we follow the ideas developed for the Stokes system in [23] (see also [7] and [9]), thus for simplicity we omit some detail of the computations. We start by applying $\nabla\nabla\mathcal{D}$ to the equation satisfied by σ. We have

$$(\nabla\nabla\mathcal{D}\sigma)_t - \Delta(\nabla\nabla\mathcal{D}\sigma) = \nabla\nabla\mathcal{D}g^\sigma, \text{ in } Q.$$

We apply sequentially to this equation :
- Lemma 4.10 with $u = \nabla\nabla\mathcal{D}\sigma$,
- Lemma 4.12 with $k = 1$ and $u = \nabla\mathcal{D}\sigma$,
- Lemma 4.12 with $k = 3$ and $u = \mathcal{D}\sigma$,

and

$$\int_\Omega |\sigma|^2\,dx \le C\int_\Omega |\mathcal{D}\sigma|^2\,dx, \tag{4.34}$$

for some C depending on Ω (this last inequality holds since Ω is bounded and $\sigma_{|\partial\Omega} = 0$). Therefore, we obtain

$$\begin{aligned} \widetilde{I}_4(\sigma) :=& \iint_Q e^{-8s\alpha}(s^{-1}\xi^{-1}|\nabla^3\mathcal{D}\sigma|^2 + s\xi|\nabla^2\mathcal{D}\sigma|^2 + s^3\xi^3|\nabla\mathcal{D}\sigma|^2 + s^5\xi^5|\mathcal{D}\sigma|^2)\mathrm{d}x\,\mathrm{d}t \\ &+ s^5\iint_Q e^{-8s\alpha^*}(\xi^*)^5|\sigma|^2\mathrm{d}x\,\mathrm{d}t \le Cs^{-2}\iint_Q e^{-8s\alpha}\xi^{-2}|\nabla^2\mathcal{D}g^\sigma|^2\mathrm{d}x\,\mathrm{d}t \\ &+ Cs^{-1/2}\left\|e^{-4s\alpha^*}(\xi^*)^{1/4}\nabla^2\mathcal{D}\sigma\right\|^2_{H^{1/4,1/2}(\Sigma)} + Cs^{-1/2}\left\|e^{-4s\alpha^*}(\xi^*)^{-1/4+1/m}\nabla^2\mathcal{D}\sigma\right\|^2_{L^2(\Sigma)} \\ &+ C\iint_{\omega_0\times(0,T)} e^{-8s\alpha}\left(s\xi|\nabla^2\mathcal{D}\sigma|^2 + s^3\xi^3|\nabla\mathcal{D}\sigma|^2 + s^5\xi^5|\mathcal{D}\sigma|^2\right)\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4.35}$$

It is not hard to prove that, considering a cut-off function supported in ω' (recall that $\omega_0 \Subset \omega' \Subset \omega\cap\mathcal{O}$) ; integration by parts and Young's inequality, we can estimate the local terms in the last inequality by

$$\frac{1}{2}\widetilde{I}_4(\sigma) + Cs^5\iint_{\omega'\times(0,T)} e^{-8s\alpha}\xi^5|\mathcal{D}\sigma|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.36}$$

To estimate the boundary terms, we use regularity results for the heat equation. We start by defining

$$\widetilde{\sigma} := s^{3/2}e^{-4s\alpha^*}(\xi^*)^{3/2-1/m}\sigma.$$

It is not difficult to check that $\widetilde{\sigma}$ satisfies the heat equation

$$\begin{cases} \widetilde{\sigma}_t - \Delta\widetilde{\sigma} = s^{3/2}e^{-4s\alpha^*}(\xi^*)^{3/2-1/m}g^\sigma + (s^{3/2}e^{-4s\alpha^*}(\xi^*)^{3/2-1/m})_t\sigma & \text{in } Q, \\ \widetilde{\sigma} = 0 & \text{on } \Sigma, \\ \widetilde{\sigma}(0) = 0 & \text{on } \Omega. \end{cases}$$

From regularity result (4.16) and (4.12), we obtain

$$\|\widetilde{\sigma}\|_{X_1}^2 \leq C\left(\left\|s^{3/2}e^{-4s\alpha^*}(\xi^*)^{3/2-1/m}g^\sigma\right\|_{X_0}^2 + \left\|s^{5/2}e^{-4s\alpha^*}(\xi^*)^{5/2}\sigma\right\|_{X_0}^2\right).$$

Applying successively regularity result (4.17), taking into account (4.12), we can prove the general formula

$$\begin{aligned}\left\|s^{5/2-n}e^{-4s\alpha^*}(\xi^*)^{5/2-n-n/m}\sigma\right\|_{X_n}^2 \leq C\sum_{k=0}^{n-1}\left\|s^{3/2-k}e^{-4s\alpha^*}(\xi^*)^{3/2-k-(k+1)/m}g^\sigma\right\|_{X_k}^2 \\ + C\left\|s^{5/2}e^{-4s\alpha^*}(\xi^*)^{5/2}\sigma\right\|_{X_0}^2,\end{aligned} \tag{4.37}$$

for $n=1,\dots,5$. Let us call in what follows

$$\mathcal{R}_n(g^\sigma) := \sum_{k=0}^{n-1}\left\|s^{3/2-k}e^{-4s\alpha^*}(\xi^*)^{3/2-k-(k+1)/m}g^\sigma\right\|_{X_k}^2.$$

Using a trace inequality (see, for instance, [66]) we have

$$\left\|e^{-4s\alpha^*}(\xi^*)^{1/4}\nabla^2\mathcal{D}\sigma\right\|_{H^{1/4,1/2}(\Sigma)}^2 \leq C\left\|e^{-4s\alpha^*}(\xi^*)^{1/4}\sigma\right\|_{L^2(0,T;H^5(\Omega))\cap H^1(0,T;H^3(\Omega))}^2.$$

Now, by an interpolation argument between the spaces X_2 and X_3, and since $m \geq 10$, we can combine this estimate with (4.37) and write

$$s^{-1/2}\left\|e^{-4s\alpha^*}(\xi^*)^{1/4}\nabla^2\mathcal{D}\sigma\right\|_{H^{1/4,1/2}(\Sigma)}^2 \leq Cs^{-1/2}\mathcal{R}_3(g^\sigma) + Cs^{-1/2}\left\|s^{5/2}e^{-4s\alpha^*}(\xi^*)^{5/2}\sigma\right\|_{X_0}^2. \tag{4.38}$$

For the other boundary term, taking into account that α^* and ξ^* do not depend on x, and $(\xi^*)^{-1/4+1/m}$ is bounded since $m \geq 10$, we can easily obtain (see (2.24))

$$\begin{aligned}&s^{-1/2}\left\|e^{-4s\alpha^*}(\xi^*)^{-1/4+1/m}\nabla^2\mathcal{D}\sigma\right\|_{L^2(\Sigma)}^2 \leq Cs^{-1/2}\left\|e^{-4s\alpha^*}\nabla^2\mathcal{D}\sigma\right\|_{L^2(\Sigma)}^2 \\ &\leq Cs^{-1/2}\left(\left\|s^{1/2}e^{-4s\alpha^*}(\xi^*)^{1/2}\nabla^2\mathcal{D}\sigma\right\|_{L^2(Q)}\left\|s^{-1/2}e^{-4s\alpha^*}(\xi^*)^{-1/2}\nabla^3\mathcal{D}\sigma\right\|_{L^2(Q)} \right. \\ &\left. +\left\|e^{-4s\alpha^*}\nabla^2\mathcal{D}\sigma\right\|_{L^2(Q)}^2\right) \leq Cs^{-1/2}\widetilde{I}_4(\sigma).\end{aligned} \tag{4.39}$$

Finally, notice that from (4.37), we have

$$J(\sigma) := \sum_{k=1}^{5}\left\|s^{5/2-k}e^{-4s\alpha^*}(\xi^*)^{5/2-k-k/m}\sigma\right\|_{X_k}^2 \leq C\mathcal{R}_5(g^\sigma) + C\widetilde{I}_4(\sigma). \tag{4.40}$$

Combining estimates (4.36), (4.38)-(4.40) in (4.35), we obtain the following Carleman estimate for σ :

$$\begin{aligned}I_4(\sigma) := \widetilde{I}_4(\sigma) + J(\sigma) \leq Cs^{-2}\iint_Q e^{-8s\alpha}\xi^{-2}|\nabla^2\mathcal{D}g^\sigma|^2\mathrm{d}x\,\mathrm{d}t + C\mathcal{R}_5(g^\sigma) \\ +Cs^5\iint_{\omega'\times(0,T)} e^{-8s\alpha}\xi^5|\mathcal{D}\sigma|^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

for every $s \geq C$. Furthermore, by (4.12), (4.22) and (4.23), we find the more compact form :

$$I_4(\sigma) \leq C \left\| e^{-7/2s\alpha^*} g^\sigma \right\|_{X_4}^2 + Cs^5 \iint_{\omega' \times (0,T)} e^{-8s\alpha} \xi^5 |\mathcal{D}\sigma|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{4.41}$$

for every $s \geq C$. Finally, we are going to estimate the local term in (4.41) in terms of g^φ, g^ψ, g^ϕ, g^σ and a local term of ϕ.

We recall the expression (4.33) for $\mathcal{D}\sigma$ valid in $\mathcal{O} \times (0,T)$ and consider a cut-off function $\zeta \in \mathcal{C}_0^8(\omega \cap \mathcal{O})$ such that $\zeta \equiv 1$ in ω'. Recall that $\omega_0 \Subset \omega' \Subset \omega \cap \mathcal{O}$. We have the following for the last term in (4.41) :

$$\begin{aligned}
&s^5 \iint_{\omega' \times (0,T)} e^{-8s\alpha} \xi^5 |\mathcal{D}\sigma|^2 \,\mathrm{d}x\,\mathrm{d}t \leq s^5 \iint_{(\omega\cap\mathcal{O}) \times (0,T)} \zeta(x) e^{-8s\alpha} \xi^5 |\mathcal{D}\sigma|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&= -s^5 \iint_{(\omega\cap\mathcal{O}) \times (0,T)} \zeta(x) e^{-8s\alpha} \xi^5 \mathcal{D}\sigma \left((\Delta g_3^\varphi)_t - \Delta^2 g_3^\varphi - \partial_3 \nabla \cdot g_t^\varphi + \Delta(\partial_3 \nabla \cdot g^\varphi) \right) \mathrm{d}x\,\mathrm{d}t \\
&- s^5 \iint_{(\omega\cap\mathcal{O}) \times (0,T)} \zeta(x) e^{-8s\alpha} \xi^5 \mathcal{D}\sigma \left(\Delta g_3^\psi - \partial_3 \nabla \cdot g^\psi \right) \mathrm{d}x\,\mathrm{d}t \\
&- s^5 \iint_{(\omega\cap\mathcal{O}) \times (0,T)} \zeta(x) e^{-8s\alpha} \xi^5 \mathcal{D}\sigma \mathcal{P} g^\phi \,\mathrm{d}x\,\mathrm{d}t \\
&+ s^5 \iint_{(\omega\cap\mathcal{O}) \times (0,T)} \zeta(x) e^{-8s\alpha} \xi^5 \mathcal{D}\sigma \left(\Delta g_t^\sigma + \Delta^2 g^\sigma \right) \mathrm{d}x\,\mathrm{d}t \\
&- s^5 \iint_{(\omega\cap\mathcal{O}) \times (0,T)} \zeta(x) e^{-8s\alpha} \xi^5 \mathcal{D}\sigma [\mathcal{P}_t + \Delta \mathcal{P}] \phi \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{4.42}$$

Similarly as in Subsections 4.3.1 and 4.3.2, we need to integrate by parts in both time and space, keeping in mind the definition of the operator $\mathcal{P}$ (see (4.33)). Let us call the five integrals in (4.42) $A_1, \ldots, A_5$, respectively. A careful analysis of each of these terms (after integration by parts), taking into account properties (4.12), (4.22), (4.23) and Young's inequality, we can prove the following estimates for every $s \geq C$:

$$|A_1| \leq \frac{1}{8C} J(\sigma) + Cs^{15} \iint_Q e^{-16s\alpha + 8s\alpha^*} \xi^{18} |g^\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{4.43}$$

$$|A_2| \leq \frac{1}{2C} s^5 \iint_Q e^{8s\alpha} \xi^5 |\mathcal{D}\sigma|^2 \,\mathrm{d}x\,\mathrm{d}t + C \left\| e^{-7/2s\alpha^*} g^\psi \right\|_{Y_1}^2, \tag{4.44}$$

$$|A_3| \leq \frac{1}{8C} J(\sigma) + Cs^{19} \iint_Q e^{-16s\alpha + 8s\alpha^*} \xi^{22} |g^\phi|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{4.45}$$

$$|A_4| \leq \frac{1}{8C} J(\sigma) + C \left\| e^{-7/2s\alpha^*} g^\sigma \right\|_{X_0}^2, \tag{4.46}$$

$$|A_5| \leq \frac{1}{8C} J(\sigma) + Cs^{23} \iint_{\omega \times (0,T)} e^{-16s\alpha + 8s\alpha^*} \xi^{26} |\phi|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{4.47}$$

From (4.41)-(4.47) we obtain

$$\begin{aligned}
I_4(\sigma) \leq C\Big(&\left\| s^{15/2} e^{-8s\alpha + 4s\alpha^*} \xi^9 g^\varphi \right\|_{Y_0}^2 + \left\| e^{-7/2s\alpha^*} g^\psi \right\|_{Y_1}^2 + \left\| s^{19/2} e^{-8s\alpha + 4s\alpha^*} \xi^{11} g^\phi \right\|_{X_0}^2 \\
&+ \left\| e^{-7/2s\alpha^*} g^\sigma \right\|_{X_4}^2 \Big) + Cs^{23} \iint_{\omega \times (0,T)} e^{-16s\alpha + 8s\alpha^*} \xi^{26} |\phi|^2 \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{4.48}$$

for every $s \geq C$.

Conclusion of the proof of (4.21)

To conclude the proof of Proposition 4.15, we go back to (4.32), which combined with (4.48), taking into account that

$$s^9 e^{-9s\alpha}\xi^9 \le Cs^{15}e^{-16s\alpha+8s\alpha^*}\xi^{18},$$

$$s^{13} e^{-9s\alpha}\xi^{13} \le Cs^{19}e^{-16s\alpha+8s\alpha^*}\xi^{22},$$

$$s^{23} e^{-16s\alpha+8s\alpha^*}\xi^{26} \le Cs^{26}e^{-18s\alpha+11s\alpha^*}\xi^{30},$$

for every $s \ge C$ and choosing ε small enough we finally obtain

$$\begin{aligned}I_1(\varphi) + I_2(\psi) + I_3(\phi) + I_4(\sigma) \le C \Big(&\left\| s^{15/2}e^{-8s\alpha+4s\alpha^*}\xi^9\, g^\varphi \right\|^2_{Y_0} + \left\| e^{-7/2s\alpha^*}\, g^\psi \right\|^2_{Y_2} \\ &+ \left\| s^{19/2}e^{-8s\alpha+4s\alpha^*}\xi^{11} g^\phi \right\|^2_{X_0} + \left\| e^{-7/2s\alpha^*} g^\sigma \right\|^2_{X_4} \Big) \\ &+ Cs^{13}\iint_{\omega\times(0,T)} e^{-9s\alpha}\xi^{13}|\varphi_1|^2 \mathrm{d}x\,\mathrm{d}t + Cs^{26}\iint_{\omega\times(0,T)} e^{-18s\alpha+11s\alpha^*}\xi^{30}|\phi|^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

for every $s \ge C$, from which we readily deduce (4.21).

This concludes the proof of Proposition 4.15.

4.4 Null controllability of the linear system

In this section we deal with the null controllability of system

$$\begin{cases} \mathcal{L}w + \nabla p_0 = f^w + v\,\mathbb{1}_\omega + re_N, & \nabla\cdot w = 0 \quad \text{in } Q, \\ \mathcal{L}^* z + \nabla p_1 = f^z + w\mathbb{1}_{\mathcal{O}}, & \nabla \cdot z = 0 \quad \text{in } Q, \\ \mathcal{L}r = f^r + v_0\,\mathbb{1}_\omega & \text{in } Q, \\ \mathcal{L}^* q = f^q + z_N + r\mathbb{1}_{\mathcal{O}} & \text{in } Q, \\ w = z = 0, \quad r = q = 0 & \text{on } \Sigma, \\ w(0) = 0, \quad z(T) = 0, \quad r(0) = 0, \quad q(T) = 0 & \text{in } \Omega. \end{cases} \tag{4.49}$$

Here, we will assume that f^w, f^z, f^r and f^q are in appropriate weighted functional spaces. We look for controls (v, v_0), such that $v_{i_0} \equiv v_N \equiv 0$, for some given $0 < i_0 < N$, such that the associated solution of (4.49) satisfies $z(0) = 0$ and $q(0) = 0$ in Ω.

To do this, let us first state a Carleman inequality with weight functions not vanishing in $t = T$. We introduce the following weight functions :

$$\begin{aligned} \beta(x,t) &:= \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\eta(x)}}{\widetilde{\ell}(t)^m}, & \gamma(x,t) &:= \frac{e^{\lambda\eta(x)}}{\widetilde{\ell}(t)^m}, \\ \beta^*(t) &:= \max_{x\in\overline{\Omega}} \beta(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - 1}{\widetilde{\ell}(t)^m}, & \gamma^*(t) &:= \min_{x\in\overline{\Omega}} \gamma(x,t) = \frac{1}{\widetilde{\ell}(t)^m}, \\ \widehat{\beta}(t) &:= \min_{x\in\overline{\Omega}} \beta(x,t) = \frac{e^{2\lambda\|\eta\|_\infty} - e^{\lambda\|\eta\|_\infty}}{\widetilde{\ell}(t)^m}, & \widehat{\gamma}(t) &:= \max_{x\in\overline{\Omega}} \gamma(x,t) = \frac{e^{\lambda\|\eta\|_\infty}}{\widetilde{\ell}(t)^m}, \end{aligned}$$

where

$$\widetilde{\ell}(t) = \begin{cases} \ell(t) & 0 \le t \le T/2, \\ \|\ell\|_\infty & T/2 < t \le T. \end{cases}$$

Lemma 4.16. *Let $0 < j_0 < N$ and let g^φ, g^ϕ, s and λ be like in Proposition 4.15. Furthermore, assume that $g^\psi \in Y_{2,0}$ and $g^\sigma \in X_{4,0}$. Then, there exists a constant $C > 0$ (depending on s and λ) such that every solution $(\varphi, \psi, \phi, \sigma)$ of (4.20) satisfies*

$$\iint_Q e^{-25/2s\beta^*} \left(|\varphi|^2 + |\psi|^2 + |\phi|^2 + |\sigma|^2\right) dx\, dt \le C \Bigg(\left\| e^{-13/4s\beta^*} g^\varphi \right\|^2_{Y_0} + \left\| e^{-13/4s\beta^*} g^\psi \right\|^2_{Y_{2,0}} + \left\| e^{-13/4s\beta^*} g^\phi \right\|^2_{X_0} + \left\| e^{-13/4s\beta^*} g^\sigma \right\|^2_{X_{4,0}} + (N-2) \iint_{\omega\times(0,T)} e^{-13/2s\beta^*} |\varphi_{j_0}|^2 dx\, dt + \iint_{\omega\times(0,T)} e^{-13/2s\beta^*} |\phi|^2 dx\, dt \Bigg). \tag{4.50}$$

To prove estimate (4.50) it suffices to combine (4.21) and classical energy estimates for the Stokes system and the heat equation satisfied by φ, ψ, ϕ and σ. For simplicity, we omit the proof. For more details on how to obtain (4.50), see [6], [7] or [49]. Notice that, in order to obtain this more compact form, we have strongly used the property (4.22) and the assumptions $g^\psi \in Y_{2,0}$ and $g^\sigma \in X_{4,0}$ (see section 4.2.1).

Remark 4.17. *Observe that the additional assumptions on g^ψ and g^σ are not needed to obtain the energy estimates, but the fact that $\varphi(T) \equiv 0$ and $\phi(T) \equiv 0$ is essential.*

Now we are ready to prove the null controllability of system (4.49). The idea is to look for a solution in an appropriate weighted functional space. To this end, we introduce, for $0 < i_0 < N$, the spaces

$$\begin{aligned} E_N^{i_0} := \{ & (w, p_0, z, p_1, r, q, v, v_0) : \, e^{13/4s\beta^*} w \in L^2(Q)^N, \quad e^{13/4s\beta^*} r \in L^2(Q), \\ & e^{13/4s\beta^*} v \mathbb{1}_\omega \in L^2(Q)^N, \quad v_{i_0} \equiv v_N \equiv 0, \quad e^{13/4s\beta^*} v_0 \mathbb{1}_\omega \in L^2(Q), \\ & e^{13/4s\beta^*} (\gamma^*)^{-1-1/m} w \in Y_1, \quad e^{13/4s\beta^*} (\gamma^*)^{-6-6/m} z \in Y_1, \quad z(T) = 0, \\ & e^{13/4s\beta^*} (\gamma^*)^{-1-1/m} r \in X_1, \quad e^{13/4s\beta^*} (\gamma^*)^{-15-15/m} q \in X_1, \quad q(T) = 0, \\ & e^{25/4s\beta^*} \left(\mathcal{L} w + \nabla p_0 - v \mathbb{1}_\omega - r\, e_N, \quad \mathcal{L}^* z + \nabla p_1 - w \mathbb{1}_\mathcal{O}\right) \in L^2(Q)^{2N}, \\ & e^{25/4s\beta^*} \left(\mathcal{L} r - v_0 \mathbb{1}_\omega, \quad \mathcal{L}^* q - z_N - r \mathbb{1}_\mathcal{O}\right) \in L^2(Q)^2 \}. \end{aligned}$$

It is clear that $E_N^{i_0}$ is a Banach space endowed with its natural norm.

Remark 4.18. *In particular, an element $(w, p_0, z, p_1, r, q, v, v_0) \in E_N^{i_0}$ satisfies $w(0) = 0$, $z(0) = 0$, $r(0) = 0$, $q(0) = 0$, $v_{i_0} \equiv v_N \equiv 0$. Moreover, since*

$$e^{-as\beta^*} (\gamma^*)^c \text{ is bounded} \tag{4.51}$$

for any $a > 0$ and $c \in \mathbb{R}$, we have that

$$e^{25/4s\beta^*} \left((w\cdot\nabla)w,\, (w\cdot\nabla)z,\, (z\cdot\nabla^t)w,\, q\nabla r,\, w\cdot\nabla r,\, w\cdot\nabla q\right) \in L^2(Q)^{4N+2}.$$

All the details are given in section 4.5.

Proposition 4.19. *Assume the hypothesis of Lemma 4.16 and*

$$e^{25/4s\beta^*} (f^w, f^z, f^r, f^q) \in L^2(Q)^{2N+2}. \tag{4.52}$$

Let also $i_0 \in \{1, \ldots, N-1\}$. Then, we can find controls $(v, v_0) \in L^2(Q)^{N+1}$ such that the associated solution $(w, p_0, z, p_1, r, q, v, v_0)$ to (4.49) belongs to $E_N^{i_0}$. In particular, $v_{i_0} \equiv v_N \equiv 0$ and $(z(0), q(0)) = (0, 0)$ in Ω.

Proof. Following the arguments in [39] and [51], we introduce the space P_0 of functions $(\varphi, \pi_\varphi, \psi, \pi_\psi, \phi, \sigma) \in \mathcal{C}^\infty(\overline{Q})^{2N+4}$ such that

– $\nabla \cdot \varphi = \nabla \cdot \psi = 0$,
– $\varphi_{|\Sigma} = \psi_{|\Sigma} = 0$, $\phi_{|\Sigma} = \sigma_{|\Sigma} = 0$,
– $\varphi(T) = \psi(0) = 0$, $\phi(T) = \sigma(0) = 0$,
– $\int_\Omega \pi_\varphi \,\mathrm{d}x = \int_\Omega \pi_\psi \,\mathrm{d}x = 0$,
– $\nabla \cdot (\mathcal{L}\psi + \nabla \pi_\psi - \sigma\, e_N) = 0$,
– $(\mathcal{L}_H^k[e^{-13/4s\beta^*}(\mathcal{L}\psi + \nabla\pi_\psi - \sigma\, e_N)])_{|\Sigma} = 0$, $k = 0, 1$,
– $(\mathcal{L}_H^k[e^{-13/4s\beta^*}(\mathcal{L}\psi + \nabla\pi_\psi - \sigma\, e_N)])(0) = 0$, $k = 0, 1$,
– $\mathcal{L}^k[e^{-13/4s\beta^*}\mathcal{L}\sigma]_{|\Sigma} = 0$, $k = 0, \dots, 3$,
– $\mathcal{L}^k[e^{-13/4s\beta^*}\mathcal{L}\sigma](0) = 0$, $k = 0, \dots, 3$.

We define the bilinear form

$$
\begin{aligned}
&a((\widetilde{\varphi}, \widetilde{\pi}_\varphi, \widetilde{\psi}, \widetilde{\pi}_\psi, \widetilde{\phi}, \widetilde{\sigma}), (\varphi, \pi_\varphi, \psi, \pi_\psi, \phi, \sigma)) \\
&:= \iint_Q e^{-13/2s\beta^*}(\mathcal{L}^*\widetilde{\varphi} + \nabla\widetilde{\pi}_\varphi - \widetilde{\psi}\mathbb{1}_\mathcal{O}) \cdot (\mathcal{L}^*\varphi + \nabla\pi_\varphi - \psi\mathbb{1}_\mathcal{O}) \,\mathrm{d}x\,\mathrm{d}t \\
&+ \iint_Q \mathcal{L}_H^2[e^{-13/4s\beta^*}(\mathcal{L}\widetilde{\psi} + \nabla\widetilde{\pi}_\psi - \widetilde{\sigma}\, e_N)] \cdot \mathcal{L}_H^2[e^{-13/4s\beta^*}(\mathcal{L}\psi + \nabla\pi_\psi - \sigma\, e_N)] \,\mathrm{d}x\,\mathrm{d}t \\
&+ \iint_Q e^{-13/2s\beta^*}(\mathcal{L}^*\widetilde{\phi} - \widetilde{\varphi}_N - \widetilde{\sigma}\,\mathbb{1}_\mathcal{O})(\mathcal{L}^*\phi - \varphi_N - \sigma\,\mathbb{1}_\mathcal{O}) \,\mathrm{d}x\,\mathrm{d}t \\
&+ \iint_Q \mathcal{L}^4[e^{-13/4s\beta^*}\mathcal{L}\widetilde{\sigma}]\mathcal{L}^4[e^{-13/4s\beta^*}\mathcal{L}\sigma] \,\mathrm{d}x\,\mathrm{d}t \\
&+ (N-2)\iint_{\omega\times(0,T)} e^{-13/2s\beta^*}\widetilde{\varphi}_{j_0}\varphi_{j_0} \,\mathrm{d}x\,\mathrm{d}t + \iint_{\omega\times(0,T)} e^{-13/2s\beta^*}\widetilde{\phi}\,\phi \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

where $j_0 \in \{1, \dots, N-1\} \setminus \{i_0\}$ and a linear form

$$
\langle G, (\varphi, \pi_\varphi, \psi, \pi_\psi, \phi, \sigma)\rangle := \iint_Q f^w \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t + \iint_Q f^z \cdot \psi \,\mathrm{d}x\,\mathrm{d}t + \iint_Q f^r\, \phi \,\mathrm{d}x\,\mathrm{d}t + \iint_Q f^q\, \sigma \,\mathrm{d}x\,\mathrm{d}t.
$$

Thanks to (4.50), we have that $a(\cdot,\cdot) : P_0 \times P_0 \mapsto \mathbb{R}$ is a symmetric, definite positive bilinear form on P_0. We denote by P the completion of P_0 for the norm induced by $a(\cdot,\cdot)$. Then, $a(\cdot,\cdot)$ is well-defined, continuous and definite positive on P. Furthermore, in view of the Carleman estimate (4.50) and the assumptions (4.52), the linear form $(\varphi, \pi_\varphi, \psi, \pi_\psi, \phi, \sigma) \mapsto \langle G, (\varphi, \pi_\varphi, \psi, \pi_\psi, \phi, \sigma)\rangle$ is well-defined and continuous on P. Hence, from Lax-Milgram's lemma, we deduce that the variational problem :

$$
\left\{
\begin{array}{l}
\text{Find } (\widetilde{\varphi}, \widetilde{\pi}_\varphi, \widetilde{\psi}, \widetilde{\pi}_\psi, \widetilde{\phi}, \widetilde{\sigma}) \in P \text{ such that} \\
a((\widetilde{\varphi}, \widetilde{\pi}_\varphi, \widetilde{\psi}, \widetilde{\pi}_\psi, \widetilde{\phi}, \widetilde{\sigma}), (\varphi, \pi_\varphi, \psi, \pi_\psi, \phi, \sigma)) = \langle G, (\varphi, \pi_\varphi, \psi, \pi_\psi, \phi, \sigma)\rangle \quad \forall\, (\varphi, \pi_\varphi, \psi, \pi_\psi, \phi, \sigma) \in P,
\end{array}
\right.
\tag{4.53}
$$

possesses exactly one solution $(\widehat{\varphi}, \widehat{\pi}_\varphi, \widehat{\psi}, \widehat{\pi}_\psi, \widehat{\phi}, \widehat{\sigma})$.

Let $\widehat{v}$ and $\widehat{v}_0$ be given by

$$
\left\{
\begin{array}{ll}
\widehat{v}_{j_0} := -(N-2)e^{-13/2s\beta^*}\widehat{\varphi}_{j_0}\mathbb{1}_\omega,\ \widehat{v}_j \equiv 0,\ j \neq j_0, & \text{in } Q, \\
\widehat{v}_0 := -e^{-13/2s\beta^*}\widehat{\phi}\mathbb{1}_\omega, & \text{in } Q.
\end{array}
\right.
\tag{4.54}
$$

It is straightforward from (4.53) and (4.54) that we have

$$
\begin{aligned}
&\iint_Q \left(|\widetilde{w}|^2 + |\widetilde{z}|^2 + |\widetilde{r}|^2 + |\widetilde{q}|^2\right) \mathrm{d}x\,\mathrm{d}t \\
&+ \iint_{\omega\times(0,T)} e^{13/2s\beta^*}\left((N-2)|\widehat{v}_{j_0}|^2 + |\widehat{v}_0|^2\right) \mathrm{d}x\,\mathrm{d}t < +\infty,
\end{aligned}
\tag{4.55}
$$

where $\widetilde{w}$, $\widetilde{z}$, $\widetilde{r}$ and $\widetilde{q}$ are given by

$$\begin{cases} \widetilde{w} := e^{-13/4s\beta^*}(\mathcal{L}^*\widehat{\varphi} + \nabla\widehat{\pi}_\varphi - \widehat{\psi}\mathbb{1}_{\mathcal{O}}), \\ \widetilde{z} := \mathcal{L}_H^2[e^{-13/4s\beta^*}(\mathcal{L}\widehat{\psi} + \nabla\widehat{\pi}_\psi - \widehat{\sigma}\, e_N)], \\ \widetilde{r} := e^{-13/4s\beta^*}(\mathcal{L}^*\widehat{\phi} - \widehat{\varphi}_N - \widehat{\sigma}\,\mathbb{1}_{\mathcal{O}}), \\ \widetilde{q} := \mathcal{L}^4[e^{-13/4s\beta^*}\mathcal{L}\widehat{\sigma}]. \end{cases} \tag{4.56}$$

In particular, $\widehat{v} \in L^2(Q)^N$, $\widehat{v}_0 \in L^2(Q)$.

Let us call $(\widehat{w},\ \widehat{z},\ \widehat{r},\ \widehat{q})$, together with some pressures $(\widehat{p}_0, \widehat{p}_1)$, the (weak) solution of (4.49) with $v = \widehat{v}$ and $v_0 = \widehat{v}_0$, that is, they solve

$$\begin{cases} \mathcal{L}\widehat{w} + \nabla\widehat{p}_0 = f^w + \widehat{v}\,\mathbb{1}_\omega + \widehat{r}e_N, & \nabla\cdot\widehat{w} = 0 & \text{in } Q, \\ \mathcal{L}^*\widehat{z} + \nabla\widehat{p}_1 = f^z + \widehat{w}\mathbb{1}_{\mathcal{O}}, & \nabla\cdot\widehat{z} = 0 & \text{in } Q, \\ \mathcal{L}\widehat{r} = f^r + \widehat{v}_0\,\mathbb{1}_\omega & & \text{in } Q, \\ \mathcal{L}^*\widehat{q} = f^q + \widehat{z}_N + \widehat{r}\mathbb{1}_{\mathcal{O}} & & \text{in } Q, \\ \widehat{w} = \widehat{z} = 0, \quad \widehat{r} = \widehat{q} = 0 & & \text{on } \Sigma, \\ \widehat{w}(0) = 0, \quad \widehat{z}(T) = 0, \quad \widehat{r}(0) = 0, \quad \widehat{q}(T) = 0 & & \text{in } \Omega. \end{cases} \tag{4.57}$$

The rest of the proof is devoted to prove the following exponential decay properties

$$\begin{aligned} &e^{13/4s\beta^*}(\gamma^*)^{-1-1/m}\widehat{w} \in Y_1, \quad e^{13/4s\beta^*}(\gamma^*)^{-6-6/m}\widehat{z} \in Y_1 \\ &e^{13/4s\beta^*}(\gamma^*)^{-1-1/m}\widehat{r} \in X_1, \quad e^{13/4s\beta^*}(\gamma^*)^{-15-15/m}\widehat{q} \in X_1, \end{aligned} \tag{4.58}$$

which will solve the null controllability problem for system (4.49) (see Remark 4.18).

First, we will prove that $(\widetilde{w}, \widetilde{z}, \widetilde{r}, \widetilde{q})$ given by (4.56) is actually the solution (in the sense of transposition) of

$$\begin{cases} e^{-13/4s\beta^*}\widetilde{w} = \widehat{w} & & \text{in } Q, \\ e^{-13/4s\beta^*}(\mathcal{L}_H^*)^2\widetilde{z} = \widehat{z}, & \nabla\cdot\widetilde{z} = 0 & \text{in } Q, \\ e^{-13/4s\beta^*}\widetilde{r} = \widehat{r} & & \text{in } Q, \\ e^{-13/4s\beta^*}(\mathcal{L}^*)^4\widetilde{q} = \widehat{q} & & \text{in } Q, \end{cases} \tag{4.59}$$

such that

$$\begin{cases} (\mathcal{L}_H^*)^\ell\widetilde{z} = 0 & \text{on } \Sigma, \quad \ell = 0, 1, \\ (\mathcal{L}_H^*)^\ell\widetilde{z}(T) = 0 & \text{in } \Omega, \quad \ell = 0, 1, \\ (\mathcal{L}^*)^k\widetilde{q} = 0 & \text{on } \Sigma, \quad k = 0, \dots, 3, \\ (\mathcal{L}^*)^k\widetilde{q}(T) = 0 & \text{in } \Omega, \quad k = 0, \dots, 3. \end{cases} \tag{4.60}$$

Now, from (4.53), (4.54), (4.56) and (4.57), we obtain for every $(\varphi, \pi_\varphi, \psi, \pi_\psi, \phi, \sigma) \in P_0$

$$\begin{aligned} &\iint_Q \widetilde{w}\cdot e^{-13/4s\beta^*}(\mathcal{L}^*\varphi + \nabla\pi_\varphi - \psi\mathbb{1}_{\mathcal{O}})\mathrm{d}x\,\mathrm{d}t + \iint_Q \widetilde{z}\cdot\mathcal{L}_H^2[e^{-13/4s\beta^*}(\mathcal{L}\psi + \nabla\pi_\psi - \sigma\, e_N)]\mathrm{d}x\,\mathrm{d}t \\ &+ \iint_Q \widetilde{r}\, e^{-13/4s\beta^*}(\mathcal{L}^*\phi - \varphi_N - \sigma\,\mathbb{1}_{\mathcal{O}})\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widetilde{q}\,\mathcal{L}^4[e^{-13/4s\beta^*}\mathcal{L}\sigma]\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad = \iint_Q \varphi\cdot(\mathcal{L}\widehat{w} + \nabla\widehat{p}_0 - \widehat{r}\, e_N)\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \psi\cdot(\mathcal{L}^*\widehat{z} + \nabla\widehat{p}_1 - \widehat{w}\mathbb{1}_{\mathcal{O}})\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \iint_Q \phi\,\mathcal{L}\widehat{r}\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \sigma\,(\mathcal{L}^*\widehat{q} - \widehat{z}_N - \widehat{r}\mathbb{1}_{\mathcal{O}})\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad = \iint_Q \widehat{w}\cdot(\mathcal{L}^*\varphi + \nabla\pi_\varphi - \psi\mathbb{1}_{\mathcal{O}})\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widehat{z}\cdot(\mathcal{L}\psi + \nabla\pi_\psi - \sigma\, e_N)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \iint_Q \widehat{r}\,(\mathcal{L}^*\phi - \varphi_N - \sigma\,\mathbb{1}_{\mathcal{O}})\,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widehat{q}\,\mathcal{L}\sigma\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

From this last equality, we obtain for all $(h^w, h^z, h^r, h^q) \in L^2(Q)^{2N+2}$

$$\begin{aligned}&\iint_Q \widetilde{w}\cdot h^w \,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widetilde{z}\cdot h^z \,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widetilde{r}\, h^r \,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widetilde{q}\, h^q \,\mathrm{d}x\,\mathrm{d}t \\ &= \iint_Q \widehat{w}\cdot \Phi^w \,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widehat{z}\cdot \Phi^z \,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widehat{r}\, \Phi^r \,\mathrm{d}x\,\mathrm{d}t + \iint_Q \widehat{q}\, \Phi^q \,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{4.61}$$

where $(\Phi^w, \Phi^z, \Phi^r, \Phi^q)$ is the solution of

$$\begin{cases} e^{-13/4s\beta^*}\Phi^w = h^w & \text{in } Q, \\ \mathcal{L}_H^2[e^{-13/4s\beta^*}\Phi^z] = h^z, \quad \nabla\cdot\Phi^z = 0 & \text{in } Q, \\ e^{-13/4s\beta^*}\Phi^r = h^r & \text{in } Q, \\ \mathcal{L}^4[e^{-13/4s\beta^*}\Phi^q] = h^q & \text{in } Q, \end{cases} \tag{4.62}$$

such that

$$\begin{cases} \mathcal{L}_H^\ell(e^{-13/4s\beta^*}\Phi^z) = 0 & \text{on } \Sigma, \quad \ell = 0,1, \\ \mathcal{L}_H^\ell(e^{-13/4s\beta^*}\Phi^z)(0) = 0 & \text{in } \Omega, \quad \ell = 0,1, \\ \mathcal{L}^k(e^{-13/4s\beta^*}\Phi^q) = 0 & \text{on } \Sigma, \quad k = 0,\dots,3, \\ \mathcal{L}^k(e^{-13/4s\beta^*}\Phi^q)(0) = 0 & \text{in } \Omega, \quad k = 0,\dots,3. \end{cases} \tag{4.63}$$

It is classical to show that (4.61)-(4.63) is equivalent to (4.59)-(4.60).

Next, we define the following functions :

$$(z_{*,0}, p_{*,0}) := e^{13/4s\beta^*}(\gamma^*)^{-3-3/m}(\widehat{z}, \widehat{p}_1), \quad f^z_{*,0} := e^{13/4s\beta^*}(\gamma^*)^{-3-3/m}(f^z + \widehat{w}\mathbb{1}_{\mathcal{O}}).$$

Observe that, from (4.52), (4.55) and (4.59), we have $f^z_{*,0} \in L^2(Q)^N$. Then, by (4.57) $z_{*,0}$ satisfies

$$\begin{cases} \mathcal{L}^* z_{*,0} + \nabla p_{*,0} = f^z_{*,0} - (e^{13/4s\beta^*}(\gamma^*)^{-3-3/m})_t\widehat{z}, \quad \nabla\cdot z_{*,0} = 0 & \text{in } Q, \\ z_{*,0} = 0 & \text{on } \Sigma, \\ z_{*,0}(T) = 0 & \text{in } \Omega, \end{cases}$$

where the last term in the right-hand side can be written as

$$(e^{13/4s\beta^*}(\gamma^*)^{-3-3/m})_t\widehat{z} = c_2(t)(\mathcal{L}_H^*)^2\widetilde{z},$$

where $c_k(t)$ denotes a generic function such that (see (4.12))

$$|c_k^{(\ell)}(t)| \le C < \infty, \quad \forall\, \ell = 0,\dots,k. \tag{4.64}$$

On the other hand, for any $h \in Y_{1,0}$ we have

$$\iint_Q z_{*,0}\cdot h \,\mathrm{d}x\,\mathrm{d}t = \iint_Q f^z_{*,0}\cdot\Phi \,\mathrm{d}x\,\mathrm{d}t - \iint_Q c_2(t)(\mathcal{L}^*)^2_H\widetilde{z}\cdot\Phi \,\mathrm{d}x\,\mathrm{d}t,$$

where Φ solves, together with some pressure π_Φ,

$$\begin{cases} \mathcal{L}\Phi + \nabla\pi_\Phi = h, \quad \nabla\cdot\Phi = 0 & \text{in } Q, \\ \Phi = 0 & \text{on } \Sigma, \\ \Phi(0) = 0 & \text{in } \Omega. \end{cases}$$

Using (4.60), we can integrate by parts to obtain

$$\begin{aligned}\iint_Q z_{*,0}\cdot h \,\mathrm{d}x\,\mathrm{d}t &= \iint_Q f^z_{*,0}\cdot\Phi \,\mathrm{d}x\,\mathrm{d}t - \iint_Q \mathcal{L}_H^*\widetilde{z}\cdot(\mathcal{L}[c_2(t)\Phi] + \nabla(c_2(t)\pi_\Phi)) \,\mathrm{d}x\,\mathrm{d}t \\ &= \iint_Q f^z_{*,0}\cdot\Phi \,\mathrm{d}x\,\mathrm{d}t - \iint_Q \widetilde{z}\cdot\mathcal{L}[c_2'(t)\Phi + c_2(t)h] \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Notice that here we have relied on the fact that $\mathcal{L}_H^*\widetilde{z}$, Φ and h belong to the space H. Since

$$\|\Phi\|_{Y_2} \leq C\|h\|_{Y_{1,0}},$$

(see regularity result (4.19)), we obtain from the last equality, together with (4.64),

$$\iint_Q z_{*,0}\cdot h \,\mathrm{d}x\,\mathrm{d}t \leq C\left[\left\|f^z_{*,0}\right\|_{L^2(Q)^N} + \|\widetilde{z}\|_{L^2(Q)^N}\right]\|h\|_{Y_{1,0}}, \quad \forall h\in Y_{1,0}. \tag{4.65}$$

Now, let

$$(z_{*,1},p_{*,1}) := e^{13/4s\beta^*}(\gamma^*)^{-5-5/m}(\widehat{z},\widehat{p}_1), \quad f^z_{*,1} := e^{13/4s\beta^*}(\gamma^*)^{-5-5/m}(f^z+\widehat{w}\mathbb{1}_{\mathcal{O}}).$$

Similarly as before, $(z_{*,1},p_{*,1})$ satisfies

$$\begin{cases} \mathcal{L}^*z_{*,1}+\nabla p_{*,1} = f^z_{*,1} - (e^{13/4s\beta^*}(\gamma^*)^{-5-5/m})_t\widehat{z}, \quad \nabla\cdot z_{*,1}=0 & \text{in } Q,\\ z_{*,1}=0 & \text{on } \Sigma,\\ z_{*,1}(T)=0 & \text{in } \Omega. \end{cases}$$

Thus, for any $h\in Y_0$ we get

$$\iint_Q z_{*,1}\cdot h\,\mathrm{d}x\,\mathrm{d}t = \iint_Q f^z_{*,1}\cdot\Phi\,\mathrm{d}x\,\mathrm{d}t - \iint_Q (e^{13/4s\beta^*}(\gamma^*)^{-5-5/m})_t\widehat{z}\cdot\Phi\,\mathrm{d}x\,\mathrm{d}t.$$

Moreover, since

$$\iint_Q (e^{13/4s\beta^*}(\gamma^*)^{-5-5/m})_t\widehat{z}\cdot\Phi\,\mathrm{d}x\,\mathrm{d}t = \iint_Q c_1(t)\Phi\cdot z_{*,0}\,\mathrm{d}x\,\mathrm{d}t,$$

using (4.65) with $c_1(t)\Phi$ instead of h (notice that $c_1(t)\Phi\in Y_{1,0}$), we get the estimate

$$\iint_Q c_1(t)\Phi\cdot z_{*,0}\,\mathrm{d}x\,\mathrm{d}t \leq C\left[\left\|f^z_{*,0}\right\|_{L^2(Q)^N} + \|\widetilde{z}\|_{L^2(Q)^N}\right]\|c_1(t)\Phi\|_{Y_{1,0}}.$$

Turning back to $z_{*,1}$, we get

$$\iint_Q z_{*,1}\cdot h\,\mathrm{d}x\,\mathrm{d}t \leq C\left[\left\|f^z_{*,0}\right\|_{L^2(Q)^N} + \|\widetilde{z}\|_{L^2(Q)^N}\right]\|\Phi\|_{Y_{1,0}},$$

where we have used (4.64) and the property $(\gamma^*)^{-5-5/m} \leq C(\gamma^*)^{-3-3/m}$. Taking into account that

$$\|\Phi\|_{Y_{1,0}} \leq C\|h\|_{Y_0},$$

(see (4.18)) we obtain

$$\iint_Q z_{*,1}\cdot h\,\mathrm{d}x\,\mathrm{d}t \leq C\left[\left\|f^z_{*,0}\right\|_{L^2(Q)^N} + \|\widetilde{z}\|_{L^2(Q)^N}\right]\|h\|_{Y_0}, \quad \forall h\in Y_0.$$

Thus, we deduce that $z_{*,1}\in L^2(Q)^N$. Following the same iterative argument we can show that $e^{13/4s\beta^*}(\gamma^*)^{-14-14/m}\widehat{q}\in L^2(Q)^N$.

Finally, to complete the proof of (4.58), let

$$(z_*,p_{1*}) := e^{13/4s\beta^*}(\gamma^*)^{-6-6/m}(\widehat{z},\widehat{p}_1), \quad f^z_* := e^{13/4s\beta^*}(\gamma^*)^{-6-6/m}(f^z+\widehat{w}\mathbb{1}_{\mathcal{O}}).$$

Then, (z_*,p_{1*}) satisfies

$$\begin{cases} \mathcal{L}^*z_*+\nabla p_{1*} = f^z_* + (e^{13/4s\beta^*}(\gamma^*)^{-6-6/m})_t\widehat{z}, \quad \nabla\cdot z_*=0 & \text{in } Q,\\ z_*=0 & \text{on } \Sigma,\\ z_*(T)=0 & \text{in } \Omega. \end{cases}$$

From (4.52), (4.55), (4.59), (4.12) and $z_{*,1}\in L^2(Q)^N$, we have that the right-hand side of this equation belongs to $L^2(Q)^N$. Using the regularity result (4.18), we deduce that $z_*\in Y_1$. Similarly, we are able to obtain the rest of the regularity properties in (4.58). This concludes the proof of Proposition 4.19. □

4.5 Proof of Theorem 4.1

Recall that we deal with the following null controllability problem : to find controls (v, v_0) verifying $v_{i_0} \equiv v_N \equiv 0$ such that the solution of the system

$$\begin{cases} \mathcal{L}w + (w\cdot\nabla)w + \nabla p_0 = f + v\mathbb{1}_\omega + r\,e_N, & \nabla\cdot w = 0 \quad \text{in } Q,\\ \mathcal{L}^* z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + q\nabla r + \nabla p_1 = w\mathbb{1}_{\mathcal{O}}, & \nabla\cdot z = 0 \quad \text{in } Q,\\ \mathcal{L}r + w\cdot\nabla r = f_0 + v_0\mathbb{1}_\omega & \text{in } Q,\\ \mathcal{L}^* q - w\cdot\nabla q = z_N + r\,\mathbb{1}_{\mathcal{O}} & \text{in } Q,\\ w = z = 0, \quad r = q = 0 & \text{on } \Sigma,\\ w(0) = 0, \quad z(T) = 0, \quad r(0) = 0, \quad q(T) = 0 & \text{in } \Omega, \end{cases} \tag{4.66}$$

satisfies $(z(0), q(0)) = (0, 0)$.

We proceed using similar arguments to those in [51] (see also [7], [35], [49] and [9]). The null controllability result for the linear system given by Proposition 4.19 will allow us to apply the following inverse mapping theorem (see [3]) :

Theorem 4.20. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two Banach spaces and let $\mathcal{A} : \mathcal{B}_2 \to \mathcal{B}_2$ satisfy $\mathcal{A} \in \mathcal{C}^1(\mathcal{B}_2; \mathcal{B}_2)$. Assume that $b_1 \in \mathcal{B}_1$, $\mathcal{A}(b_1) = b_2$ and that $\mathcal{A}'(b_1) : \mathcal{B}_1 \to \mathcal{B}_2$ is surjective. Then, there exists $\delta > 0$ such that, for every $b' \in \mathcal{B}_2$ satisfying $\|b' - b_2\|_{\mathcal{B}_2} < \delta$, there exists a solution of the equation*

$$\mathcal{A}(b) = b', \quad b \in \mathcal{B}_1.$$

Let us set the framework to apply Theorem 4.20 to the problem at hand. Let

$$\mathcal{B}_1 := E_N^{i_0},$$

$$\mathcal{B}_2 := L^2(e^{25/4s\beta^*}(0,T); L^2(\Omega)^{2N+2})$$

and the operator

$$\begin{aligned} \mathcal{A}(w, p_0, z, p_1, r, q, v, v_0) := (&\mathcal{L}w + (w\cdot\nabla)w + \nabla p_0 - v\mathbb{1}_\omega - r\,e_N,\\ &\mathcal{L}^* z + (z\cdot\nabla^t)w - (w\cdot\nabla)z + q\nabla r + \nabla p_1 - w\mathbb{1}_{\mathcal{O}},\\ &\mathcal{L}r + w\cdot\nabla r - v_0\mathbb{1}_\omega, \quad \mathcal{L}^* q - w\cdot\nabla q - z_N - r\,\mathbb{1}_{\mathcal{O}}) \end{aligned}$$

for $(w, p_0, z, p_1, r, q, v, v_0) \in \mathcal{B}_1$. Here, $u \in L^2(e^{25/4s\beta^*}(0,T); L^2(\Omega))$ means $e^{25/4s\beta^*}u \in L^2(Q)$.

It only remains to check that the operator $\mathcal{A}$ is of class $\mathcal{C}^1(\mathcal{B}_1; \mathcal{B}_2)$. To do this, we notice that all the terms in $\mathcal{A}$ are linear, except for $(w\cdot\nabla)w$, $(z\cdot\nabla^t)w - (w\cdot\nabla)z$, $q\nabla r$, $w\cdot\nabla r$ and $w\cdot\nabla q$. So it will suffice to prove that the bilinear operator

$$((w^1, p_0^1, z^1, p_1^1, r^1, q^1, v^1, v_0^1), (w^2, p_0^2, z^2, p_1^2, r^2, q^2, v^2, v_0^2)) \to (w^1\cdot\nabla)w^2$$

is continuous from $\mathcal{B}_1 \times \mathcal{B}_1$ to $L^2(e^{25/4s\beta^*}(0,T); L^2(\Omega)^N)$. Since $Y_1 \subset L^\infty(0,T;V)$, we have that

$$e^{13/4s\beta^*}(\gamma^*)^{-1-1/m} w \in L^2(0,T; H^2(\Omega)^N) \cap L^\infty(0,T;V)$$

for any $(w, p_0, z, p_1, r, q, v, v_0) \in \mathcal{B}_1$. Consequently

$$e^{13/4s\beta^*}(\gamma^*)^{-1-1/m} w \in L^2(0,T; L^\infty(\Omega)^N)$$

and

$$\nabla(e^{13/4s\beta^*}(\gamma^*)^{-1-1/m} w) \in L^\infty(0,T; L^2(\Omega)^{N\times N}).$$

Thus, we obtain

$$\begin{aligned}
&\|e^{13/2s\beta^*}(\gamma^*)^{-2-2/m}(w^1\cdot\nabla)w^2\|_{L^2(Q)^N}\\
&\leq C\|(e^{13/4s\beta^*}(\gamma^*)^{-1-1/m}w^1\cdot\nabla)e^{13/4s\beta^*}(\gamma^*)^{-1-1/m}w^2\|_{L^2(Q)^N}\\
&\leq C\|e^{13/4s\beta^*}(\gamma^*)^{-1-1/m}w^1\|_{L^2(0,T;L^\infty(\Omega)^N)}\,\|e^{13/4s\beta^*}(\gamma^*)^{-1-1/m}w^2\|_{L^\infty(0,T;V)},
\end{aligned}$$

and the continuity follows since $25/4 < 13/2$ and thanks to (4.51). The terms $(z\cdot\nabla^t)w$, $(w\cdot\nabla)z$ are treated analogously.

Finally, we can prove in the same way that the bilinear operator

$$((w^1,p_0^1,z^1,p_1^1,r^1,q^1,v^1,v_0^1),(w^2,p_0^2,z^2,p_1^2,r^2,q^2,v^2,v_0^2))\to(w^1\cdot\nabla r^2,w^1\cdot\nabla q^2)$$

is continuous from $\mathcal{B}_1\times\mathcal{B}_1$ to $L^2(e^{25/4s\beta^*}(0,T);L^2(\Omega)^2)$ just by taking into account that

$$e^{13/4s\beta^*}\left((\gamma^*)^{-1-1/m}\,r,(\gamma^*)^{-15-15/m}\,q\right)\in L^\infty(0,T;H_0^1(\Omega)^2),$$

for any $(w,p_0,z,p_1,r,q,v,v_0)\in\mathcal{B}_1$.

It is readily seen that $\mathcal{A}'(0):\mathcal{B}_1\to\mathcal{B}_2$ is given by

$$\begin{aligned}
\mathcal{A}'(0)(w,p_0,z,p_1,r,q,v,v_0)=\,&(\mathcal{L}w+\nabla p_0-v\mathbb{1}_\omega-r\,e_N,\quad \mathcal{L}^*z+\nabla p_1-w\mathbb{1}_{\mathcal{O}},\\
&\mathcal{L}r-v_0\mathbb{1}_\omega,\quad \mathcal{L}^*q-z_N-r\,\mathbb{1}_{\mathcal{O}})\,,
\end{aligned}$$

for all $(w,p_0,z,p_1,r,s,v,v_0)\in\mathcal{B}_1$. It follows that this functional is surjective in view of the null controllability result for the linear system given by Proposition 4.19.

Now, we are in condition to apply Theorem 4.20. By taking $b_1=0$ and $b_2=0$, it gives the existence of $\delta>0$ such that, if $\|e^{C/t^m}(f,f_0)\|_{L^2(Q)^{N+1}}\leq\delta$, for some $C>0$, then we can find $(w,p_0,z,p_1,r,q,v,v_0)\in\mathcal{B}_1$ solution of (4.66). In particular, $v_{i_0}\equiv v_N\equiv 0$ and $(z(0),q(0))=(0,0)$ (see Remark 4.18). Therefore, the proof of Theorem 4.1 is complete.

Deuxième partie

On the null controllability of a linear KdV equation with Colin-Ghidaglia boundary conditions in the vanishing dispersion limit

Chapitre 5

On the null controllability of a linear KdV equation in the vanishing dispersion limit

In this chapter we improve an existent null controllability result for the linear KdV equation with Colin-Ghidaglia boundary conditions. Furthermore, we study the uniform controllability in the vanishing dispersion limit.

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5.1 Introduction

Let $T > 0$ and $Q := (0, T) \times (0, 1)$. We consider the following controlled Korteweg-de Vries (KdV) equation posed in a finite domain :

$$\begin{cases} y_t + \varepsilon y_{xxx} - M y_x = 0 & \text{in } Q, \\ y_{|x=0} = v, \quad y_{x|x=1} = 0, \quad y_{xx|x=1} = 0 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,1), \end{cases} \tag{5.1}$$

where $\varepsilon > 0$ is the dispersion coefficient, $M \in \mathbb{R}$ is the transport coefficient, $y_0 \in L^2(0,1)$ is the initial condition and $v = v(t)$ stands for the control. Here and all along the chapter we use the notation, for a given function $f = f(t, x)$,

$$f_{|x=x_0} := f(\cdot, x_0) \quad \text{and} \quad f_{|t=t_0} := f(t_0, \cdot).$$

The boundary conditions in (5.1) were proposed by T. Colin and J.-M. Ghidaglia in [15] (see also [14]) as a model for propagation of surface water waves in the situation where a wave maker is putting energy in a finite-length channel from the left extremity and the right one is free.

Most results for (5.1) are related to its well-posedness of its nonlinear version, for instance, [57, 55, 71]. As for the controllability problem, in [13] the authors considered controls in all the boundary conditions and all the possible combinations.

Our first result states an improvement of the size of the control with respect to the one in [50]. From this work, one can deduce that there exists $v \in L^2(0,T)$ such that the associated solution of (5.1) satisfies $y_{|t=T} = 0$ and

$$\|v\|_{L^2(0,T)} \leq C \exp(C\varepsilon^{-1}) \|y_0\|_{L^2(0,1)}. \tag{5.2}$$

The first main result of this chapter is the following :

Theorem 5.1. *Let $T > 0$, $M \in \mathbb{R}$ and $\varepsilon > 0$ be three fixed real numbers. Then, for any $y_0 \in L^2(0,1)$, there exists a control $v \in L^2(0,T)$ such that the associated solution of* (5.1) *satisties $y_{|t=T} = 0$. Furthermore, we have the estimate*

$$\|v\|_{L^2(0,T)} \leq \bar{C} \exp\Big(C(\varepsilon^{-1/2}T^{-1/2} + M^{1/2}\varepsilon^{-1/2} + MT)\Big) \|y_0\|_{L^2(0,1)}, \tag{5.3}$$

if $M > 0$, and

$$\|v\|_{L^2(0,T)} \leq \bar{C} \exp\Big(C(\varepsilon^{-1/2}T^{-1/2} + |M|^{1/2}\varepsilon^{-1/2})\Big) \|y_0\|_{L^2(0,1)}, \tag{5.4}$$

if $M < 0$, where $C > 0$ is a constant independent of T, M and ε and $\bar{C} > 0$ depends at most polynomially on ε^{-1}, T^{-1} and $|M|^{-1}$.

We remark that (5.3)-(5.4) say that the size of the control is of order $\exp(C\varepsilon^{-1/2})$, whereas in (5.2) is of order $\exp(C\varepsilon^{-1})$. This difference becomes of great importance when one compares with a dissipation estimate for (5.1) in the limit $\varepsilon \to 0$. Thus, this would be helpful when trying to prove the uniform null controllability of (5.1) with respect to ε, as it is explained below.

Before stating the second main result of this chapter, let us consider the transport equation

$$y_t - My_x = 0 \quad \text{in } Q. \tag{5.5}$$

Since (5.5) is controllable if and only if $T \geq 1/|M|$ (see, for instance, [20, Theorem 2.6, page 29]), with a control $y_{|x=0} = v_1$ if $M < 0$ and a control $y_{|x=1} = v_2$ if $M > 0$. Furthermore, the cost of null controllability is equal to zero. Indeed, the solution of (5.5) can be brought to zero at time T just by taking $v_1 \equiv 0$ when $M < 0$, and by taking $v_2 \equiv 0$ when $M > 0$. Thus, we should expect that the cost would decrease to zero in this case as $\varepsilon \to 0$, or at least if the final time T is large enough. On the other hand, if $T < 1/|M|$ it is expected that the cost of the control would explode as ε tends to zero.

In [43], the authors consider this problem for the classical boundary conditions

$$y_{|x=0} = v(t), \quad y_{|x=1} = 0, \quad y_{x|x=1} = 0 \quad \text{in } (0,T), \tag{5.6}$$

and in [42] with controls in all the boundary terms. We refer also to [22] and [47] for the case of vanishing viscosity in one and arbitrary space dimension, respectively.

The strategy relies on the combination of a suitable Carleman inequality, which gives an observability constant that explodes with ε, with an exponential dissipation estimate for the adjoint equation such that for T large enough counteracts the previous constant.

It has been pointed out in [42] and [43] that such a result can only be expected for (5.1) when $M > 0$ due to the asymmetric effect of the dispersion term.

In our case, the adjoint equation of (5.1) is given by

$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xxx} + M\varphi_x = 0 & \text{in } Q, \\ \varphi_{|x=0} = 0, \quad \varphi_{x|x=0} = 0, \quad (\varepsilon\varphi_{xx} - M\varphi)_{|x=1} = 0 & \text{in } (0,T), \\ \varphi_{|t=T} = \varphi_T & \text{in } (0,1), \end{cases} \tag{5.7}$$

with $\varphi_T \in L^2(0,1)$. Although we can obtain an optimal observability constant with respect to ε for (5.7) (given by (5.3)), we do not know how to prove an exponential dissipation result.

However, we are able to obtain the expected result when T is small enough with respect to $1/|M|$.

Theorem 5.2. *Let $M \neq 0$. Then, there exists $T_0 < 1/|M|$ such that for every $T \in (0, T_0)$ there exist a constant $C > 0$ (independent of ε), $\varepsilon_0 > 0$ and initial conditions $y_0 \in L^2(0,1)$ such that, if $v \in L^2(0,T)$ is a control such that the solution y of* (5.1) *satisfies $y_{|t=T} = 0$, then, for every $\varepsilon \in (0, \varepsilon_0)$,*

$$\|v\|_{L^2(0,T)} \geq \exp\left(\frac{C}{\varepsilon^{1/2}}\right)\|y_0\|_{L^2(0,1)}. \tag{5.8}$$

Furthermore, if $M < 0$, we can take $T_0 = 1/|M|$.

The rest of the chapter is organized as follows. In Section 5.2, we prove Theorem 5.1. We prove an observability inequality for the adjoint system (5.7) and then applying the Hilbert Uniqueness Method (HUM). The observability inequality is proved by means of a suitable Carleman estimate. In Section 5.3, we prove Theorem 5.2. Finally, we give the proof of the Carleman estimate in Section 5.4.

5.2 Proof of Theorem 5.1

In this section, we will prove Theorem 5.1 by applying the *Hilbert Uniqueness Method* (H.U.M.) (see Paragraph 1.1.1 or, for instance, [62]), that is, we prove the following observability inequality :

$$\|\varphi_{|t=0}\|_{L^2(0,1)} \leq C_{obs}\|\varphi_{xx|x=0}\|_{L^2(0,T)}. \tag{5.9}$$

Here, φ is the solution of the adjoint equation (5.7) with $\varphi_T \in L^2(0,1)$ and $C_{obs} > 0$ is a constant independent of φ. Indeed, (5.9) is equivalent to prove that for every $y_0 \in L^2(0,T)$, there exists a control $v \in L^2(0,T)$ such that $y_{|t=T} = 0$ satisfying

$$\|v\|_{L^2(0,T)} \leq \frac{C_{obs}}{\varepsilon}\|y_0\|_{L^2(0,1)} \tag{5.10}$$

where y is the solution of (5.1). Thus, once an inequality like (5.9) is established, the proof of Theorem 5.1 is finished.

The observability inequality (5.9) is proved by means of Carleman and dissipation estimates, which are the goals of the following sections.

5.2.1 A change of variables

A relevant system associated to (5.7) will be

$$\begin{cases} -\phi_t - \varepsilon\phi_{xxx} + M\phi_x = 0 & \text{in } Q, \\ \phi_{x|x=0} = 0, \quad \phi_{xx|x=0} = 0, \quad \phi_{|x=1} = 0 & \text{in } (0,T). \end{cases} \tag{5.11}$$

Notice that (5.11) comes from the change of variables

$$\phi := \varepsilon\varphi_{xx} - M\varphi. \tag{5.12}$$

Furthermore, we notice that from (5.12) and the boundary conditions on $x = 0$ in (5.7), for every $t \in (0,T)$ we have the following initial value ordinary differential equation

$$\begin{cases} \varphi_{xx} - M\varepsilon^{-1}\varphi = \varepsilon^{-1}\phi & \text{in (0,1)}, \\ \varphi_{|x=0} = 0, \quad \varphi_{x|x=0} = 0. \end{cases}$$

We can actually find an explicit formula for φ in terms of ϕ, but we need to distinguish the cases $M > 0$ and $M < 0$:

Case $M > 0$

It is not difficult to show that the solution is given by

$$\varphi(t,x) = \frac{1}{\varepsilon^{1/2}M^{1/2}} \int_0^x \sinh(M^{1/2}\varepsilon^{-1/2}(x-s))\phi(t,s)\,\mathrm{d}s.$$

Thus,

$$|\varphi(t,x)| \leq \frac{1}{\varepsilon^{1/2}M^{1/2}} \left(\int_0^1 \sinh^2(M^{1/2}\varepsilon^{-1/2}(1-s))\,\mathrm{d}s \right)^{1/2} \left(\int_0^1 |\phi(t,s)|^2\,ds \right)^{1/2}$$

Taking the $L^2(0,1)$-norm in this inequality, we get

$$\int_0^1 |\varphi(t,x)|^2\,\mathrm{d}x \leq \frac{1}{4\varepsilon^{1/2}M^{3/2}} \exp\left(2M^{1/2}\varepsilon^{-1/2}\right) \int_0^1 |\phi(t,x)|^2\,\mathrm{d}x. \tag{5.13}$$

Moreover, since

$$\varphi_x(t,x) = \frac{1}{\varepsilon} \int_0^x \cosh(M^{1/2}\varepsilon^{-1/2}(x-s))\phi(t,s)\,\mathrm{d}s,$$

same computations show that

$$\int_0^1 |\varphi_x(t,x)|^2\,\mathrm{d}x \leq \frac{1}{\varepsilon^{3/2}M^{1/2}} \exp\left(2M^{1/2}\varepsilon^{-1/2}\right) \int_0^1 |\phi(t,x)|^2\,\mathrm{d}x. \tag{5.14}$$

Using directly (5.12) and (5.13), we obtain

$$\int_0^1 |\varphi_{xx}(t,x)|^2\,\mathrm{d}x \leq \left(\frac{2}{\varepsilon^2} + \frac{M^{1/2}}{2\varepsilon^{5/2}} \exp\left(2M^{1/2}\varepsilon^{-1/2}\right)\right) \int_0^1 |\phi(t,x)|^2\,\mathrm{d}x. \tag{5.15}$$

Case $M < 0$

In this case, φ is given by

$$\varphi(t,x) = \frac{1}{\varepsilon^{1/2}|M|^{1/2}} \int_0^x \sin(|M|^{1/2}\varepsilon^{-1/2}(x-s))\phi(s)\,\mathrm{d}s.$$

The same computations as for the case $M > 0$ show that

$$\int_0^1 |\varphi(t,x)|^2\,\mathrm{d}x \leq \frac{1}{\varepsilon|M|} \int_0^1 |\phi(t,x)|^2\,\mathrm{d}x, \tag{5.16}$$

$$\int_0^1 |\varphi_x(t,x)|^2\,\mathrm{d}x \leq \frac{1}{\varepsilon^2} \int_0^1 |\phi(t,x)|^2\,\mathrm{d}x, \tag{5.17}$$

and

$$\int_0^1 |\varphi_{xx}(t,x)|^2\,\mathrm{d}x \leq \Big(\frac{2}{\varepsilon^2} + \frac{2|M|}{\varepsilon^3}\Big) \int_0^1 |\phi(t,x)|^2\,\mathrm{d}x. \tag{5.18}$$

5.2.2 Carleman estimates

To establish the Carleman estimate, we introduce some weight functions. Let

$$\alpha(t,x) = \frac{p(x)}{t^m(T-t)^m}, \tag{5.19}$$

where $m \geq 1/2$ and p is a positive, strictly increasing and concave polynomial in $(0,1)$ of degree 2, for example, $p(x) = -x^2 + 4x + 1$. Notice that there exist positive constants C_0 and C_1 that do not depend on T such that

$$C_0 \leq T^{2m}\alpha, \quad C_0\alpha \leq \alpha_x \leq C_1\alpha, \quad C_0\alpha \leq -\alpha_{xx} \leq C_1\alpha, \tag{5.20}$$

and

$$|\alpha_t| + |\alpha_{xt}| + |\alpha_{xxt}| \leq C_1 T\alpha^{1+1/m}, \quad |\alpha_{tt}| \leq C_1 T^2\alpha^{1+2/m}, \tag{5.21}$$

for every $(t,x) \in Q$. This kind of weights has been introduced in [39] and widely used in the literature (in particular, in [42, 43, 50]).

We now are in position to present our Carleman inequality whose proof is given at the end of the chapter (section 5.4)

Proposition 5.3. *Let $\varepsilon, T > 0$, $M \in \mathbb{R}$ and $m = 1/2$. There exists a positive constant C independent of T, ε and M such that, for any solution ϕ of (5.11), we have*

$$\iint_Q e^{-2s\alpha}\Big(s^5\alpha^5|\phi|^2 + s^3\alpha^3|\phi_x|^2 + s\alpha|\phi_{xx}|^2\Big)\,dx\,dt \leq Cs^5 \int_0^T e^{-2s\alpha_{|x=0}}\alpha^5_{|x=0}|\phi_{|x=0}|^2\,dt, \tag{5.22}$$

for all $s \geq C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$.

Furthermore, we can deduce from Proposition 5.3 and (5.13)-(5.18) a Carleman estimate for the solutions of (5.7).

Proposition 5.4. *Let $\varepsilon, T > 0$, $M \in \mathbb{R}\backslash\{0\}$ and $m = 1/2$. There exists a positive constant C independent of T, ε and M such that, for any solution φ of (5.7), we have*

$$\iint_Q e^{-2s\alpha_{|x=1}}\Big(s^5\alpha^5_{|x=0}|\varphi|^2 + s^3\alpha^3_{|x=0}|\varphi_x|^2 + s\alpha_{|x=0}|\varphi_{xx}|^2\Big)\,dx\,dt$$
$$\leq \bar{C}\exp\big(C|M|^{1/2}\varepsilon^{-1/2}\big)s^5 \int_0^T e^{-2s\alpha_{|x=0}}\alpha^5_{|x=0}|\varphi_{xx|x=0}|^2\,dt, \tag{5.23}$$

for all $s \geq C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$ and $\bar{C}$ depends at most polynomially on ε^{-1} and $|M|^{-1}$.

Remark 5.5. *The lack of homogeneous Dirichlet condition on $x = 1$ plays an important role in the choice of the power m of the weight function to prove* (5.23)*. Indeed, a similar inequality was proved in [50] where $m \geq 1$ was needed to estimate a trace term on $x = 1$. This would imply that the cost of the control would be of order* $\exp(C\varepsilon^{-1})$*.*

Here, by means of the change of variable (5.12)*, which satisfies $\phi_{|x=1} = 0$, we manage to take the optimal power $m = 1/2$ as in [42, 43].*

It would be interesting to know if a Carleman estimate can be obtained for the solutions of (5.7) *for $m = 1/2$ and without using this change of variables.*

5.2.3 Dissipation estimates

To prove (5.9), we will combine (5.22) with a dissipation estimate that we will prove in the following. Let us observe that the usual dissipation estimate independent of ε can be obtained for the solutions of (5.7) in the case $M < 0$. Indeed, if $M < 0$, we multiply (5.7) by φ and integrate in $(0, 1)$. Integration by parts gives

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 |\varphi|^2\,\mathrm{d}x + \frac{\varepsilon}{2}|\varphi_{x|x=1}|^2 - \frac{M}{2}|\varphi_{|x=1}|^2\,\mathrm{d}x = 0,$$

and thus, neglecting the positive terms on the left-hand side and integrating between t_1 and t_2 we have

$$\int_0^1 |\varphi_{|t=t_1}|^2\,\mathrm{d}x \leq \int_0^1 |\varphi_{|t=t_2}|^2\,\mathrm{d}x, \tag{5.24}$$

for every $0 \leq t_1 \leq t_2 \leq T$.

As long as the case $M > 0$ is concerned, we could not come up with such an estimate. We could only obtain partial results, which are included in Paragraph 5.3 below. Nevertheless, we can prove a dissipation result similar to (5.24) for the solutions of (5.11).

More precisely, we prove

Proposition 5.6. *Let $M > 0$ and ε be two fixed parameters. Then, for every pair (t_1, t_2) such that $0 \leq t_1 < t_2 \leq T$ and every solution ϕ of* (5.11) *the following inequality is satisfied*

$$\int_0^1 |\phi_{|t=t_1}|^2\,dx + \int_0^\delta |\phi_{x|t=t_1}|^2\,dx \leq C\exp\Big(CM(t_2-t_1)\Big)\int_0^1 \left(|\phi_{|t=t_2}|^2 + |\phi_{x|t=t_2}|^2\right)dx, \tag{5.25}$$

where $\delta \in (0,1)$ and $C > 0$ is a constant independent of M, T and ε.

Proof. We proceed in two steps. First, we multiply (5.11) by $(2-x)\phi$ and integrate in $(0, 1)$. We obtain after some integration by parts

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (2-x)|\phi|^2\,\mathrm{d}x + \frac{3\varepsilon}{2}\int_0^1 |\phi_x|^2\,\mathrm{d}x + \frac{\varepsilon}{2}|\phi_{x|x=1}|^2 + \frac{M}{2}\int_0^1 |\phi|^2\,\mathrm{d}x = M|\phi_{|x=0}|^2. \tag{5.26}$$

Next, we take the derivative with respect to x of (5.11), multiply by $\frac{(1-x)^3}{2}\phi_x$ and proceed as before. Straightforward computations lead to

$$\begin{aligned}
-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{(1-x)^3}{2}|\phi_x|^2\,\mathrm{d}x + \frac{9\varepsilon}{4}\int_0^1 (1-x)^2|\phi_{xx}|^2\,\mathrm{d}x& \\
-\frac{3\varepsilon}{2}\int_0^1 |\phi_x|^2\,\mathrm{d}x + \frac{3M}{4}\int_0^1 (1-x)^2|\phi_x|^2\,\mathrm{d}x = 0.&
\end{aligned} \tag{5.27}$$

Adding (5.26) and (5.27) we obtain, after neglecting the positive terms on the left-hand side,

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (2-x)|\phi|^2\,\mathrm{d}x - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{(1-x)^3}{2}|\phi_x|^2\,\mathrm{d}x \le M|\phi_{|x=0}|^2.$$

Now, notice that

$$|\phi_{|x=0}|^2 \le \hat{C}\|\phi\|^2_{H^1(0,\frac{1}{2})},$$

and therefore we obtain

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (2-x)|\phi|^2\,\mathrm{d}x - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{(1-x)^3}{2}|\phi_x|^2\,\mathrm{d}x \\ \le 16\hat{C}M\Big(\int_0^1 (2-x)|\phi|^2\,\mathrm{d}x + \int_0^1 \frac{(1-x)^3}{2}|\phi_x|^2\,\mathrm{d}x\Big).$$

Estimate (5.25) can be easily deduced by integrating in (t_1, t_2). □

5.2.4 Observability inequality

In this section we combine the Carleman inequality (5.22) and the dissipation estimate (5.25) to finish the proof of (5.9) and therefore the proof of Theorem 5.1. Let us separate the cases $M > 0$ and $M < 0$.

Case $M > 0$

Notice that from (5.22) we have

$$\int_{T/4}^{3T/4}\int_0^1 \Big(|\phi|^2 + |\phi_x|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \le Cs^2T^{-2}\exp\left(sC/T\right)\int_0^T |\phi_{|x=0}|^2\,\mathrm{d}t.$$

From the dissipation estimate (5.25) with $t_1 = 0$ and integrating between $T/4$ and $3T/4$ we obtain

$$\frac{T}{2}\int_0^1 |\phi_{|t=0}|^2\,\mathrm{d}x \le C\exp\Big(CMT\Big)\int_{T/4}^{3T/4}\int_0^1 \left(|\phi|^2 + |\phi_x|^2\right)\mathrm{d}x\,\mathrm{d}t.$$

Combining these two inequalities and fixing $s = C(T + \varepsilon^{-1/2}T^{1/2} + M^{1/2}\varepsilon^{-1/2}T)$, we get

$$\int_0^1 |\phi_{|t=0}|^2\,\mathrm{d}x \le \bar{C}\exp\Big(CMT\Big)\exp\Big(C(1 + \varepsilon^{-1/2}T^{-1/2} + M^{1/2}\varepsilon^{-1/2})\Big)\int_0^T |\phi_{|x=0}|^2\,\mathrm{d}t, \tag{5.28}$$

where $\bar{C}$ depends at most polynomially on T^{-1}, M^{-1} and ε^{-1}.

Let us now go back to φ. From (5.12), we obtain directly $\phi_{|x=0} = \varepsilon\varphi_{xx|x=0}$. On the other hand, from (5.13) with $t = 0$, we find

$$\int_0^1 |\varphi_{|t=0}|^2\,\mathrm{d}x \le \frac{C}{\varepsilon^{1/2}M^{3/2}}\exp\left(CM^{1/2}\varepsilon^{-1/2}\right)\int_0^1 |\phi_{|t=0}|^2\,\mathrm{d}x.$$

These two elements combined with (5.28) give (5.9) with

$$C_{obs} = \hat{C}\exp\Big(C(\varepsilon^{-1/2}T^{-1/2} + M^{1/2}\varepsilon^{-1/2} + MT)\Big),$$

where $\hat{C}$ depends polynomially on T^{-1}, M^{-1} and ε. Consequently (5.3) is deduced.

Case $M < 0$

This case is actually simpler. From (5.22) we have

$$\int_{T/4}^{3T/4}\int_0^1 |\phi|^2 \,\mathrm{d}x\,\mathrm{d}t \leq C\exp\left(sC/T\right)\int_0^T |\phi_{|x=0}|^2\,\mathrm{d}t.$$

It is easy to check that

$$\int_0^1 |\phi_{|t=t_1}|^2\,\mathrm{d}x \leq \int_0^1 |\phi_{|t=t_2}|^2\,\mathrm{d}x,$$

for every $0 \leq t_1 \leq t_2 \leq T$ (just proceed as for (5.24)). Using this dissipation estimate instead of (5.25) and the same arguments as for the previous case yield

$$\int_0^1 |\phi_{|t=0}|^2\,\mathrm{d}x \leq \frac{C}{T}\exp\Big(C(\varepsilon^{-1/2}T^{-1/2} + |M|^{1/2}\varepsilon^{-1/2})\Big)\int_0^T |\phi_{|x=0}|^2\,\mathrm{d}t.$$

We recover φ same as before from (5.16) instead of (5.13) and obtain (5.9) with

$$C_{obs} = \hat{C}\exp\Big(C(\varepsilon^{-1/2}T^{-1/2} + |M|^{1/2}\varepsilon^{-1/2})\Big),$$

where $\hat{C}$ depends polynomially on ε, $|M|^{-1}$ and T^{-1}. This gives (5.4)

This completes the proof of Theorem 5.1.

5.3 Proof of Theorem 5.2

The proof of Theorem 5.2 relies on finding a particular solution $\widehat{\varphi}$ of (5.7) such that $\|\widehat{\varphi}_{xx|x=0}\|_{L^2(0,T)}$ decays exponentially as $\varepsilon \to 0^+$ and $\|\widehat{\varphi}_{|t=0}\|_{L^2(0,1)}$ behaves like a constant in ε. The proof we perform here is inspired by [43, Theorem 1.4]. The main difference is the boundary condition on $x = 1$ which is not homogeneous. When $M < 0$, we will see that no major changes are needed to be made with respect to [43]. However, the case $M > 0$ is more challenging since we do not have a good dissipation estimate. This is the reason why we will need T to be sufficiently small.

5.3.1 Case $M < 0$

In this case, we can look at (5.7) as

$$\begin{cases} -\varphi_t - \varepsilon\varphi_{xxx} - |M|\varphi_x = 0 & \text{in } Q,\\ \varphi_{|x=0} = 0, \quad \varphi_{x|x=0} = 0, \quad (\varepsilon\varphi_{xx} + |M|\varphi)_{|x=1} = 0 & \text{in } (0,T),\\ \varphi_{|t=T} = \varphi_T & \text{in } (0,1). \end{cases} \tag{5.29}$$

First, notice that we have the dissipation estimate (see (5.24))

$$\|\varphi_{|t=t_1}\|_{L^2(0,1)} \leq \|\varphi_{|t=t_2}\|_{L^2(0,1)} \quad \text{for every } 0 \leq t_1 \leq t_2 \leq T. \tag{5.30}$$

Now, we choose $R > 0$ such that

$$0 < 7R < 1 - |M|T, \tag{5.31}$$

and a non-negative function $\widehat{\varphi}_T \in \mathcal{C}_0^\infty(0,1)$ such that

$$\mathrm{Supp}(\widehat{\varphi}_T) \subset (1-2R, 1-R) \quad \text{and} \quad \|\widehat{\varphi}_T\|_{L^2(0,1)} = 1. \tag{5.32}$$

Let $\widehat{\varphi}$ be the solution of (5.29) associated to $\widehat{\varphi}_T$ as initial condition. We will prove that

$$\|\widehat{\varphi}_{|t=0}\|_{L^2(0,1)} \geq c > 0 \tag{5.33}$$

and

$$\|\widehat{\varphi}_{xx|x=0}\|_{L^2(0,T)} \leq C(\varepsilon) \exp\left[-\frac{R^{3/2}}{3^{3/2}\varepsilon^{1/2}T^{1/2}}\right] \|\widehat{\varphi}_T\|_{L^2(0,1)}, \tag{5.34}$$

where $C(\varepsilon) > 0$ depends on ε^{-1} at most polynomially.

Let us explain how (5.33) and (5.34) allow us to conclude Theorem 5.2. Let $v \in L^2(0,T)$ be a control which drives the solution y of (5.1) from y_0 to 0 (we know such a v exists by Theorem 5.1 and [50]). We multiply (5.1) by $\widehat{\varphi}$ and integrate by parts to get

$$-\int_0^1 y_0\, \widehat{\varphi}_{|t=0}\, \mathrm{d}x = \varepsilon \int_0^T v\, \widehat{\varphi}_{xx|x=0}\, \mathrm{d}t \leq \varepsilon \|v\|_{L^2(0,T)} \|\widehat{\varphi}_{xx|x=0}\|_{L^2(0,T)}.$$

Setting $y_0 := -\widehat{\varphi}_{|t=0}$ and using (5.32), (5.33) and (5.34) in this last inequality we obtain (5.8).

Proof of (5.33). Let us define $\theta(t,x) := \widehat{\varphi}_T(x + |M|(T-t))$ in Q. It follows from (5.31) and (5.32) that $\theta(t,\cdot) \in \mathcal{C}_0^\infty(0,1)$ for all $t \in [0,T]$.

We multiply (5.29) by θ and after integration by parts we obtain

$$\int_0^1 \widehat{\varphi}_T\, \theta_{|t=T}\, \mathrm{d}x = \int_0^1 \widehat{\varphi}_{|t=0}\, \theta_{|t=0}\, \mathrm{d}x + \varepsilon \iint_Q \widehat{\varphi}\, \theta_{xxx}\, \mathrm{d}x\, \mathrm{d}t. \tag{5.35}$$

From (5.30) with $t_2 = T$, we find

$$\|\widehat{\varphi}\|_{L^2(Q)} \leq T^{1/2} \|\widehat{\varphi}_T\|_{L^2(0,1)}.$$

On the other hand, from the definition of θ, (5.31) and (5.32), it is easy to see that

$$\|\theta_{|t=0}\|_{L^2(0,1)} = \|\widehat{\varphi}_T\|_{L^2(0,1)}$$

and

$$\|\theta_{xxx}\|_{L^2(Q)} \leq T^{1/2} R^{1/2} \|\widehat{\varphi}_T'''\|_{L^\infty(0,1)}.$$

Using these elements in (5.35), together with Young's inequality, we obtain (5.33) for ε small enough depending on T and R. ☐

Proof of (5.34). The objective is to prove that

$$\|\widehat{\varphi}\|_{L^2(0,T;H^3(0,R))} \leq C(\varepsilon) \exp\left[-\frac{R^{3/2}}{3^{3/2}\varepsilon^{1/2}T^{1/2}}\right] \|\widehat{\varphi}_T\|_{L^2(0,1)}, \tag{5.36}$$

from where (5.34) will readily follow. This is done by proving the estimate

$$\|\widehat{\varphi}(t)\|_{L^2(0,3R)} \leq C \exp\left[-\frac{R^{3/2}}{3^{3/2}\varepsilon^{1/2}T^{1/2}}\right] \|\widehat{\varphi}_T\|_{L^2(0,1)}, \tag{5.37}$$

and then applying an internal regularity result proved in [43] to conclude.

We consider a cut-off function $\gamma \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\gamma \geq 0, \quad \gamma' \leq 0, \quad \gamma = 1 \text{ in } (-\infty, 1-3R), \quad \gamma = 0 \text{ in } (1-2R, +\infty). \tag{5.38}$$

Now we proceed as usual. We set $\beta(t,x) := -|M|(T-t) - x$ and multiply (5.29) by $\gamma(-\beta)e^{r\beta}\widehat{\varphi}$, where $r > 0$ is to be chosen later on. We perform several integrations by parts, but observe that from (5.31) and (5.38), we have that $\gamma(-\beta(t,x)) = 0$ for all $(t,x) \in [0,T] \times [1-2R, 1]$, so there are no boundary terms. We get, after neglecting the positive terms,

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \gamma(-\beta)e^{r\beta}|\widehat{\varphi}|^2\,\mathrm{d}x - \frac{\varepsilon r^3}{2}\int_0^1 \gamma(-\beta)e^{r\beta}|\widehat{\varphi}|^2\,\mathrm{d}x$$
$$\leq \varepsilon C(r)(\|\gamma'\|_\infty + \|\gamma''\|_\infty + \|\gamma'''\|_\infty)\int_{1-3R-|M|(T-t)}^{1-2R-|M|(T-t)} e^{r\beta}|\widehat{\varphi}|^2\mathrm{d}x,$$

where $C(r)$ is a polynomial function of degree 2 in r and we have used (5.38) to restrict the limits in the integral. Multiplying by $\exp\left(-\varepsilon r^3(T-t)\right)$ and using that β is decreasing, we have

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(-\varepsilon r^3(T-t)\right)\int_0^1 \gamma(-\beta)e^{r\beta}|\widehat{\varphi}|^2\,\mathrm{d}x\right)$$
$$\leq \varepsilon C(r)\exp\left(-\varepsilon r^3(T-t) + r(3R-1)\right)\int_0^1 |\widehat{\varphi}|^2\,\mathrm{d}x.$$

By (5.30), we obtain

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(-\varepsilon r^3(T-t)\right)\int_0^1 \gamma(-\beta)e^{r\beta}|\widehat{\varphi}|^2\,\mathrm{d}x\right) \leq \varepsilon C(r)\exp\left(r(3R-1)\right)\int_0^1 |\widehat{\varphi}_T|^2\,\mathrm{d}x.$$

Integrating in (t,T), we get :

$$\int_0^1 \gamma(-\beta)e^{r\beta}|\widehat{\varphi}|^2\,\mathrm{d}x \leq \varepsilon C(r)T\exp\left(\varepsilon r^3(T-t) + r(3R-1)\right)\int_0^1 |\widehat{\varphi}_T|^2\,\mathrm{d}x,$$

where we have used the fact that $\gamma(s)\widehat{\varphi}_T(s) = 0$ for all $s \in \mathbb{R}$.

Now, notice that $\gamma(-\beta(t,x)) = 1$ for all $(t,x) \in [0,T] \times [0,3R]$ thanks to (5.31), so we have

$$\exp\left(-r(|M|(T-t)\right)\int_0^{3R} |\widehat{\varphi}|^2\,\mathrm{d}x \leq \varepsilon C(r)\exp\left(\varepsilon r^3(T-t) - r(1-6R)\right)\int_0^1 |\widehat{\varphi}_T|^2\,\mathrm{d}x,$$

and thus

$$\int_0^{3R} |\widehat{\varphi}|^2\,\mathrm{d}x \leq \varepsilon C(r)\exp\left(\varepsilon r^3 T - r(1-|M|T-6R)\right)\int_0^1 |\widehat{\varphi}_T|^2\,\mathrm{d}x.$$

Again from (5.31), we obtain

$$\int_0^{3R} |\widehat{\varphi}|^2\,\mathrm{d}x \leq \varepsilon C(r)\exp\left(\varepsilon r^3 T - Rr\right)\int_0^1 |\widehat{\varphi}_T|^2\,\mathrm{d}x.$$

We finish the proof of (5.37) by choosing $r > 0$ such that it minimises the expression inside the exponential, that is.

$$r := \frac{R^{1/2}}{3^{1/2}\varepsilon^{1/2}T^{1/2}}.$$

To prove (5.36), we will use the following lemma, which corresponds to Proposition 3.3 in [43]

Lemma 5.7. *Let $\varepsilon \in (0,1]$ and $M \in \mathbb{R}$. Consider a solution w of*

$$\begin{cases} -w_t - \varepsilon w_{xxx} + M w_x = 0 & \text{in } Q, \\ w_{|x=0} = 0, \quad w_{x|x=0} = 0, \quad w_{|x=1} = u(t) & \text{in } (0,T), \\ w_{|t=T} = w_T & \text{in } (0,1), \end{cases} \tag{5.39}$$

for some $u \in L^2(0,T)$ and $w_T \in H^3(0,1) \cap H_0^2(0,1)$. Then, $w \in L^2(0,T;H^4(0,1/2))$, with the estimate

$$\|w\|_{L^2(0,T;H^4(0,1/2))} \leq C(\varepsilon)\big(\|w_T\|_{H^3(0,1)} + \|u\|_{L^2(0,T)}\big), \tag{5.40}$$

for some constant $C(\varepsilon)$ depending at most polynomially in $1/\varepsilon$ and $|M|$.

Let $w := \widehat{\varphi}_{|[0,2R]}$ and apply Lemma 5.7 with $(0,2R)$ and $(0,R)$ instead of $(0,1)$ and $(0,1/2)$, respectively. Notice that with this setting we have $w_T = 0$ and $u = \widehat{\varphi}_{|x=2R} \in L^2(0,T)$. Thus, from (5.40) we have

$$\|\widehat{\varphi}\|_{L^2(0,T;H^4(0,R))} \leq C(\varepsilon)\|\widehat{\varphi}_x\|_{L^2(0,T;L^2(0,2R))}. \tag{5.41}$$

Now, we estimate the term in the right-hand side in a slightly larger interval. To do this, we multiply (5.29) by $(3R-x)^3\widehat{\varphi}$ and integrate in $(0,3R)$. We obtain

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{3R}(3R-x)^3|\widehat{\varphi}|^2\,\mathrm{d}x + \frac{9\varepsilon}{2}\int_0^{3R}(3R-x)^2|\widehat{\varphi}_x|^2\,\mathrm{d}x$$
$$= 3\varepsilon\int_0^{3R}|\widehat{\varphi}|^2\,\mathrm{d}x + \frac{3|M|}{2}\int_0^{3R}(3R-x)^2|\widehat{\varphi}|^2\,\mathrm{d}x.$$

Since $\widehat{\varphi}_T = 0$ in $(0,3R)$, we get by integrating between t and T

$$\|(3R-\cdot)\widehat{\varphi}_x\|_{L^2(0,T;L^2(0,3R))} \leq \frac{C}{\varepsilon^{1/2}}\|\widehat{\varphi}\|_{L^2(0,T;L^2(0,3R))}. \tag{5.42}$$

Combining this with (5.41) and (5.37), we obtain (5.36).

□

5.3.2 Case $M > 0$

In this case, we no longer have the dissipation estimate (5.30). However, we will prove that

$$\int_0^1 e^{r(M(T-t)-x)}|\varphi(t)|^2\,\mathrm{d}x \leq \exp\left[5\varepsilon T r^3 + \frac{M^2T}{2\varepsilon r}\right]\int_0^1 e^{-rx}|\varphi_T|^2\,\mathrm{d}x. \tag{5.43}$$

for every $t \in (0,T)$ and $r > 0$.

Proof of (5.43). Inspired by [43, section 3.2], we multiply now (5.7) by $e^r\varphi$, where $\beta(t,x) := M(T-t) - x$ and r is a positive parameter to be chosen later, and integrate in space :

$$-\frac{1}{2}\int_0^1 e^{r\beta}\partial_t|\varphi|^2\,\mathrm{d}x - \varepsilon\int_0^1 e^{r\beta}\varphi_{xxx}\varphi\,\mathrm{d}x + \frac{M}{2}\int_0^1 e^{r\beta}\partial_x|\varphi|^2\,\mathrm{d}x = 0.$$

We integrate by parts :

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 e^{r\beta}|\varphi|^2\,\mathrm{d}x - \frac{rM}{2}\int_0^1 e^{r\beta}|\varphi|^2\,\mathrm{d}x + \frac{\varepsilon}{2}\int_0^1 e^{r\beta}\partial_x|\varphi_x|^2\,\mathrm{d}x$$
$$-\varepsilon r\int_0^1 e^{r\beta}\varphi_{xx}\varphi\,\mathrm{d}x - \varepsilon e^{r\beta_{|x=1}}\varphi_{xx|x=1}\varphi_{|x=1} + \frac{rM}{2}\int_0^1 e^{r\beta}|\varphi|^2\,\mathrm{d}x + \frac{M}{2}e^{r\beta_{|x=1}}|\varphi_{|x=1}|^2 = 0.$$

Using $M\varphi_{|x=1} = \varepsilon\varphi_{xx|x=1}$, we get after some integration by parts

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 e^{r\beta}|\varphi|^2\,\mathrm{d}x - \frac{M}{2}e^{r\beta_{|x=1}}|\varphi_{|x=1}|^2 + \frac{3\varepsilon r}{2}\int_0^1 e^{r\beta}|\varphi_x|^2\,\mathrm{d}x + \frac{\varepsilon}{2}e^{r\beta_{|x=1}}|\varphi_{x|x=1}|^2 - \frac{\varepsilon r^3}{2}\int_0^1 e^{r\beta}|\varphi|^2\,\mathrm{d}x - \frac{\varepsilon r^2}{2}e^{r\beta_{|x=1}}|\varphi_{|x=1}|^2 - \varepsilon r e^{r\beta_{|x=1}}\varphi_{x|x=1}\varphi_{|x=1} = 0.$$

Rearranging :

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 e^{r\beta}|\varphi|^2\,\mathrm{d}x - \frac{\varepsilon r^3}{2}\int_0^1 e^{r\beta}|\varphi|^2\,\mathrm{d}x + \frac{3\varepsilon r}{2}\int_0^1 e^{r\beta}|\varphi_x|^2\,\mathrm{d}x + \frac{\varepsilon}{2}e^{r\beta_{|x=1}}|\varphi_{x|x=1}|^2 = \frac{M}{2}e^{r\beta_{|x=1}}|\varphi_{|x=1}|^2 + \frac{\varepsilon r^2}{2}e^{r\beta_{|x=1}}|\varphi_{|x=1}|^2 + \varepsilon r e^{r\beta_{|x=1}}\varphi_{x|x=1}\varphi_{|x=1}.$$

We estimate the three terms in the right-hand side :

$$\varepsilon r e^{r\beta_{|x=1}}\varphi_{x|x=1}\varphi_{|x=1} \leq \frac{\varepsilon}{2}e^{r\beta_{|x=1}}|\varphi_{x|x=1}|^2 + \frac{\varepsilon r^2}{2}e^{r\beta_{|x=1}}|\varphi_{|x=1}|^2$$

$$\varepsilon r^2 e^{r\beta_{|x=1}}|\varphi_{|x=1}|^2 = \varepsilon r^2\int_0^1 e^{r\beta_{|x=1}}\partial_x|\varphi|^2\,\mathrm{d}x = 2\varepsilon r^2\int_0^1 e^{r\beta_{|x=1}}\varphi\varphi_x\,\mathrm{d}x \leq 2\varepsilon r^2\int_0^1 e^{r\beta}|\varphi||\varphi_x|\,\mathrm{d}x \leq 2\varepsilon r^3\int_0^1 e^{r\beta}|\varphi|^2\,\mathrm{d}x + \frac{1}{2}\varepsilon r\int_0^1 e^{r\beta}|\varphi_x|^2\,\mathrm{d}x$$

$$\frac{M}{2}e^{r\beta_{|x=1}}|\varphi_{|x=1}|^2 = \frac{M}{2}\int_0^1 e^{r\beta_{|x=1}}\partial_x|\varphi|^2\,\mathrm{d}x = M\int_0^1 e^{r\beta_{|x=1}}\varphi\varphi_x\,\mathrm{d}x \leq M\int_0^1 e^{r\beta}|\varphi||\varphi_x|\,\mathrm{d}x \leq \frac{M^2}{4\varepsilon r}\int_0^1 e^{r\beta}|\varphi|^2\,\mathrm{d}x + \varepsilon r\int_0^1 e^{r\beta}|\varphi_x|^2\,\mathrm{d}x$$

We get

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 e^{r\beta}|\varphi|^2\,\mathrm{d}x - \frac{1}{2}\Big(5\varepsilon r^3 + \frac{M^2}{2\varepsilon r}\Big)\int_0^1 e^{r\beta}|\varphi|^2\,\mathrm{d}x \leq 0.$$

Using Gronwall's Lemma, we obtain :

$$\int_0^1 e^{r(M(T-t_1)-x)}|\varphi_{|t=t_1}|^2\,\mathrm{d}x \leq \exp\Big[5\varepsilon(t_2-t_1)r^3 + \frac{M^2(t_2-t_1)}{2\varepsilon r}\Big]\int_0^1 e^{r(M(T-t_2)-x)}|\varphi_{|t=t_2}|^2\,\mathrm{d}x, \tag{5.44}$$

for every $0 \leq t_1 \leq t_2 \leq T$. In particular, taking $t_2 = T$ in (5.44), we obtain (5.43).

□

We fix $T_0 < 1/M$ and, as before, we choose $R > 0$ (which will also remain fixed) such that

$$0 < 6R < 1 - MT_0. \tag{5.45}$$

Notice that (5.45) is still true for any $T < T_0$. We consider a non-negative function $\widehat{\varphi}_T \in \mathcal{C}_0^\infty(0,1)$ such that

$$\mathrm{Supp}(\widehat{\varphi}_T) \subset (5R, 6R) \quad \text{and} \quad \|\widehat{\varphi}_T\|_{L^2(0,1)} = 1. \tag{5.46}$$

Let $\widehat{\varphi}$ be the solution of (5.7) associated to $\widehat{\varphi}_T$ as initial condition. We will be able to prove that

$$\|\widehat{\varphi}_{xx|x=0}\|_{L^2(0,T)} \leq C(\varepsilon)\exp\Big[-\frac{5R^{3/2}}{144\,\varepsilon^{1/2}T^{1/2}}\Big]\|\widehat{\varphi}_T\|_{L^2(0,1)}, \tag{5.47}$$

where $C(\varepsilon) > 0$ depends on ε^{-1} at most polynomially and T is small enough with respect to R and M, but independent of ε.

We will also prove

$$\|\widehat{\varphi}_{|t=0}\|_{L^2(0,1)} \geq c > 0, \tag{5.48}$$

which in this case will be a consequence of (5.47). Once (5.47) and (5.48) are proved, we can conclude as in the previous paragraph.

Proof of (5.47). Since Lemma 5.7 and (5.42) are still valid, it is sufficient to prove

$$\|\widehat{\varphi}(t)\|_{L^2(0,3R)} \leq C(\varepsilon)\exp\Big[-\frac{5R^{3/2}}{144\,\varepsilon^{1/2}T^{1/2}}\Big]\|\widehat{\varphi}_T\|_{L^2(0,1)}. \tag{5.49}$$

We redefine the cut-off function $\gamma \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\gamma \geq 0, \quad \gamma' \leq 0, \quad \gamma = 1 \text{ in } (-\infty, 4R), \quad \gamma = 0 \text{ in } (5R, +\infty). \tag{5.50}$$

Now we proceed as before by setting $\beta(t,x) := M(T-t) - x$ and multiplying (5.7) by $\gamma(-\beta)e^{r\beta}\widehat{\varphi}$, where $r > 0$ is to be chosen later on. We perform the same computations as before, but observe that from (5.45) and (5.50), we have that $\gamma(-\beta(t,x)) = 0$ for all $(t,x) \in [0,T] \times [MT+5R, 1]$, so there are no boundary terms. We get, after neglecting the positive terms,

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \gamma(-\beta)e^{r\beta}|\widehat{\varphi}|^2\,\mathrm{d}x - \frac{\varepsilon r^3}{2}\int_0^1 \gamma(-\beta)e^{r\beta}|\widehat{\varphi}|^2\,\mathrm{d}x \\ \leq \varepsilon C(r)(\|\gamma'\|_\infty + \|\gamma''\|_\infty + \|\gamma'''\|_\infty)\int_{4R+M(T-t)}^{5R+M(T-t)} e^{r\beta}|\widehat{\varphi}|^2\mathrm{d}x,$$

where $C(r)$ is a polynomial function of degree 2 in r and we have used (5.50) to restrict the limits in the integral. By (5.43) and multiplying by $\exp\big(-\varepsilon r^3(T-t)\big)$ we obtain

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\exp\big(-\varepsilon r^3(T-t)\big)\int_0^1 \gamma(-\beta)e^{r\beta}|\widehat{\varphi}|^2\,\mathrm{d}x\Big) \\ \leq \varepsilon C(r)\exp\big(-5rR\big)\exp\Big[5\varepsilon T r^3 + \frac{M^2T}{2\varepsilon r}\Big]\int_{5R}^{6R}|\widehat{\varphi}_T|^2\,\mathrm{d}x,$$

and we can integrate in (t,T) :

$$\int_0^1 \gamma(-\beta)e^{r\beta}|\widehat{\varphi}|^2\,\mathrm{d}x \leq \varepsilon C(r)T\exp\big(\varepsilon r^3(T-t) - 5rR\big)\exp\Big[5\varepsilon T r^3 + \frac{M^2T}{2\varepsilon r}\Big]\int_0^1 |\widehat{\varphi}_T|^2\,\mathrm{d}x,$$

where we have used the fact that $\gamma(s)\widehat{\varphi}_T(s) = 0$ for all $s \in \mathbb{R}$.

Now, notice that $\gamma(-\beta(t,x)) = 1$ for all $(t,x) \in [0,T]\times[0,3R]$, so we have

$$\exp\big(r(M(T-t)\big)\int_0^{3R}|\widehat{\varphi}|^2\,\mathrm{d}x \\ \leq \varepsilon C(r)\exp\big(\varepsilon r^3(T-t) - 2rR)\big)\exp\Big[5\varepsilon T r^3 + \frac{M^2T}{2\varepsilon r}\Big]\int_0^1 |\widehat{\varphi}_T|^2\,\mathrm{d}x,$$

and thus

$$\int_0^{3R} |\widehat{\varphi}|^2 \, \mathrm{d}x \leq \varepsilon C(r) \exp\Big[6\varepsilon T r^3 + \frac{M^2 T}{2\varepsilon r} - 2rR\Big] \int_0^1 |\widehat{\varphi}_T|^2 \, \mathrm{d}x.$$

Here, we need to take $T < C_0 R/M$, where $C_0 < 1/\sqrt{3}$. Notice that this condition is independent with respect to ε. To fix ideas, we take $C_0 = 1/2$. With this choice, the previous estimate becomes

$$\int_0^{3R} |\widehat{\varphi}|^2 \, \mathrm{d}x \leq \varepsilon C(r) \exp\Big[6\varepsilon T r^3 + \frac{R^2}{8\varepsilon T r} - 2rR\Big] \int_0^1 |\widehat{\varphi}_T|^2 \, \mathrm{d}x.$$

Now we take $r > 0$ to be

$$r := \frac{R^{1/2}}{3\,\varepsilon^{1/2} T^{1/2}}.$$

With this choice of r, we find that

$$\int_0^{3R} |\widehat{\varphi}|^2 \, \mathrm{d}x \leq C \exp\Big[\frac{R^{3/2}}{\varepsilon^{1/2} T^{1/2}}\Big(\frac{6}{27} + \frac{3}{8} - \frac{2}{3}\Big)\Big] \int_0^1 |\widehat{\varphi}_T|^2 \, \mathrm{d}x,$$

which gives (5.49). □

Proof of (5.48). The proof of (5.48) follows a different approach with respect to the proof of (5.33), since we do not have a good dissipation for (5.7) and therefore we cannot treat the last term in (5.35). Here, we use the boundary condition on the right of (5.7) to our advantage.

We integrate (5.7) with respect to x :

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \widehat{\varphi} \, \mathrm{d}x + \varepsilon \widehat{\varphi}_{xx|x=0} = 0 \quad \text{for every } t \in (0,T).$$

Now, we integrate between 0 and T :

$$\int_0^1 \widehat{\varphi}_{|t=0} \, \mathrm{d}x = \int_0^1 \widehat{\varphi}_T \, \mathrm{d}x - \varepsilon \int_0^T \widehat{\varphi}_{xx|x=0} \, \mathrm{d}t.$$

By (5.47) and (5.46), we find that

$$\int_0^1 \widehat{\varphi}_{|t=0} \, \mathrm{d}x \geq \int_0^1 \widehat{\varphi}_T \, \mathrm{d}x - \varepsilon C(\varepsilon) T^{1/2} \exp\Big[-\frac{5R^{3/2}}{144\,\varepsilon^{1/2} T^{1/2}}\Big],$$

from where, taking ε small enough, and together with

$$\int_0^1 \widehat{\varphi}_{|t=0} \, \mathrm{d}x \leq \|\widehat{\varphi}_{|t=0}\|_{L^2(0,1)}$$

we obtain (5.48). □

The proof of Theorem 5.2 is complete.

5.4 Proof of Proposition 5.3

We now follow the steps of [42] and [43]. Let $\psi := e^{-s\alpha}\phi$. Using equation (5.11), we get

$$L_1\psi + L_2\psi = L_3\psi,$$

where we have denoted

$$L_1\psi := \varepsilon\psi_{xxx} + \psi_t + 3\varepsilon s^2\alpha_x^2\psi_x - M\psi_x,$$

$$L_2\psi := (\varepsilon s^3\alpha_x^3 + s\alpha_t - Ms\alpha_x)\psi + 3\varepsilon s\alpha_{xx}\psi_x + 3\varepsilon s\alpha_x\psi_{xx}$$

and

$$L_3\psi := -3\varepsilon s^2\alpha_x\alpha_{xx}\psi.$$

Notice that we have the following boundary values for ψ :

$$\psi_{|x=1} = 0 \tag{5.51}$$

$$\psi_{x|x=0} = -s\alpha_{x|x=0}\psi_{|x=0} \tag{5.52}$$

$$\psi_{xx|x=0} = (s^2\alpha_x^2 - s\alpha_{xx})_{|x=0}\psi_{|x=0} \tag{5.53}$$

Taking the L^2-norm we have

$$\|L_1\psi\|^2_{L^2(Q)} + \|L_2\psi\|^2_{L^2(Q)} + 2(L_1\psi,\, L_2\psi)_{L^2(Q)} = \|L_3\psi\|^2_{L^2(Q)}. \tag{5.54}$$

In the following, our efforts will be devoted to computing the double product in the previous equation. Let us denote by $(L_i\psi)_j$ the j-th term of $L_i\psi$.

Computing $((L_1\psi)_1,\, L_2\psi)_{L^2(Q)}$.
For the first term, we integrate by parts twice in space :

$$\begin{aligned}
((L_1\psi)_1,\, (L_2\psi)_1)_{L^2(Q)} &= -\frac{1}{2}\varepsilon \iint_Q (\varepsilon s^3\alpha_x^3 + s\alpha_t - Ms\alpha_x)\partial_x|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t \\
&-\varepsilon \iint_Q (3\varepsilon s^3\alpha_x^2\alpha_{xx} + s\alpha_{xt} - Ms\alpha_{xx})\psi\psi_{xx}\,\mathrm{d}x\,\mathrm{d}t \\
&-\varepsilon \int_0^T (\varepsilon s^3\alpha_x^3 + s\alpha_t - Ms\alpha_x)_{|x=0}\psi_{|x=0}\psi_{xx|x=0}\,\mathrm{d}t \\
&= \frac{3}{2}\varepsilon \iint_Q (3\varepsilon s^3\alpha_x^2\alpha_{xx} + s\alpha_{xt} - Ms\alpha_{xx})|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t \\
&-\frac{1}{2}\varepsilon \int_0^T (\varepsilon s^3\alpha_x^3 + s\alpha_t - Ms\alpha_x)_{|x=1}|\psi_{x|x=1}|^2\mathrm{d}t \\
&-3\varepsilon^2 s^3 \iint_Q \alpha_{xx}^3|\psi|^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\varepsilon \int_0^T (6\varepsilon s^3\alpha_x\alpha_{xx}^2 + s\alpha_{xxt})_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t \\
&+\frac{1}{2}\varepsilon \int_0^T (\varepsilon s^3\alpha_x^3 + s\alpha_t - Ms\alpha_x)_{|x=0}|\psi_{x|x=0}|^2\,\mathrm{d}t \\
&+\varepsilon \int_0^T (3\varepsilon s^3\alpha_x^2\alpha_{xx} + s\alpha_{xt} - Ms\alpha_{xx})_{|x=0}\psi_{|x=0}\psi_{x|x=0}\,\mathrm{d}t \\
&-\varepsilon \int_0^T (\varepsilon s^3\alpha_x^3 + s\alpha_t - Ms\alpha_x)_{|x=0}\psi_{|x=0}\psi_{xx|x=0}\,\mathrm{d}t
\end{aligned}$$

Using the properties (5.20), (5.21), (5.52) and (5.53), we obtain

$$((L_1\psi)_1,\,(L_2\psi)_1)_{L^2(Q)} \geq \frac{9}{2}\varepsilon^2 s^3 \iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t - C\varepsilon s(T+|M|T^2)\iint_Q \alpha^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$-\varepsilon^2\frac{s^3}{2}\int_0^T \alpha^3_{x|x=1}|\psi_{x|x=1}|^2\,\mathrm{d}t - C\varepsilon s(T+|M|T^2)\int_0^T \alpha^3_{x|x=1}|\psi_{x|x=1}|^2\,\mathrm{d}t - C\varepsilon^2 s^3T^2\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$-C\varepsilon\Big(\varepsilon s^5+\varepsilon s^4T+s^3(T+\varepsilon T^2+|M|T^2)+s^2(T^2+|M|T^3)+sT^3\Big)\int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t.$$

We integrate by parts again in space in the second term, and using the boundary values for $\psi_{x|x=0}$ and $\psi_{xx|x=0}$:

$$\begin{aligned}((L_1\psi)_1,\,(L_2\psi)_2)_{L^2(Q)} &= -3\varepsilon^2 s\iint_Q \alpha_{xx}|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t + 3\varepsilon^2 s\int_0^T \alpha_{xx|x=1}\psi_{x|x=1}\psi_{xx|x=1}\,\mathrm{d}t\\
&\quad - 3\varepsilon^2 s\int_0^T \alpha_{xx|x=0}\psi_{x|x=0}\psi_{xx|x=0}\,\mathrm{d}t\\
&\geq -3\varepsilon^2 s\iint_Q \alpha_{xx}|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\varepsilon^2 s\int_0^T \alpha_{x|x=1}|\psi_{xx|x=1}|^2\,\mathrm{d}t\\
&\quad - C\varepsilon^2 sT^2\int_0^T \alpha^3_{x|x=1}|\psi_{x|x=1}|^2\,\mathrm{d}t - C\varepsilon^2(s^4T+s^3T^2)\int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t.\end{aligned}$$

Here, we have used the boundary values for $\psi_x|_{x=0}$ and $\psi_{xx}|_{x=0}$, the properties in (5.20) and Young's inequality.

For the third term, we proceed in a similar manner :

$$\begin{aligned}((L_1\psi)_1,\,(L_2\psi)_3)_{L^2(Q)} &= -\frac{3}{2}\varepsilon^2 s\iint_Q \alpha_{xx}|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{3}{2}\varepsilon^2 s\int_0^T \alpha_{x|x=1}|\psi_{xx|x=1}|^2\,\mathrm{d}t\\
&\quad - \frac{3}{2}\varepsilon^2 s\int_0^T \alpha_{x|x=0}|\psi_{xx|x=0}|^2\,\mathrm{d}t\\
&\geq -\frac{3}{2}\varepsilon^2 s\iint_Q \alpha_{xx}|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{3}{2}\varepsilon^2 s\int_0^T \alpha_{x|x=1}|\psi_{xx|x=1}|^2\,\mathrm{d}t\\
&\quad - C\varepsilon^2(s^5+s^3T^2)\int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t.\end{aligned}$$

Putting together these calculations, we obtain

$$\begin{aligned}((L_1\psi)_1,\,L_2\psi)_{L^2(Q)} &\geq \frac{9}{2}\varepsilon^2 s^3\iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t - \frac{9}{2}\varepsilon^2 s\iint_Q \alpha_{xx}|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t\\
&+\varepsilon^2 s\int_0^T \alpha_{x|x=1}|\psi_{xx|x=1}|^2\,\mathrm{d}t - \varepsilon^2\frac{s^3}{2}\int_0^T \alpha^3_{x|x=1}|\psi_{x|x=1}|^2\,\mathrm{d}t\\
&- C\varepsilon s(T+|M|T^2)\iint_Q \alpha^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t - C\varepsilon^2 s^3T^2\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&- C\varepsilon s(T+\varepsilon T^2+|M|T^2)\int_0^T \alpha^3_{x|x=1}|\psi_{x|x=1}|^2\,\mathrm{d}t\\
&- C\varepsilon\Big(\varepsilon s^5+\varepsilon s^4T+s^3(T+\varepsilon T^2+|M|T^2)+s^2T(T+|M|T^2)+sT^3\Big)\int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t.\end{aligned} \tag{5.55}$$

Computing $((L_1\psi)_2,\,L_2\psi)_{L^2(Q)}$.

For the first term, we integrate by parts in time :

$$((L_1\psi)_2,\,(L_2\psi)_1)_{L^2(Q)} = -\frac{1}{2}\iint_Q (3\varepsilon s^3\alpha_x^2\alpha_{xt} + s\alpha_{tt} - Ms\alpha_{xt})|\psi|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\geq -C(\varepsilon s^3 + s(T+|M|T^2))T\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t.$$

The second terms gives :

$$((L_1\psi)_2,\,(L_2\psi)_2)_{L^2(Q)} = 3\varepsilon s\iint_Q \alpha_{xx}\psi_x\psi_t\,\mathrm{d}x\,\mathrm{d}t.$$

In the third term, we integrate by parts first in space and then in time. We obtain

$$\begin{aligned}((L_1\psi)_2,\,(L_2\psi)_3)_{L^2(Q)} &= -\frac{3}{2}\varepsilon s\iint_Q \alpha_x\partial_t|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t - 3\varepsilon s\iint_Q \alpha_{xx}\psi_x\psi_t\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - 3\varepsilon s\int_0^T \alpha_{x|x=0}\psi_{x|x=0}\psi_{t|x=0}\,\mathrm{d}t\\
&= \frac{3}{2}\varepsilon s\iint_Q \alpha_{xt}|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t - 3\varepsilon s\iint_Q \alpha_{xx}\psi_x\psi_t\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - 3\varepsilon s^2\int_0^T \alpha_{x|x=0}\alpha_{xt|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t\\
&\geq -3\varepsilon s\iint_Q \alpha_{xx}\psi_x\psi_t\,\mathrm{d}x\,\mathrm{d}t - C\varepsilon sT\iint_Q \alpha^3|\psi_x|^2\\
&\quad - C\varepsilon s^2T^2\int_0^T \alpha_{|x=0}^5|\psi_{|x=0}|^2\,\mathrm{d}t,\end{aligned}$$

where we have used the boundary value (5.52).

Putting together this inequalities, we have

$$\begin{aligned}((L_1\psi)_2,\,L_2\psi)_{L^2(Q)} &\geq -C(\varepsilon s^3 + s(T+|M|T^2))T\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - C\varepsilon sT\iint_Q \alpha^3|\psi_x|^2 - C\varepsilon s^2T^2\int_0^T \alpha_{|x=0}^5|\psi_{|x=0}|^2\,\mathrm{d}t.\end{aligned} \tag{5.56}$$

Computing $((L_1\psi)_3,\,L_2\psi)_{L^2(Q)}$.

We integrate by parts in space and the properties (5.20)-(5.21) to treat the first term :

$$\begin{aligned}&((L_1\psi)_3,\,(L_2\psi)_1)_{L^2(Q)}\\
&= -\frac{1}{2}\varepsilon\iint_Q (15\varepsilon s^5\alpha_x^4\alpha_{xx} + 6s^3\alpha_x\alpha_{xx}\alpha_t + 3s^3\alpha_x^2\alpha_{xt} - 9Ms^3\alpha_x^2\alpha_{xx})|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&- \frac{1}{2}\varepsilon\int_0^T (3\varepsilon s^5\alpha_x^5 + 3s^3\alpha_x^2\alpha_t + 3Ms^3\alpha_x^3)_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t\\
&\geq \frac{15}{2}C_0^5\varepsilon^2 s^5\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t - C\varepsilon s^3(T+|M|T^2)\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&- C\varepsilon(\varepsilon s^5 + s^3(T+|M|T^2))\int_0^T \alpha_{|x=0}^5|\psi_{|x=0}|^2\,\mathrm{d}t.\end{aligned}$$

The second term simply gives :

$$((L_1\psi)_3,\,(L_2\psi)_2)_{L^2(Q)} = 9\varepsilon^2 s^3\iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t.$$

As for the third term, we integrate by parts in space once and use the boundary value (5.52). We obtain :

$$\begin{aligned}((L_1\psi)_3,\,(L_2\psi)_3)_{L^2(Q)} =& -\frac{27}{2}\varepsilon^2 s^3 \iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{9}{2}\varepsilon^2 s^3 \int_0^T \alpha^3_{x|x=1}|\psi_{x|x=1}|^2\,\mathrm{d}t \\ & - \frac{9}{2}\varepsilon^2 s^3 \int_0^T \alpha^3_{x|x=0}|\psi_{x|x=0}|^2\,\mathrm{d}t \\ =& -\frac{27}{2}\varepsilon^2 s^3 \iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{9}{2}\varepsilon^2 s^3 \int_0^T \alpha^3_{x|x=1}|\psi_{x|x=1}|^2\,\mathrm{d}t \\ & - \frac{9}{2}\varepsilon^2 s^5 \int_0^T \alpha^5_{x|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t.\end{aligned}$$

Putting together these estimates, we get :

$$\begin{aligned}((L_1\psi)_3,\,L_2\psi)_{L^2(Q)} \geq & -\frac{9}{2}\varepsilon^2 s^3 \iint_Q \alpha_x^2\alpha_{xx}|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{15}{2}C_0^5\varepsilon^2 s^5 \iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\ & + \frac{9}{2}\varepsilon^2 s^3 \int_0^T \alpha^3_{x|x=1}|\psi_{x|x=1}|^2\,\mathrm{d}t - C\varepsilon s^3(T+|M|T^2) \iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\ & - C\varepsilon(\varepsilon s^5 + s^3(T+|M|T^2)) \int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t.\end{aligned} \tag{5.57}$$

Computing $((L_1\psi)_4,\,L_2\psi)_{L^2(Q)}$.
For the first term, we integrate by parts in space :

$$\begin{aligned}((L_1\psi)_4,\,(L_2\psi)_1)_{L^2(Q)} =& \frac{M}{2} \iint_Q (3\varepsilon s^3\alpha_x^2\alpha_{xx} + s\alpha_{xt} - Ms\alpha_{xx})|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\ & + \frac{M}{2} \int_0^T (\varepsilon s^3\alpha_x^3 + s\alpha_t - Ms\alpha_x)_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t \\ \geq & -C(\varepsilon s^3 + s(T+|M|T^2))|M|T^2 \iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t \\ & - C(\varepsilon s^3 + s(T+|M|T^2))|M|T^2 \int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t.\end{aligned}$$

The second term gives directly :

$$((L_1\psi)_4,\,(L_2\psi)_2)_{L^2(Q)} \geq -C|M|\varepsilon s T^2 \iint_Q \alpha^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t.$$

The third and final term gives, after integration by parts :

$$\begin{aligned}((L_1\psi)_4,\,(L_2\psi)_3)_{L^2(Q)} =& \frac{3}{2}M\varepsilon s \iint_Q \alpha_{xx}|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t - \frac{3}{2}M\varepsilon s \int_0^T \alpha_{x|x=1}|\psi_{x|x=1}|^2\,\mathrm{d}t \\ & + \frac{3}{2}M\varepsilon s \int_0^T \alpha_{x|x=0}|\psi_{x|x=0}|^2\,\mathrm{d}t \\ \geq & -C|M|\varepsilon s T^2 \iint_Q \alpha^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t - C|M|\varepsilon s T^2 \int_0^T \alpha^3_{x|x=1}|\psi_{x|x=1}|^2\,\mathrm{d}t \\ & - C|M|\varepsilon s^3 T^2 \int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t.\end{aligned}$$

Putting together these expressions, we obtain :

$$
\begin{aligned}
((L_1\psi)_4,\, L_2\psi)_{L^2(Q)} \geq & -C(\varepsilon s^3 + s(T+|M|T^2))|M|T^2 \iint_Q \alpha^5|\psi|^2 \,\mathrm{d}x\,\mathrm{d}t \\
& - C|M|\varepsilon s T^2 \iint_Q \alpha^3 |\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t - C|M|\varepsilon s T^2 \int_0^T \alpha^3_{x|x=1} |\psi_{x|x=1}|^2 \,\mathrm{d}t \\
& - C(\varepsilon s^3 + s(T+|M|T^2))|M|T^2 \int_0^T \alpha^5_{|x=0} |\psi_{|x=0}|^2 \,\mathrm{d}t.
\end{aligned}
\tag{5.58}
$$

The entire product $(L_1\psi,\, L_2\psi)_{L^2(Q)}$.
Adding inequalities (5.55)-(5.58), we find four positive terms, namely :

$$
A_1 := \frac{15}{2} C_0^5 \varepsilon^2 s^5 \iint_Q \alpha^5 |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t, \quad A_2 := \frac{9}{2} C_0 \varepsilon^2 s \iint_Q \alpha |\psi_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}t,
$$

$$
A_3 := 4\varepsilon^2 s^3 \int_0^T \alpha^3_{x|x=1} |\psi_{x|x=1}|^2 \,\mathrm{d}t, \quad A_4 := \varepsilon^2 s \int_0^T \alpha_{x|x=1} |\psi_{xx|x=1}|^2 \,\mathrm{d}t.
$$

In the following, we explain how to estimate the nonpositive integrals coming from the addition of (5.55)-(5.58) in terms of A_i.

Let us start with the terms concerning $|\psi|^2$ in Q. We can easily check that they can all be bounded by

$$
C\big(s^3(\varepsilon T + \varepsilon^2 T^2 + |M|\varepsilon T^2) + s(T^2 + |M|T^3 + |M|^2 T^4)\big) \iint_Q \alpha^5 |\psi|^2 \,\mathrm{d}x\,\mathrm{d}t,
$$

which by taking $s \geq C(T + T^{1/2}\varepsilon^{-1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$ can be absorbed by A_1.

The integral of $|\psi_{x|x=1}|^2$, can be bounded by

$$
C\varepsilon s(T + \varepsilon T^2 + |M|T^2) \int_0^T \alpha^3_{x|x=1} |\psi_{x|x=1}|^2 \,\mathrm{d}t.
$$

Taking $s \geq C(T + T^{1/2}\varepsilon^{-1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$, this term can be absorbed by A_3.

Furthermore, taking $s \geq C(T + T^{1/2}\varepsilon^{-1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$ shows that all the integrals concerning $|\psi_{|x=0}|^2$ can be estimated by

$$
C\varepsilon^2 s^5 \int_0^T \alpha^5_{|x=0} |\psi_{|x=0}|^2 \,\mathrm{d}t. \tag{5.59}
$$

Finally, let us treat the terms containing $|\psi_x|^2$ in Q. They can be estimated by

$$
C\varepsilon s(T + |M|T^2) \iint_Q \alpha^3 |\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t.
$$

Now, similarly as the previous steps, integration by parts in space shows that :

$$\begin{aligned}
&C\varepsilon s(T+|M|T^2)\iint_Q \alpha^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t = \frac{3}{2}C\varepsilon s(T+|M|T^2)\iint_Q (2\alpha\alpha_x^2+\alpha^2\alpha_{xx})|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&+\frac{3}{2}C\varepsilon s(T+|M|T^2)\int_0^T \alpha^2_{|x=0}\alpha_{x|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t - C\varepsilon s(T+|M|T^2)\iint_Q \alpha^3\psi\psi_{xx}\,\mathrm{d}x\,\mathrm{d}t\\
&-C\varepsilon s(T+|M|T^2)\int_0^T \alpha^3_{|x=0}\psi_{|x=0}\psi_{x|x=0}\,\mathrm{d}t\\
&\geq -C\big(\varepsilon sT^2(T+|M|T^2)+\varepsilon^{1/2}s^2(T^{3/2}+|M|^{3/2}T^3)\big)\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&-C\varepsilon^{3/2}(T^{1/2}+|M|^{1/2}T)\iint_Q \alpha|\psi_{xx}|^2\mathrm{d}x\,\mathrm{d}t\\
&-C\varepsilon\big(sT^2(T+|M|T^2)+s^2T(T+|M|T^2)\big)\int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t.
\end{aligned} \tag{5.60}$$

Notice that here we have used (5.52) and Young's inequality. By taking $s\geq C(T+\varepsilon^{-1/2}T^{1/2}+|M|^{1/2}\varepsilon^{-1/2}T)$, the first two integrals can be absorbed by A_1 and A_2, respectively, and the last one can be estimated by (5.59).

Finally, all these estimations give

$$\begin{aligned}
(L_1\psi,\,L_2\psi)_{L^2(Q)} \geq & C\varepsilon^2s^5\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + C\varepsilon^2 s\iint_Q \alpha|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t\\
&+\varepsilon^2s^3\int_0^T \alpha^3_{x|x=1}|\psi_{x|x=1}|^2\,\mathrm{d}t + C\varepsilon^2 s\int_0^T \alpha_{x|x=1}|\psi_{xx|x=1}|^2\,\mathrm{d}t\\
&-C\varepsilon^2s^5\int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t,
\end{aligned}$$

for every $s\geq C(T+\varepsilon^{-1/2}T^{1/2}+|M|^{1/2}\varepsilon^{-1/2}T)$.

Coming back to (5.54), and together with the fact that

$$\|L_3\psi\|^2_{L^2(Q)} \leq C\varepsilon^2s^4T\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t,$$

we obtain

$$\begin{aligned}
&\varepsilon^2s^5\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s\iint_Q \alpha|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2s^3\int_0^T \alpha^3_{x|x=1}|\psi_{x|x=1}|^2\,\mathrm{d}t\\
&\qquad+\varepsilon^2 s\int_0^T \alpha_{x|x=1}|\psi_{xx|x=1}|^2\,\mathrm{d}t \leq C\varepsilon^2s^5\int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t,
\end{aligned} \tag{5.61}$$

for every $s\geq C(T+\varepsilon^{-1/2}T^{1/2}+|M|^{1/2}\varepsilon^{-1/2}T)$.

Coming back to the original variable.

Let us now go back to the original variable ϕ. First, we point out that the same computations made in (5.60), show that

$$\begin{aligned}
\varepsilon^2s^3\iint_Q \alpha^3|\psi_x|^2\,\mathrm{d}x\,\mathrm{d}t \leq & C\varepsilon^2s^5\iint_Q \alpha^5|\psi|^2\,\mathrm{d}x\,\mathrm{d}t + C\varepsilon^2 s\iint_Q \alpha|\psi_{xx}|^2\,\mathrm{d}x\,\mathrm{d}t\\
&+C\varepsilon^2s^5\int_0^T \alpha^5_{|x=0}|\psi_{|x=0}|^2\,\mathrm{d}t,
\end{aligned}$$

as long as $s \geq CT$. This means that we can add this term to the left-hand side of (5.61), and together with $\psi = e^{-s\alpha}\phi$, we have directly from (5.61) that

$$\begin{aligned} \varepsilon^2 s^5 \iint_Q e^{-2s\alpha}\alpha^5|\phi|^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s^3 \iint_Q \alpha^3|\psi_x|^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s \iint_Q \alpha|\psi_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}t \\ + \varepsilon^2 s^3 \int_0^T \alpha^3_{x|x=1}|\psi_{x|x=1}|^2 \,\mathrm{d}t + \varepsilon^2 s \int_0^T \alpha_{x|x=1}|\psi_{xx|x=1}|^2 \,\mathrm{d}t \\ \leq C\varepsilon^2 s^5 \int_0^T e^{-2s\alpha_{|x=0}}\alpha^5_{|x=0}|\phi_{|x=0}|^2 \,\mathrm{d}t, \end{aligned} \tag{5.62}$$

for every $s \geq C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$.

Now, from $\psi = e^{-s\alpha}\phi$, we find that

$$s^{3/2}e^{-s\alpha}\alpha^{3/2}\phi_x = s^{3/2}\alpha^{3/2}\psi_x + s^{5/2}e^{-s\alpha}\alpha^{3/2}\alpha_x\phi,$$

and taking the $L^2(Q)$-norm, we see that we can add

$$\varepsilon^2 s^3 \iint_Q e^{-2s\alpha}\alpha^3|\phi_x|^2 \,\mathrm{d}x\,\mathrm{d}t$$

to the left-hand side of (5.62). Similarly, from

$$s^{1/2}e^{-s\alpha}\alpha^{1/2}\phi_{xx} = s^{1/2}\alpha^{1/2}\psi_{xx} + s^{3/2}e^{-s\alpha}\alpha^{1/2}\alpha_{xx}\phi + 2s^{3/2}\alpha^{1/2}\alpha_x\psi_x + s^{5/2}e^{-s\alpha}\alpha^{1/2}\alpha_x^2\phi,$$

we can add

$$\varepsilon^2 s \iint_Q e^{-2s\alpha}\alpha|\phi_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}t$$

to the left-hand side of (5.62) if $s \geq CT$. Finally, using (5.51), we can add to the left-hand side of (5.62) the respective boundary integrals and obtain

$$\begin{aligned} \varepsilon^2 s^5 \iint_Q e^{-2s\alpha}\alpha^5|\phi|^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s^3 \iint_Q e^{-2s\alpha}\alpha^3|\phi_x|^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon^2 s \iint_Q e^{-2s\alpha}\alpha|\phi_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}t \\ + \varepsilon^2 s^3 \int_0^T e^{-2s\alpha_{|x=1}}\alpha^3_{x|x=1}|\phi_{x|x=1}|^2 \,\mathrm{d}t + \varepsilon^2 s \int_0^T e^{-2s\alpha_{|x=1}}\alpha_{x|x=1}|\phi_{xx|x=1}|^2 \,\mathrm{d}t \\ \leq C\varepsilon^2 s^5 \int_0^T e^{-2s\alpha_{|x=0}}\alpha^5_{|x=0}|\phi_{|x=0}|^2 \,\mathrm{d}t, \end{aligned} \tag{5.63}$$

for every $s \geq C(T + \varepsilon^{-1/2}T^{1/2} + |M|^{1/2}\varepsilon^{-1/2}T)$. The proof of Proposition 5.3 is complete.

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Résumé

Ce travail est consacré à l'étude de quelques problèmes de contrôlabilité concernant le système de Navier-Stokes, le système de Boussinesq et l'équation de Korteweg-de Vries (KdV). Nous avons choisi de le diviser en deux parties.

Dans la première et principale partie, on est concerné par la contrôlabilité à zéro de plusieurs modèles non linéaires issues de la mécanique des fluides. Plus précisément, on s'intéresse aux contrôles ayant un nombre réduit de composantes. Tout d'abord, dans le Chapitre 2, on obtient la contrôlabilité locale à zéro du système de Navier-Stokes avec contrôles distribués ayant une composante nulle. La nouveauté la plus importante est l'absence de conditions géometriques sur le domaine de contrôle, ce qui améliore et étend des résultats précedents. Dans le Chapitre 3, on étend ce résultat pour le système de Boussinesq, où le couplage avec l'équation de la température nous permet d'avoir jusqu'à deux composantes nulles dans le contrôle agissant sur l'équation du fluide. Dans le même esprit, le Chapitre 4 traite l'existence de contrôles insensibilisants pour le système de Boussinesq avec des contrôles et dans l'équation du fluide et dans l'équation de la température. En particulier, on montre la contrôlabilité à zéro d'un système en cascade issu de la reformulation du problème d'insensibilisation où le contrôle dans l'équation du fluide possède deux composantes nulles. Pour tous ces problèmes, on suit une approche classique. On établit la contrôlabilité à zéro du système linéalisé autour de l'origine par une inégalité de Carleman appropriée pour le système adjoint avec des termes source. Puis, on obtient le résultat pour le système non linéaire par un argument d'inversion locale.

Dans la deuxième partie, on étudie quelques aspects de la contrôlabilité à zéro d'une équation de KdV linéaire avec conditions au bord de type Colin-Ghidaglia et avec un coefficient de dispersion qui tend vers zéro. D'une part, on obtient une estimation du coût de la contrôlabilité à zéro de cette équation qui est optimal par rapport au coefficient de dispersion. Ce résultat améliore les résultats précédents en ce sujet. Sa preuve repose sur l'obtention d'une inégalité de Carleman avec un comportement optimal en temps. D'autre part, on montre que le coût de la contrôlabilité à zéro explose exponentiellement par rapport au coefficient de dispersion lorsque le temps de contrôlabilité est suffisamment petit.

Mots-clés : Système de Navier-Stokes, système de Boussinesq, systèmes non linéaires couplés, contrôlabilité à zéro, contrôles insensibilisants, équation de KdV linéaire, dispersion évanescente, coût de la contrôlabilité à zéro, inégalités de Carleman.

Abstract

This work is devoted to the study of some controllability problems concerning the Navier-Stokes system, the Boussinesq system and the Korteweg-de Vries (KdV) equation. It is divided into two parts.

In the first and main part, we are concerned with the null controllability of several nonlinear models from fluid mechanics. More precisely, we are interested in controls having a reduced number of components. First, in Chapter 2, we obtain the local null controllability of the Navier-Stokes system with distributed controls having one vanishing component. The main novelty is that no geometric condition is imposed on the control domain, which improves and extends previous results. In Chapter 3, we extend this result for the Boussinesq system, where the coupling with the temperature equation allows us to have up to two vanishing components in the control acting on the fluid equation. In the same spirit, Chapter 4 deals with the existence of insensitizing controls for the Boussinesq system with controls on both the fluid and temperature equations. In particular, we prove the null controllability of the cascade system arising from the reformulation of the insensitizing problem where the control on the fluid equation has two vanishing components. For all these problems, we follow a classical approach. We establish the null controllability of the linearized system around the origin by means of a suitable Carleman inequality for the adjoint system with source terms. Then, we obtain the result for the nonlinear system by a local inversion argument.

In the second part, we study some null controllability aspects of a linear KdV equation with Colin-Ghidaglia boundary conditions and with a dispersion coefficient tending to zero. On the one hand, we obtain an estimation of the cost of null controllability of this equation which is optimal with respect to the dispersion coefficient. This result improves previous results on this matter. Its proof relies on obtaining a Carleman estimate with an optimal behavior in time. On the other hand, we prove that the cost of null controllability blows up exponentially with respect to the dispersion coefficient provided that the time of controllability is small enough.

Keywords : Navier-Stokes system, Boussinesq system, nonlinear coupled systems, null controllability, insensitizing controls, linear KdV equation, vanishing dispersion, cost of null controllability, Carleman estimates.